Analysis of Capacitive Heat Exchangers
Part I

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Governing equations are developed for a simple capacitive heat exchanger. This type of heat exchanger consists of hot spherical particles falling through an ascending cold gas stream. The assumptions made in deriving the continuity, momentum and energy equations are clearly stated. The analysis yields a system of first-order, ordinary, nonlinear equations which form a complex boundary-value problem. Part II of this report will present the method of solution together with a comparison between the performance of capacitive heat exchangers and conventional counter flow ones.

I. Introduction

Recently published works (Refs. 1, 2 and 3) propose a new concept in heat exchangers. The main idea is to have particles of a "hot" fluid fall through an ascending "cold" fluid. The heat exchange between the hot and cold streams takes place inside a duct which physically contains the two flows. In most analyses, the "cold" fluid is a gas stream. The origin of the hot particles depends on the applications. For example, when the particles are solid, they can originate in a packed bed; when they are fluid droplets, they can represent the cooling of a melt. Because the hot material consists of discrete particles of a different phase than that of the cooling steam, these heat exchangers have been labeled as capacitive.

Three potential advantages are claimed for capacitive heat exchangers. First, the two streams are in direct contact with each other. This eliminates the resistance to heat transfer which occurs when the fluids are separated by a tube wall. Second, the contact area between the two streams is greatly magnified because the heat exchange takes place on the entire surface of each of the particles. Finally, friction between the two fluids is less than when a solid interface is present. This promises reduced pumping power in capacitive heat exchangers.

Numerous heat exchangers exist throughout the Deep Space Network facilities. They range from those heat exchangers serving the hydrostatic oil bearings of the antennas to a cooling tower for the heating, ventilation and air conditioning system. The large number of heat exchangers and their importance in maintaining the performance of the DSN systems require that possible improvements in their efficiency be thoroughly analyzed. As a result, a systematic investigation of capacitive heat exchangers has been performed and the present report gives some of the results of this investigation.

The physical phenomena that occur in capacitive heat exchangers are quite complex. Because of this complexity, the corresponding mathematical models are based on simplifying assumptions. With few exceptions (Ref. 4) the assumptions are not explicitly stated in the literature and the reader is presented with a set of governing equations whose origin is dubious at best. Thus, one of the aims of this report is to present a detailed and explicit derivation of the basic equations that govern fluid-solid processes. In addition a thermal analysis of a capacitive heat exchanger under a given set of assumptions will also be presented.

In the second part of the report, a numerical solution of the equations will be presented. The numerical model allows a sensitivity analysis to be performed for a certain type of capacitive heat exchanger. Part II of this report will also suggest future lines of research in this field.

The system to be analyzed is shown in Fig. 1. The hot stream is made up of solid particles that fall through an ascending cold gas. The gas forms the continuous phase while the solid particles occupy only discrete volume elements and form the dispersed phase. The two phases (particles and gas) form a heterogeneous mixture with each phase occupying only part of the volume of the mixture.[1] Examples of other possible heterogeneous mixtures are given in Table 1 (Ref. 6).

The particles occupy only a fraction κ out of a unit volume of the mixture. Therefore, the gas occupies the remaining $(1 - \kappa)$ fraction. κ is generally a function of both space and time, $\kappa = \kappa\,(x, y, z, t)$, and is made up of a number of particles per unit volume of the mixture, N_D. If the volume of each particle is denoted by V_D, then

$$\kappa = N_D V_D$$

and N_D is also a function of space and time. The particles are of a material whose density is denoted by ρ_D. Therefore, the particles have a mass per unit volume (of the mixture) given by

$$\lambda_D = \kappa \rho_D = N_D V_D \rho_D$$

λ_D is sometimes called the species (or phase) density of the particles.

The continuous phase occupies $(1 - \kappa)$ out of a unit volume of the mixture and it is made up of a material whose density is denoted by ρ_c. Therefore, the species density for the continuous phase is given by $\lambda_c = (1 - \kappa)\,\rho_c = (1 - N_D V_D)\,\rho_c$. It is important to realize that in the analysis of multiphase flows the species densities are the counterparts of material density in single phase flows.

The mathematical model of capacitive heat exchangers can be considerably simplified by assuming that the flow and heat transfer processes are one-dimensional and steady state. The one-dimensional assumption proposes uniformity of the variables in any given y-plane. There is only limited experimental evidence to back these hypotheses.[2] However, the advantage of these two assumptions is that they reduce the governing set of partial differential equations into ordinary ones. Furthermore, as the purpose of this paper is to analyze the simplest model of capacitive heat exchangers, these assumptions seem warranted at this juncture.

Many authors make the additional assumption that the length scale in which the flow parameters change substantially is much larger than the interparticle distance (Refs. 4, 5, 8). This implies a sufficiently large number of particles in a unit volume of the mixture. When N_D is "large enough," the dispersed phase can be treated as a pseudofluid and its governing equations can be written in the usual continuum form. The precise meaning of "large enough" is unclear even though it has important consequences for the physics of capacitive heat exchangers. For example, a very large number of particles tends to reduce radiative heat transfer between the container wall and the particles (Ref. 9). Furthermore, the question of partial pressures in such mixtures is not quite clear (Ref. 4, 5, 10). Examples of systems where it is justified to assume that N_D is "large enough" are fluidized beds and nuclear reactor accident analysis.

In contrast, an alternative approach is to assume that the volume fraction occupied by the particles is small and therefore the interactions between the particles can be neglected. Marble (Ref. 11) points out that even in this case, continuum relations can be written for the solid phase. For such situations, he notes that the velocities, temperatures and densities which appear in the equations are "values averaged over a gas volume containing several particles." This approach has been shown to give adequate results for the dynamics of dusty gases (Refs. 11, 12, 13). Such continuum conservation equations are especially useful when one discusses complex phenomena such as shock waves, for example (Refs. 11, 14). A similar approach, although more simplified, will be used in the next section.

II. Conservation Equations

The phenomena considered in this report are caused by the interactions between the continuous and dispersed phases.

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[1] This is in contrast with homogeneous mixtures (e.g., solutions, alloys, mixtures of gases) in which each component can be considered as occupying the entire volume of the mixture (Ref. 5).

[2] Counter flow experiments reported by El'Kin and Timofeyev (Ref. 7) indicate that at low particle loading ratios ($\eta \leqslant 7$) the distribution of solids over the cross-sectional area is almost uniform except in the entrance region. However, for higher loading ratios, large nonuniformities in the distribution appear both axially and in cross section.

Therefore, a mathematical analysis of the problem must include equations regarding the conservation of mass, momentum and energy of each of the phases. For both continuous and dispersed phases the equations are written in terms of a unit volume of the mixture. The volume fraction occupied by the particles is assumed to be small and therefore the partial pressure of the particles can be neglected. All particles are assumed to be spheres of identical radius r_D; this radius is assumed to be a known quantity. Closure relations for the problem are given by boundary conditions on the temperatures and velocities.

A. Continuity

1. Dispersed phase. No sources or sinks of mass are assumed to exist within the conduit. Therefore, the mass flow rate is given by:

$$\dot{m} = \rho_D \mathrm{v}_D \kappa A$$

where A is the cross-sectional area of the conduit. With A a constant and with the one-dimensional and steady-state assumptions, the continuity equation is:

$$\frac{d\dot{m}}{dy} = \frac{d}{dy}(\rho_D \mathrm{v}_D \kappa) = \frac{d}{dy}(\rho_D \mathrm{v}_D N_D V_D) = 0$$

If there is no mass transfer, V_D is constant and the equation reduces to:

$$\frac{d\dot{m}}{dy} = \frac{d}{dy}(\rho_D \mathrm{v}_D N_D) = 0 \tag{1}$$

When the material of the particles is such that its density is only a weak function of the temperature, then the continuity equation can be further simplified to:

$$\frac{d}{dy}(\mathrm{v}_D N_D) = 0 \tag{2}$$

In these equations, v_D and N_D are quantities averaged over a unit volume of the mixture at a given location y. Therefore, Eqs. (1) and (2) can be written respectively as:

$$(\rho_D \mathrm{v}_D N_D)_y = (\rho_D \mathrm{v}_D N_D)_{y+\Delta y} \tag{3}$$

$$(\mathrm{v}_D N_D)_y = (\mathrm{v}_D N_D)_{y+\Delta y} \tag{4}$$

2. Continuous phase. In terms of the species densities of the gas, λ_c, the continuity equation is:

$$\frac{\partial \lambda_c}{\partial t} + \nabla \cdot (\lambda_c \mathrm{v}_c) = 0 \tag{5}$$

With the steady-state and one dimensional assumptions this becomes:

$$\frac{d}{dy}(\lambda_c \mathrm{v}_c) = \frac{d}{dy}[(1-\kappa)\rho_c \mathrm{v}_c] = 0 \tag{6}$$

Alternative forms are:

$$\frac{d}{dy}[(1 - N_D V_D)\rho_c \mathrm{v}_c] = 0 \tag{7}$$

and

$$[(1-\kappa)\rho_c \mathrm{v}_c]_y = [(1-\kappa)\rho_c \mathrm{v}_c]_{y+\Delta y} \tag{8}$$

B. Conservation of Momentum

1. Dispersed phase. In terms of the species density, λ_D, conservation of momentum per unit volume of the mixture is given by[3]

$$\lambda_D \frac{D\mathrm{v}_D}{Dt} = \sum F \tag{9}$$

Equation (9) implies that the acceleration of the particles is balanced by the sum of the forces acting on them. The possibility also exists that after an initial acceleration the particles will reach a steady-state velocity. For this case the momentum equation reduces to:

$$\sum F = 0 \tag{10}$$

Both cases will be considered here.

The left-hand side of Eq. (9) is a product between a mass and an acceleration. When an object accelerates through a fluid it must overcome not only the fluid's resistance, but also the inertia of the fluid accelerated by the object. This is known as the added mass effect (Ref. 16). Since this "extra" mass is due to the fluid, it is based on the fluid's density. Therefore, the added mass can be neglected when $\rho_D \gg \rho_c$. This condition is met in gas-particle flows (Ref. 17) such as that discussed here.

The right-hand side of Eqs. (9) and (10) represents the sum of forces acting on the assembly of particles in a unit volume

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[3] A distinction is necessary between "(rate of change of momentum) per unit volume" and "rate of change of (momentum per unit volume)." The former is expressed as $\lambda_D\ (D\mathrm{v}_D/Dt)$ while the latter by $(D/Dt)\ (\lambda_D \mathrm{v}_D)$. For a discussion see Ref. 15.

of the mixture. These forces can be classified as (a) body forces, (b) surface forces and (c) interaction forces. The interaction forces can also be divided into (and lumped with) the body and surface forces. They are kept separated here to indicate their different origin, i.e., that they are due to the continuous phase. The only body forces considered in this report are due to gravitational effects and are given by

$$F_{\text{body}} = \rho_D \kappa g$$

A possible surface force is due to electric charges which reside on the particles (Ref. 18). Since in this report all interactions between particles are neglected, electric forces are neglected:

$$F_{\text{electric}} = 0$$

Three principal phase-interaction forces can occur in fluid-particles flows: the Magnus force, the buoyancy force and the drag force. The Magnus force is due to the rotation acquired by the particles while moving through the fluid.[4] Torobin and Gauvin (Ref. 16) divide this rotation into "screw motion," in which the rotation is perpendicular to the air flow direction, and "top spin," in which the rotation parallels the air flow. The latter is said to be more common and yet to have a negligible effect on the resistance to the motion. Experiments conducted at the Von Karman Institute for Fluid Dynamics (Ref. 20, 21) confirm that the Magnus force can be neglected relative to the other forces. The buoyancy force per particle is given by $\rho_c V_D g$. For an assembly of particles in a unit volume of the mixture, the buoyancy force becomes:

$$F_{\text{buoyancy}} = \rho_c \kappa g$$

The most important interaction force is the drag force. This is a product between a characteristic area of the particle, the kinetic energy of the flow per unit volume and the drag coefficient, C_d. The characteristic area is the surface that the body projects to the flow; for the assembly of particles this is $N_D \pi r_D^2$. The kinetic energy term is given by:

$$\frac{1}{2} \lambda_c v_c^2 = \frac{1}{2}(1 - \kappa)\rho_c v_{net}^2 = \frac{1}{2}(1 - N_D V_D)\rho_c v_{net}^2$$

where v_{net} is the net velocity difference between the two phases. To account for the different directions of the velocities, this can be written as

$$v_{net}^2 = (v_c - v_D)\,|v_c - v_D|$$

In general, the drag coefficient for an assembly of spherical particles is very difficult to determine and quite different from that for single spheres. The results given by Brinkman (Ref. 22) and Tam (Ref. 23) are valid only for low Reynolds numbers ($N_{Re} < 10$). Experiments have shown (Ref. 24) that for $N_{Re} < 300$ and for low values of κ, the drag coefficient is not much different from that of single spheres. For higher Reynolds numbers, the data compiled by Soo (Ref. 25) show a wide variety of results. Figure 2 (Ref. 26) illustrates the variation of the drag coefficient vs the Reynolds number for various values of κ. The large uncertainty in the data has led Rudinger (Ref. 14) to observe that "in numerical evaluations of specific flows, at least the sensitivity of the results to various assumptions for the drag coefficient should be assessed."

In this report the drag coefficient is written as $\overline{C}_d$ to denote that it applies to the sum of the particles in a unit volume of the mixture.

By adding all the terms considered in Eq. (9) and after some algebraic manipulation, the following result is obtained

$$v_D \frac{dv_D}{dy} = \frac{3}{8}\overline{C}_d (1 - N_D V_D)\frac{\rho_c}{\rho_D}\frac{(v_c - v_D)\,|v_c - v_D|}{r_D} - g\left(1 - \frac{\rho_c}{\rho_D}\right) \qquad (11)$$

For the nonaccelerating case (Eq. 10) the sum of the forces yields an expression for the steady-state velocity:

$$(v_c - v_D)\,|v_c - v_D| = \frac{8 r_D\, g\,(\rho_D - \rho_c)}{3\rho_c \overline{C}_d} \qquad (12)$$

Except for the nature of the drag coefficient, Eq. (12) is analogous to the settling velocity of single spheres (Ref. 27).

2. Continuous phase. The momentum balance for the continuous phase can be written as

$$\lambda_c \frac{Dv_c}{Dt} = \sum F$$

In the sum of the forces term, the buoyancy and viscous stresses can be neglected. The latter are considered to be small when compared with the drag produced by the particles. When κ is small, it can be assumed that the partial pressure of the particles can be neglected, and therefore the pressure gradient term, dp/dy, refers to the gas pressure. Thus, the sum of the

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[4] The effect has been mentioned by Newton as early as 1672 and analyzed by Robins in 1805. Magnus worked unsuccessfully on the problem around 1877 (Ref. 19).

forces term includes only gravitational, pressure and drag forces. For this case the momentum balance can be written as:

$$v_c \frac{dv_c}{dy} = g - \frac{1}{\lambda_c}\frac{dp}{dy} - \frac{1}{2}\bar{C}_d (v_c - v_D)\,|v_c - v_D|\,N_D\,\pi r_D^2 \tag{13}$$

C. Conservation of Energy

The general energy equation for a single phase continuum can be written as:

$$\rho \frac{DE}{Dt} = \rho Q - \frac{\partial q_i}{\partial x_i} - p\frac{\partial v_i}{\partial x_i} + \tau_{ij}\frac{\partial v_i}{\partial x_j} \tag{14}$$

The left-hand side of this equation represents the net change of internal energy, E, within a control volume. It includes that part of the internal energy which is added through the system's boundaries by material motion. The terms on the right are, respectively, the contribution of energy sources within the control volume, the net rate of heat addition independent of material motion, the thermodynamic work done by pressure forces against volume changes and finally the irreversible rate of energy dissipation into heat due to viscosity (τ_{ij} is the viscous part of the stress tensor).

In deriving the energy equations for capacitive heat exchangers, it is assumed that no energy sources on sinks exist within the fluid ($\rho Q = 0$). Heat can be added to a system independently of material motion by conduction and/or radiation. The relative importance of radiative heat transfer can be judged from the system's characteristic temperature. At temperatures above 2700°R (1500 K), radiation is the most important mode of heat transfer (Ref. 28); below 1400°R (830 K) radiation effects can be neglected (Ref. 29). For the purpose of this paper, radiative heat transfer can be neglected. Conduction between the gas and the particles is also considered negligible. Therefore, the energy equation is reduced to

$$\rho \frac{DE}{Dt} = -p\frac{\partial v_i}{\partial x_i} + \tau_{ij}\frac{\partial v_i}{\partial x_j} \tag{15}$$

In carrying out practical calculations, temperatures are more useful than internal energy. By invoking the first and second laws of thermodynamics, Eq. (15) can be transformed into a temperature equation. It can be shown (Ref. 30, 31) that two such equations are possible:

$$\rho c_p \frac{DT}{Dt} = \mathscr{D} + \frac{Dp}{Dt} \tag{16}$$

$$\rho c_v \frac{DT}{Dt} = \mathscr{D} - p\frac{\partial v_i}{\partial x_i} \tag{17}$$

In both (16) and (17) the terms ρQ and $\partial q_i/\partial x_i$ have been discarded due to the assumptions previously made. For a Newtonian fluid $\mathscr{D}$ stands for:

$$\mathscr{D} = \left[\mu\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\right]\frac{\partial v_i}{\partial x_k} \tag{18}$$

where the term in brackets is equivalent to τ_{ij} in Eq. (15). Flow velocities in capacitive heat exchangers are relatively low and therefore, from the small Mach number assumption, $\mathscr{D}$ can be considered negligible.

The question is which of the static pressure terms in (16) and (17) is also small? Order-of-magnitude analysis (Ref. 31) shows that Eq. (16) is the one to use for the small Mach number approximation.

1. **Dispersed phase.** Equation (7) is valid for a single phase continuum. For a multiphase situation an additional heat transfer term must be included on the right side of the equation. This term represents the interphase heat transfer due to temperature differences. Since no radiation and conduction are considered in this model, the interphase heat transfer is due to convection only. Hence, under the previously stated assumptions the energy equation for the dispersed phase is

$$\rho_D N_D V_D c_{p_D} \frac{DT}{Dt} = \beta (T_c - T_D) \tag{19}$$

where β is a heat transfer parameter having units of (heat)/(time × temperature × length^3). With the addition of the steady-state and one-dimensional assumptions. Eq. (19) becomes

$$\rho_D N_D V_D c_{p_D} v_D \frac{dT}{dt} = \beta (T_c - T_D) \tag{20}$$

For convective heat transfer

$$\beta = 4\pi r_D^2 N_D \tag{21}$$

i.e., the convective heat transfer coefficient times the surface area of one particle times the number of particles in a unit volume of the mixture. After some algebra the heat transfer equation for the dispersed phase is:

$$\rho_D c_{p_D} v_D \frac{dT_D}{dy} = \frac{3h}{r_D}(T_c - T_D) \tag{22}$$

A simpler derivation of Eq. (22) can be made through the familiar heat exchanger equations (Ref. 32):

$$q = U \Delta S (T_{HOT} - T_{COLD}) = -\dot{m}_{HOT} c_{p_{HOT}} \Delta T_{HOT} \tag{23}$$

where U is the overall heat transfer coefficient and S the heat transfer surface. The total mass of the particles within a unit volume of the mixture is

$$m_{HOT} = \rho_D V_D N_D = \rho_D \frac{4}{3} r_D^3 N_D \tag{24}$$

The mass within this unit volume moves in a time scale given by:

$$t = \frac{\Delta y}{v_D} \tag{25}$$

Therefore, the mass flow rate (per unit volume of the mixture) is given by:

$$\dot{m}_{HOT} = \rho_D \frac{4}{3} \pi r_D^3 N_D \frac{v_D}{\Delta y} \tag{26}$$

Since only convection is considered, $U = h$, and $\Delta S = N_D \, 4 \pi r_D^2$. Therefore, Eq. (23) becomes in the limit:

$$\rho_D c_{p_D} v_D \frac{dT_D}{dy} = \frac{3h}{r_D} (T_c - T_D) \tag{27}$$

2. **Continuous phase.** In terms of the heat exchanger formulas:

$$h \Delta s (T_{HOT} - T_{COLD}) = -\dot{m}_{COLD} c_{p_c} \Delta T_{COLD} \tag{28}$$

After some algebra this becomes

$$\rho_c v_c c_{p_c} \frac{dT_c}{dy} = h \left(\frac{3N_D \pi r_D^2}{\pi r_D^3 N_D - 1} \right) (T_D - T_c) \tag{29}$$

III. Governing System of Equations

Equations (1), (7), (11), (13), (22) and (29) express the conservation of mass, momentum and energy for the dispersed and continuous phases. These six equations contain 12 unknowns as follows:

$$\rho_c, \rho_D, N_D, v_c, v_D, T_c, T_D, p, c_{p_c}, c_{p_D}, h, \bar{C}_d$$

Obviously, additional relations are needed. If one assumes that the continuous phase is made up a perfect gas, then one such additional equation is given by

$$P = \rho_c R T_c \tag{30}$$

Three other relationships must be determined experimentally:

$$c_{p_c} = f_1 (T_c) \tag{31}$$

$$c_{p_D} = f_2 (T_D) \tag{32}$$

$$\rho_D = f_3 (T_D) \tag{33}$$

Note that if ρ_D can be assumed to be constant, Eq. (33) is not needed and Eq. (2) can be used in place of (1).

Relationships for $\bar{C}_d$ and h are more difficult to develop. It is general practice to express these parameters as experimental functions of the dimensionless Reynolds and Prandtl numbers. In a biphasic flow the Reynolds number can be defined as:

$$N_{Re} = \frac{2 \rho_c |v_c - v_D| r_D}{\mu_c} \tag{34}$$

and the Prandtl number as:

$$N_{Pr} = \frac{c_{p_c} \mu_c}{k_c} \tag{35}$$

The heat transfer coefficient, h can be found from the Nusselt number:

$$N_{Nu} = \frac{2h r_D}{k_c} \tag{36}$$

The standard drag curve has been fitted by a number of authors and Clift et al. (Ref. 33) give an up-to-date list of such correlations. In this paper the following references will be used:

From Ref. 34:

$$N_{Re} \leqslant 10, \quad , \quad \bar{C}_d = C_d = 2 + \frac{24}{N_{Re}} \tag{37}$$

From Ref. 35:

$$10 < N_{Re} \leqslant 2000, \quad \bar{C}_d = \frac{27}{N_{Re} \, 0.84} \tag{38}$$

From Ref. 16:

$$2000 < N_{Re} \leqslant 3000, \ \overline{C}_d = \frac{145}{N_{Re}\ 1.16} \tag{39}$$

These relations are for spheres entrained by the flow, but it is assumed that they can also be used for the counter-current case discussed here. The sensitivity of the results to variations in $\overline{C}_d$ will also be analyzed in the second part of this report.

A reliable heat transfer correlation for single sphere is given in Whitaker (Ref. 36):

$$N_{Nu} = 2 + (0.4 N_{Re}^{0.5} + 0.06 N_{Re}^{0.67}) N_{Pr}^{0.4} \left(\frac{\mu_\infty}{\mu_s}\right)^{0.25} \tag{40}$$

where

$$N_{Nu} \equiv \frac{Q}{4\pi D^2 (\Delta T)}$$

For air Eq. (40) can be approximated by

$$N_{Nu} = 2 + (0.4 N_{Re}^{0.5} + 0.06 N_{Re}^{0.67}) \tag{41}$$

because $N_{Pr} = 1$ for air and the viscosity ratio is also about unit. Other relationships are given by Ranz and Marshall (Ref. 37)

$$N_{Nu} = 2.0 + 0.60 N_{Pr}^{1/3} N_{Re}^{1/2} \tag{42}$$

and Bandrowski and Kaczmarzyk (Ref. 38)

$$N_{Nu} = 0.00114\, \kappa^{-0.5984} N_{Re}^{0.8159} \tag{43}$$

Equation (42) is derived from experiments on liquid drops and Eq. (43) from experiments on the pneumatic conveying of small spheres. In Eq. (43) $0.00025 < \kappa < 0.05$ and $180 < N_{Re} < 18{,}000$. The sensitivity of the results to the Nusselt number correlation will also be analyzed in the second part of this report.

One final equation is needed to complete the set:

$$\mu_c = f_4 (T_c) \tag{44}$$

This last equation is needed to evaluate the additional unknown, μ_c, which appears in the Reynolds number. Therefore, the complete set includes 13 equations and 13 unknowns.

IV. Method of Solution

The equations and the unknowns are summarized in Table 2. Boundary and/or initial conditions must be established for these equations before any solution is attempted. In order to do this, the hypothetical performance of a capacitive heat exchanger is illustrated in Fig. 3. If all the parameters were known either at $y = 0$ or at $y = H$, this would be a classic initial value problem. However, because of the countercurrent nature of the heat exchanger, only some of the parameters can be assumed known at $y = 0$, while the others are known only at $y = H$. Therefore, this is a two-point boundary value problem.

Initial value problems are relatively easy to solve by numerical methods. They have been used successfully to analyze particle-gas flows in rocket nozzles (Ref. 39). Unfortunately there are no known algorithms for solving an arbitrary two-point boundary value problem. The so-called shooting methods reduce the solution of a boundary value problem to the iterative solution of an initial value problem (Refs. 40, 41, 42, 43, 44). These methods are based essentially on assuming some of the missing information at the initial point ($y = 0$ for example) and iterating this "trial" solution until the end point ($y = H$ for example). This end point value is obtained by a Runge-Kutta process. The value at the end point is then compared with the actual value, and on the basis of this comparison adjustments are made for a new trial solution. By this process, the true end point value is bracketed by trial solutions until a match is made. In the case of linear equations a particularly efficient method exists.

The governing equations present the added difficulty that some of them have derivatives implicitly defined in terms of other derivatives. It can be shown (Appendix A) that derivative implicitness will remain in the system unless

$$\frac{dN_D}{dy} = \text{constant} \tag{45}$$

Equation (45) implies that N_D is a linear function of y. This assumption, if justified, represents an important simplification in the mathematical solution method. Results published by El'Kin and Timofeyev (Ref. 7) indicate that for loading ratios less than 7, κ (and therefore N_D) is practically constant with y; i.e.,

$$N_D = N_D \big|_{y=0} \left(1 + \epsilon \frac{y}{H}\right) \tag{46}$$

where $\epsilon \ll 1$. When Eq. (46) is valid, in terms of the notation developed in Appendix A,

$$\frac{dx_3}{dy} = \frac{\epsilon x_3|_{y=0}}{H} \qquad (47)$$

and the system of equations becomes:

$$\frac{dx_1}{dy} = MP + N \qquad (48)$$

$$\frac{dx_2}{dy} = CP + DH \qquad (49)$$

$$\frac{dx_3}{dy} = P \qquad (50)$$

$$\frac{dx_4}{dy} = E + FK\,(MP + N) + FLI + G \qquad (51)$$

$$\frac{dx_5}{dy} = H \qquad (52)$$

$$\frac{dx_6}{dy} = I \qquad (53)$$

$$\frac{dx_7}{dy} = J \qquad (54)$$

$$\frac{dx_8}{dy} = K(MP + N) + LI \qquad (55)$$

where none of the terms on the right-hand side contain derivatives.

Numerical techniques for solving such systems have been developed at JPL (Ref. 45). These techniques are described in Part II of this report in which the solution of the system of equations is given. In Part II the results obtained for capacitive heat exchangers will be compared with those obtained for conventional counterflow heat exchangers.

References

1. Decher, R., "The Falling Bead Dry Cooling Tower," Paper 809476 15th Intersociety Energy Conversion Engineering Conference (IECEC), 1980.
2. Shaw, D., Bruckner, A. P., and Hertzberg, A., "A New Method of Efficient Heat Transfer and Storage at Very High Temperatures," Paper 809029, 15th IECEC, 1980.
3. Cease, M. E., "A Model of Direct Contact Heat Transfer for Latent Heat Energy Storage," Paper 809118, 15th IECEC, 1980.
4. Pai, S. I., "A Critical Review of the Fundamental Equations of a Mixture of a Gas and Small Solid Particles," *Z. Flugwiss.*19 (8/9), pp. 353-360, 1971.
5. Nigmatulim, R. I., "Methods of Mechanics of a Continuous Medium for the Description of Multiphase Mixtures," *Prikladnaia Matematika i Mekhanika*, 34(6): 1097-1112, 1970.
6. Kuentzman, P., "French Contribution to Aerodynamics of Gas-Particles Mixtures," in *Flow of Solid Particles in Gases* by G. Rudinger (A. Auriol ed.) AGARD-AG-222, 1976.
7. El'Kin, G. L., and Timofeyev, Yu. B., "Experimental Study of the Distribution of Solid Particles and of the Local Heat Transfer Between the Gas and Solids in a Two-Phase Flow (with One Particle Size)," *Heat Transfer-Sov. Res.*, 8(5), pp. 122-126, 1976.

8. Kraiko, A. N., and Sternin, L. E., "Theory of Flows of a Two-Velocity Continuous Medium Containing Solid or Liquid Particles," *Prikladnaia Matematika i Mekhanika*, 29(3), pp. 418-429, 1965.

9. Guletskaia, I. F., and Lisin, F. N., "Combined Heat Transfer of a Turbulent Dispersed Flow in a Duct," *Zhurnal Prikladnoi Mekhaniki i Technicheskoi Fiziki*, Jul/Aug, pp. 69-79, 1980.

10. Sha, W. T., and Soo, S. L., "Multidomain Multiphase Fluid Mechanics," *Int. J. Heat Mass Transfer*, 21, pp. 1581-1595, 1978.

11. Marble, R. E., "Dynamics of Dusty Gases," *Annual Review of Fluid Mechanics*, Vol. 2, pp. 337-446, 1970.

12. Saffman, P. G., "On the Stability of Laminar Flow of a Dusty Gas," *J. Fluid Mech.*, 13, pp. 120-128, 1962.

13. Genchev, Z. D., and Karpuzov, D. S., "Effects of the Motion of Dust Particles on Turbulence Transport Equations," *J. Fluid Mech.*, 101 (4), pp. 833-842, 1980.

14. Rudinger, G., "Fundamentals and Applications of Gas-Particle Flow", in *Flow of Solid Particles in Gases* by G. Rudinger (A. Auriol ed.), AGARD-AG-222, 1976.

15. Tritton, D. J., *Physical Fluid Dynamics*, Van Nostrand Reinhold, 1977.

16. Torobin, L. B., and Gauvin, W. H., "Fundamental Aspects of Solids-Gas Flow", *Can. J. Chem. Eng.*, 37, pp. 129-141; 37, pp. 167-176; 38, pp. 224-236 (1959); 38, 142-153; 38, pp. 189-200 (1960).

17. Murray, J. D., "On the Mathematics of Fluidization – Part 1. Fundamental Equations and Wave Propagation," *J. Fluid Mech.*, 21(3), pp. 465-493, 1965.

18. Soo, S. L., "Heat Transfer Processes of Particulate Suspensions," in *Advanced Heat Transfer*, (B. T. Chao ed.), U. of Illinois Press, Urbana, 1969.

19. Barkla, H. M., and Auchterlonie, L. J., "The Magnus or Robins Effect on Rotating Spheres," *J. Fluid Mech.*, 47(3), pp. 437-447, 1971.

20. Ribas, R., "Kinetic Model for a Gas-Particle Flow Through a Horizontal Pipe," VKI Project Report, 1977.

21. Ribas, R., Laurenco, L., and Riethmuller, M. L., "A Kinetic Model for a Gas-Particle Flow," VKI preprint 1980-4, Pneumotransport 5 Conference, 1980.

22. Brinkman, H. C., "A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles", Appl. Sci. Res., A1, p 27ff, 1947.

23. Tam, C.K.W., "The drag on a cloud of spherical particles in low Reynolds number flow," *J. Fluid Mech.*, 38(3), pp. 537-546, 1969.

24. Johnstone, H. F., Pigford, R. L., and Chapin, J. H., "Heat Transfer to Clouds of Falling Particles," *Am. Inst. Chem. Eng.*, pp. 95-133, 1941.

25. Soo, S. L., *Fluid Dynamics of Multiphase Systems*, Blaisdell, Waltham, Mass., 1967.

26. Schetz, J. A., "Two Phase Flows," Ch. 6 in *Injection and Mixing in Turbulent Flow*, Vol. 68, Progress in Astronautics and Aeronautics, AIAA, 1980.

27. Wallis, G. B., *One-dimensional Two-phase Flow*, McGraw-Hill, N.Y., 1969.

28. Schneider, W., "Radiant Interchange Among Suspended Particles and Its Effect on Thermal Relaxation in Gas-Particle Mixtures," 23rd Proc. Heat Transfer and Fluid Mechanics Institute, pp. 353-370, 1972.

29. Tien, C. L., "Heat Transfer by a Turbulently Flowing Fluid-Solids Mixture in a Pipe," ASME., *J. Heat Transfer*, pp. 183-188, 1961.

30. Chao, B. T., "Selected Topics on Convective Heat Transfer," in *Advanced Heat Transfer,* (B. T. Chao ed.), U. of Illinois Press, Urbana, 1969.

31. Corrsin, S., *Notes on Fluid Mechanics,* Johns Hopkins Univ., Baltimore, 1968.

32. Kays, W. M., and London, A. L., *Compact Heat Exchangers*, 2nd ed. McGraw-Hill Book Co., N. Y., 1964.

33. Clift, R., Grace, J. R., and Weber, M. E., *Bubbles, Drops and Particles,* Academic Press, N. Y., 1978.

34. Kurten, H., Raasch, J., and Rumpf, H., *Chem. Eng. Tech.*, Vol 38, pp. 941-948, 1966 (also in Ref. 33).

35. Ingebo, R. D., *Drag Coefficients for Droplets and Solid Spheres in Clouds Accelerated in Air Streams*, NACA TN 3762, 1956.

36. Whitaker, S., "Forced Convection Heat Transfer Correlations for Flow in Pipes, Past Flat Plates, Single Cylinders, Single Spheres, and for Flow in Packed Beds and Tube Bundles," A. I. Ch. E. J., vol. 18(2), pp. 361-371, 1972.

37. Ranz, W. E., and Marshall, Jr., W. R., "Evaporation from Drops," *Chemical Engineering Progress*, vol. 48(4), pp. 173-179, 1952.

38. Bandrowski, F., and Kaczmarzyk, G., "Gas-to-Particle Heat Transfer in Vertical Pneumatic Conveying of Granular Materials," *Chem. Eng. Sci.*, Vol. 33, pp. 1303-1310, 1978.

39. Neilson, J. H., and Gilchrist, A., "An analytical and experimental investigation of the velocities of particles entrained by the gas flow in nozzles," *J. Fluid Mech.*, Vol. 33 (1), pp. 131-149, 1968.

40. Osborne, M. R., "On Shooting Methods for Boundary Value Problems," *J. Math. Anal. Applic.*, Vol. 27, pp. 417-433, 1969.

41. Stoer, J., and Bulirsch, R., *Introduction to Numerical Analysis*, Springer-Verlag, N.Y., 1980.

42. Carnahan, B., Luther, H. A., and Wilkes, J. O., *Applied Numerical Methods*, J. Wiley, N.Y., 1969.

43. Roberts, S. M., and Shipman, J. S., *Two-Point Boundary Value Problems: Shooting Methods*, Elsevier, N.Y., 1972.

44. Keller, H. B., *Numerical Methods for Two-Point Boundary Value Problems* Blaisdell, London, 1968.

45. Krogh, F. T., Personal Communication, May 1981.

Definition of Symbols

A	cross-sectional area of the conduit
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
$C_d, \overline{C}_d$	drag coefficient
$\mathscr{D}$	kinetic energy dissipation rate
E	internal energy
g	gravitational acceleration
H	maximum length (height) of the heat exchanger
h	convective heat transfer coefficient
k	conductive heat transfer coefficient
$\dot{m}$	mass flow rate
N_D	number of particles per unit volume of mixture
p	pressure
Q	energy sources per unit volume
q	heat per unit volume
R	gas constant
r	radius
T	temperature
t	time
V	volume
v	velocity
y	axial distance for the heat exchanger
β	heat transfer parameter
η	loading ratio, equal to $\lambda_D \mathrm{v}_D / \lambda_c \mathrm{v}_c$
κ	volume fraction occupied by particles
λ	species density, equal to volume fraction times material density
μ	coefficient of viscosity
ρ	material density
τ_{ij}	stress tensor

Suffixes

c	pertains to continuous phase (gas)
D	pertains to dispersed phase (particles)

Table 1. Examples of heterogeneous mixtures

Liquid fluid phase:

- Solid dispersed phase: suspension
- Liquid dispersed phase: emulsion
- Gaseous dispersed phase: foam

Gaseous fluid phase:

- Solid dispersed phase: suspension, smoke, fluidized bed
- Liquid dispersed phase: mist, fog, aerosol

Table 2. Summary of equations and unknowns

Equation type	Equation	Unknowns
Continuity (continuous phase)	$\frac{d}{dy}\left[(1-N_D V_D)\,\rho_c\,\mathrm{v}_c\right]=0$	N_D, ρ_c, ν_c
Continuity (dispersed phase)	$\frac{d}{dy}\left(\rho_D N_D \mathrm{v}_D\right)=0$	ρ_D, ν_D
Momentum (continuous phase)	$\mathrm{v}_c\frac{d\mathrm{v}_c}{dy}=-g-\frac{1}{\rho_c(1-N_D V_D)}\frac{dp}{dy}+\frac{1}{2}\overline{C}_d\,\mathrm{v}_{net}^2 N_D\,\pi r_D^2$	$p, \overline{C}_d$
Momentum (dispersed phase)	$\mathrm{v}_D\frac{d\mathrm{v}_D}{dy}=\frac{3}{8}\overline{C}_d(1-N_D V_D)\frac{\rho_c}{\rho_D}\frac{\mathrm{v}_{net}^2}{r_D}-g\left(1-\frac{\rho_c}{\rho_D}\right)$	–
Energy (continuous phase)	$\rho_c\mathrm{v}_c c_{p_c}\frac{dT_c}{dy}=-h\left(\frac{4\pi N_D r_D^2}{1-N_D V_D}\right)(T_D-T_c)$	T_D, T_c, h, c_{p_c}
Energy (dispersed phase)	$\rho_D\mathrm{v}_D c_{p_D}\frac{dT_D}{dy}=\frac{3h}{r_D}(T_c-T_D)$	c_{p_D}
State	$p=\rho_c R T_c$	–
Drag coefficient correlation	$\overline{C}_d=f_1(N_{Re})$	μ_c
Nusselt Nr. correlation	$h=f_2(N_{Re})$	–
Viscosity correlation	$\mu=f_3(T_c)$	–
c_{p_c} correlation	$c_{p_c}=f_4(T_c)$	–
c_{p_D} correlation	$c_{p_D}=f_5(T_D)$	–
Density correlation	$\rho_D=f_6(T_D)$	–

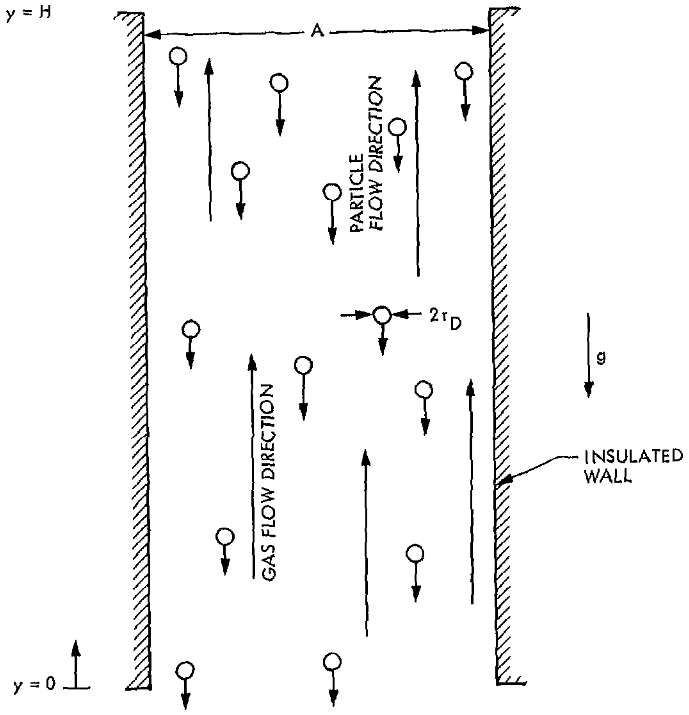

Fig. 1. *Capacitive heat exchanger model*

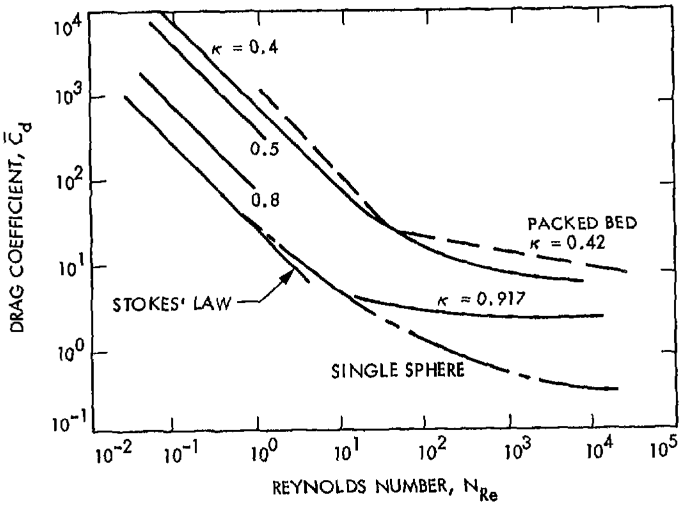

Fig. 2. Drag coefficient of spheres in a cloud

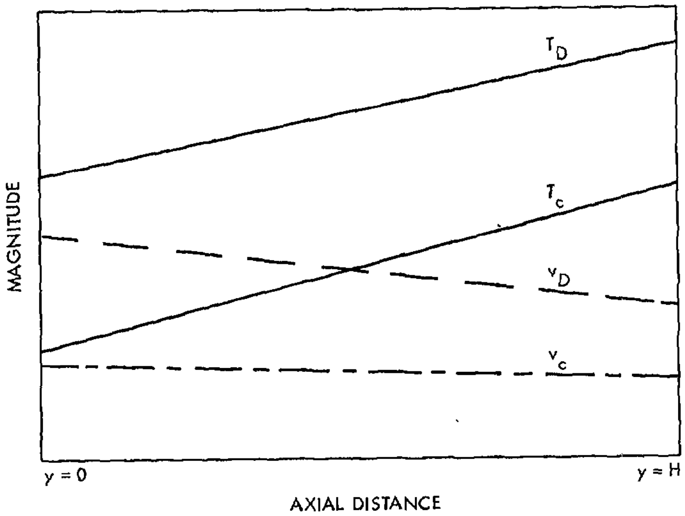

Fig. 3. Hypothetical performance of a capacitive heat exchanger

Appendix A

Elimination of Derivative Implicitness for the System Equations

Consider the following symbol transformation:

$$x_1 = \rho_c$$

$$\alpha_1 = V_D = \frac{4}{3}\pi r_D^3$$

$$x_2 = \rho_D$$

$$x_3 = N_D$$

$$\alpha_2 = g$$

$$x_4 = \mathrm{v}_c$$

$$x_5 = \mathrm{v}_D$$

$$\alpha_3 = \pi r_D^2$$

$$x_6 = T_c$$

$$x_7 = T_D$$

$$\alpha_4 = r_D$$

$$x_8 = p$$

Then, the first seven equations from Table 2 can be written respectively as:

$$\frac{dx_1}{dy} = \frac{\alpha_1 x_1 x_4}{x_4 - \alpha_1 x_3 x_4}\frac{dx_3}{dy} - \frac{x_1 - \alpha_1 x_1 x_3}{x_4 - \alpha_1 x_3 x_4}\frac{dx_4}{dy} \qquad \text{(A-1)}$$

$$\frac{dx_2}{dy} = \frac{-x_2}{x_3}\frac{dx_3}{dy} - \frac{x_2}{x_5}\frac{dx_5}{dy} \qquad \text{(A-2)}$$

$$\frac{dx_4}{dy} = \frac{-\alpha_2}{x_4} - \frac{1}{x_1 x_4 (1 - \alpha_1 x_3)}\frac{dx_8}{dy} + \frac{1}{2}\bar{C}_d (x_4 + x_5)^2 \frac{x_3}{x_4}\alpha_3 \qquad \text{(A-3)}$$

$$\frac{dx_5}{dy} = \frac{3}{8}\bar{C}_d (1 - \alpha_1 x_3)\frac{x_1}{x_2 x_5}\left(\frac{x_4 + x_5}{\alpha_4}\right)^2 - \frac{\alpha_2}{x_5}\left(1 - \frac{x_1}{x_2}\right) \qquad \text{(A-4)}$$

$$\frac{dx_6}{dy} = \frac{h}{c_{p_c}}\left(\frac{4\alpha_3 x_3}{\alpha_1 x_3 - 1}\right)\frac{(x_7 - x_6)}{x_1 x_4} \qquad \text{(A-5)}$$

$$\frac{dx_7}{dy} = \frac{3h}{\alpha_4 x_2 x_5 c_{p_D}}(x_6 - x_7) \qquad \text{(A-6)}$$

$$\frac{dx_8}{dy} = Rx_6\frac{dx_1}{dy} + Rx_1\frac{dx_6}{dy} \qquad \text{(A-7)}$$

Equation (A-7) was obtained by differentiating (17) with respect to y. Equations (A-1), (A-2), (A-3) and (A-7) define derivatives implicitly in terms of other derivatives. Although the nature of the system is such that this implicitness cannot be eliminated completely, some simplification is possible as follows. Let:

$$A = \frac{\alpha_1 x_1 x_4}{x_4 - \alpha_1 x_3 x_4} \qquad \text{(A-8)}$$

$$B = \frac{x_1 - \alpha_1 x_1 x_3}{x_4 - \alpha_1 x_3 x_4} \qquad \text{(A-9)}$$

$$C = \frac{x_2}{x_3} \qquad \text{(A-10)}$$

$$D = \frac{x_2}{x_5} \qquad \text{(A-11)}$$

$$E = -\frac{\alpha_2}{x_4} \qquad \text{(A-12)}$$

$$F = -\frac{1}{x_1 x_4 (1 - \alpha_1 x_3)} \qquad \text{(A-13)}$$

$$G = \frac{1}{2}\bar{C}_d (x_4 + x_5)^2 \frac{x_3}{x_4}\alpha_3 \qquad \text{(A-14)}$$

$$H = \frac{3}{8}\,\overline{C}_d\,(1-\alpha_1 x_3)\,\frac{x_1}{x_2 x_5}\left(\frac{x_4+x_5}{\alpha_4}\right)^2 - \frac{\alpha_2}{x_5}\left(1-\frac{x_1}{x_2}\right) \tag{A-15}$$

$$I = \frac{h}{c_{p_c}}\left(\frac{4\,\alpha_3 x_3}{\alpha_1 x_3 - 1}\right)\frac{(x_7-x_6)}{x_1 x_4} \tag{A-16}$$

$$J = \frac{3h}{\alpha_4 x_2 x_5\; c_{p_D}}\,(x_6-x_7) \tag{A-17}$$

$$K = Rx_6 \tag{A-18}$$

$$L = Rx_1 \tag{A-19}$$

Then, Eqs. (A-1) through (A-7) can be written as

$$\frac{dx_1}{dy} = A\,\frac{dx_3}{dy} + B\,\frac{dx_4}{dy} \tag{A-20}$$

$$\frac{dx_2}{dy} = C\,\frac{dx_3}{dy} + D\,\frac{dx_5}{dy} \tag{A-21}$$

$$\frac{\mathrm{d}x_4}{dy} = E + F\,\frac{dx_8}{dy} + G \tag{A-22}$$

$$\frac{dx_5}{dy} = H \tag{A-23}$$

$$\frac{dx_6}{dy} = I \tag{A-24}$$

$$\frac{dx_7}{dy} = J \tag{A-25}$$

$$\frac{dx_8}{dy} = K\,\frac{dx_1}{dy} + L\,\frac{dx_6}{dy} \tag{A-26}$$

From (A-24) and (A-26)

$$\frac{dx_8}{dy} = K\,\frac{dx_1}{dy} + LI \tag{A-27}$$

From (A-22) and (A-27)

$$\frac{dx_4}{dy} = E + FK\,\frac{dx_1}{dy} + FLI + G \tag{A-28}$$

From (A-20) and (A-28)

$$\frac{dx_1}{dy} = A\,\frac{dx_3}{dy} + BE + BFK\,\frac{dx_1}{dy} + BFLI + BG \tag{A-29}$$

From (A-21) and (A-23)

$$\frac{dx_2}{dy} = C\,\frac{dx_3}{dy} + DH \tag{A-30}$$

Equation (A-29) can be written as

$$\frac{dx_1}{dy} = \frac{A}{1-BFK}\,\frac{dx_3}{dy} + \frac{BE+BFLI+BG}{1-BFK} = M\,\frac{dx_3}{dy} + N \tag{A-31}$$

where

$$M = \frac{A}{1-BFK} \text{ and } N = \frac{BE+BFLI+BG}{1-BFK}$$

Then, the two equations

$$\frac{dx_1}{dy} = M\,\frac{dx_3}{dy} + N \tag{A-31}$$

$$\frac{dx_2}{dy} = C\,\frac{dx_3}{dy} + DH \tag{A-30}$$

Equations (A-30) and (A-31) indicate that derivative implicitness will remain in the system unless further assumptions are made with regard to dx_3/dy.