Capacity Limit of the Noiseless, Energy-Efficient Optical PPM Channel

J. R. Lesh
Communications Systems Research Section

We examine the power-efficient capacity of the noiseless optical PPM channel. It is shown that even though the capacity per photon can be made to increase without bound, the capacity per channel use (for best power efficiency) is always less than 2 nats per symbol. Furthermore, it approaches 2 nats per symbol as the bandwidth expansion factor goes to infinity.

I. Introduction

In Ref. 1 a method was described for maximizing the energy efficiency of a noiseless PPM optical channel subject to simultaneous constraints on the throughput capacity and bandwidth. Specifically, we considered a Q-ary erasure channel often used to model a Q-ary PPM optical communication system. This channel is known to have a capacity C_S per channel use of

$$C_S = \left(1 - e^{-N_S \Delta T}\right) \ln Q \quad \text{nats/channel use} \tag{1}$$

where N_S is the intensity of the optical source as seen by the receiver during the PPM pulse (measured in photons/second) and ΔT is the PPM slot width in seconds. Since there are $Q\Delta T$ seconds required for each channel use and each PPM symbol contains, on the average, $N_S \Delta T$ photons, then one can specify the channel capacity in terms of

$$C_T = \frac{C_S}{Q\Delta T} = \frac{1 - e^{-N_S \Delta T}}{\Delta T} \frac{\ln Q}{Q} \quad \text{nats/second} \tag{2}$$

and

$$C_{Ph} = \frac{C_S}{N_S \Delta T} = \frac{1 - e^{-N_S \Delta T}}{N_S \Delta T} \ln Q \quad \text{nats/photon.} \tag{3}$$

In Ref. 1 we simultaneously fixed the throughput capacity C_T and the slot width (or equivalently the system bandwidth) ΔT. Then, C_{Ph} was plotted as a function of Q. From such plots one could determine Q^*, the value of Q for which the capacity/photon, C_{Ph}, was maximized. In Ref. 1 we also observed that the product $C_T \Delta T Q^*$ was approximately constant. In this article we will examine this latter topic in more detail. This will lead to what can effectively be interpreted as a Shannon limit for the noiseless, power-efficient optical PPM channel.

II. Analysis

We must first determine the requirements on the optimizing value of Q. A necessary condition for Q^* can be obtained by setting the derivative of C_{Ph} with respect to Q equal to zero.

However, before this can be done note that if C_T and ΔT are fixed, any variation of Q must be offset by compensating changes in N_S (see Eq. 2). Thus, we must explicitly show the dependence of N_S on Q. From (2) we have that

$$N_S = -\frac{1}{\Delta T} \ln \left(1 - \frac{C_T \Delta T Q}{\ln Q}\right). \tag{4}$$

Now, substituting (4) into (3) and differentiating we obtain the necessary condition

$$\left.\frac{Z(Q)}{1 - Z(Q)} \left[1 - \frac{1}{\ln Q}\right] + \ln\left[1 - Z(Q)\right]\right|_{Q=Q^*} = 0 \tag{5}$$

where

$$Z(Q) = \frac{C_T \Delta T Q}{\ln Q} . \tag{6}$$

Thus, once we specify the product $\alpha \stackrel{\Delta}{=} C_T \Delta T$ (which is equivalently the capacity per slot), Q^* can be determined by solving Eq. (5) numerically.

Table 1 shows the results of such calculations for a wide range of α's. Both Q^* and $C_T \Delta T Q^*$ are shown for each α. Clearly, as α decreases there is a compensating increase in Q^* such that for most values of α, $C_T \Delta T Q^*$ is slightly less than 2.

We will now interpret these results in an even more interesting way. From Eqs. (1) and (2) we recognize that $C_T \Delta T Q$ is simply C_S, the channel capacity in nats/channel use. Thus, $C_T \Delta T Q^*$ is nothing more than C_S evaluated at the most energy-efficient operating point. We shall denote this quantity C_S^*. Additionally, we note that if α decreases, either ΔT decreases (bandwidth used increases) for fixed C_T or the required throughput capacity decreases for fixed bandwidth. Both statements are equivalent to saying that the ratio of available bandwidth to information bandwidth is increasing. In Fig. 1 C_S^* is plotted as a function of this bandwidth expansion. In the appendix we prove that $C_S^* < 2$ and approaches 2 in the limit. Thus, we have established the following fundamental property:

For the noiseless optical PPM channel, the most energy-efficient use of the channel results in an information per channel use rate C_S^* of less than 2 nats per channel use. Furthermore, C_S^* approaches 2 nats per channel use as the bandwidth expansion factor approaches ∞.

III. Discussion

It is well known that the capacity of the noiseless PPM channel *measured in nats/photon* can be made infinite by allowing the word size (and hence the bandwidth expansion) to go to infinity. It is therefore surprising that the capacity *in nats/channel use* limits out at 2. This is because, as α decreases, Q^* increases, Z^* goes to zero and the average energy per pulse (i.e., per channel use) also goes to zero as seen by Eq. (4). Thus

$$C_{Ph}^* = \frac{C_S^*}{N_S^* \Delta T}$$

goes to infinity because $N_S^* \Delta T$ goes to zero.

The formulation of C_S^* bears a striking resemblance to the Shannon Limit for the additive white gaussian noise channel. For the AWGN Shannon showed that the energy required to transmit reliably one bit of information, E_b, normalized by the one-sided noise power spectral density N_0 is lower bounded by

$$\frac{E_b}{N_0} > \ln 2$$

and that E_b/N_0 can be made to approach this bound as the bandwidth expansion of the signal approaches ∞. For the noiseless optical PPM Channel, N_0 vanishes (quantum noise does not) so the energy required per bit can be made arbitrarily small. However, the capacity per power efficient use of the channel is still limited and approaches 2 nats/channel use as the bandwidth expands. As in the gaussian channel case, the optical channel limiting behavior is obtained by a sequence of increasing complexity orthogonal (PPM in this case) modulation schemes.

Reference

1. Butman, S. A., Katz, J., and Lesh, J. R., "Practical Limitation on Noiseless Optical Channel Capacity," *TDA Progress Report 42-55*, Jet Propulsion Laboratory, Pasadena, Calif., Feb. 15, 1980.

Table 1. Optimized values of Q and $C_T\Delta TQ$

α	Q^*	$C_T\Delta TQ^*$
10^0	0	0
$10^{-0.7}$	7.13	1.42
10^{-1}	15.7	1.57
10^{-2}	176	1.76
10^{-3}	1.83×10^3	1.83
10^{-4}	1.87×10^4	1.87
10^{-5}	1.89×10^5	1.89
10^{-6}	1.91×10^6	1.91
10^{-7}	1.92×10^7	1.92
10^{-8}	1.93×10^8	1.93
10^{-9}	1.94×10^9	1.94
10^{-10}	1.94×10^{10}	1.94
10^{-11}	1.95×10^{11}	1.95
10^{-12}	1.95×10^{12}	1.95
10^{-14}	1.96×10^{14}	1.96
10^{-20}	1.97×10^{20}	1.97
10^{-30}	1.98×10^{30}	1.98

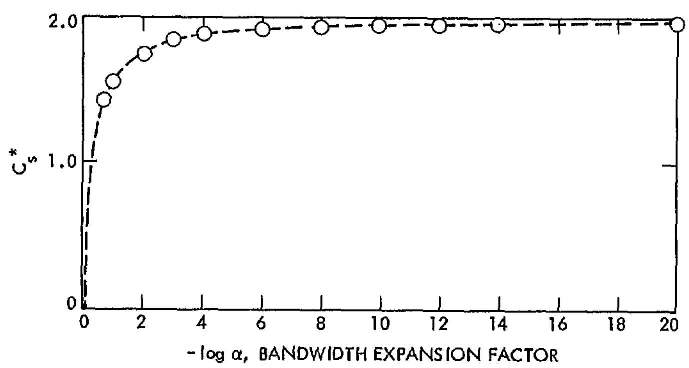

Fig. 1. Variation of energy-efficient channel capacity/channel use with bandwidth expansion

Appendix

We wish to show that $C_S^* < 2$ and asymptotically approaches 2 as $\alpha \to 0$. From (5) we have that the optimizing Q for any α satisfies the relation

$$\frac{Z^*}{1 - Z^*}\left[1 - \frac{1}{\ln Q^*}\right] + \ln(1 - Z^*) = 0$$

where

$$Z^* = \frac{C_T \Delta T Q^*}{\ln Q^*} = \frac{C_S^*}{\ln Q^*} .$$

Using this last equality we can rewrite the optimality condition as

$$C_S^* = f(Z^*)$$

where

$$f(Z) = \frac{Z^2}{Z + (1 - Z)\ln(1 - Z)}$$

From (2) we see that

$$Z = 1 - e^{-N_S \Delta T}$$

so that $0 < Z < 1$. ($Z = 1$ corresponds to infinite energy per pulse whereas $Z = 0$ corresponds to no energy per pulse.) We now claim that $f(Z) < 2$ for all $Z \in (0, 1)$. To prove this we ask if

$$\frac{Z^2}{Z + (1 - Z)\ln(1 - Z)} \overset{?}{<} 2 \qquad \text{(A-1)}$$

We will prove in a moment that $Z + (1 - Z)\ln(1 - Z) > 0$ for $Z \in (0, 1)$. Thus, (A-1) can be rewritten as

$$\frac{\left(\frac{Z^2}{2} - Z\right)}{1 - Z} \overset{?}{<} \ln(1 - Z)$$

Now, by expanding $1/1 - Z$ and $\ln(1 - Z)$ in their corresponding power series representations on the open interval of Z yields

$$\left(\frac{Z^2}{2} - Z\right) \sum_{n=0}^{\infty} Z^n \overset{?}{<} - \sum_{n=1}^{\infty} \frac{Z^n}{n}$$

Expanding and cancelling common terms gives

$$\frac{Z^3}{2} + \frac{Z^4}{2} + \frac{Z^5}{2} + \cdots \overset{?}{>} \frac{Z^3}{3} + \frac{Z^4}{4} + \frac{Z^5}{5} + \cdots$$

Each term on the left dominates the corresponding term on the right, so the assertion is clearly true whenever

$$Z + (1 - Z)\ln(1 - Z) > 0.$$

By the same procedure, this is true if

$$\frac{Z}{1 - Z} > - \ln(1 - Z)$$

Expanding both sides gives

$$Z + Z^2 + Z^3 + Z^4 + \cdots > Z + \frac{Z^2}{2} + \frac{Z^3}{3} + \frac{Z^4}{4} + \cdots$$

which is clearly true.

Finally, we note that as the bandwidth expansion factor increases, α decreases and $Z \to 0$. This is true whether or not Q^* increases (which it does) by virtue of the fact that $C_T \Delta T Q^*$ is bounded. For this reason we are interested in $f(0)$. By applying L'Hospital's rule twice to $f(Z)$ we see that $f(0)$ does indeed equal 2. Thus, C_S^* is less than 2 and approaches 2 in the limit as $\alpha \to 0$.