An Efficient Sampling Technique for Sums of Bandpass Functions

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A well known sampling theorem states that a bandlimited function can be completely determined by its values at a uniformly placed set of points whose density is at least twice the highest frequency component of the function (Nyquist rate). A less familiar but important sampling theorem states that a bandlimited narrowband function can be completely determined by its values at a properly chosen, nonuniformly placed set of points whose density is at least twice the passband width. This allows for efficient digital demodulation of narrowband signals, which are common in sonar, radar and radio interferometry, without the side effect of signal group delay from an analog demodulator. This paper extends this theorem by developing a technique which allows a finite sum of bandlimited narrowband functions to be determined by its values at a properly chosen, nonuniformly placed set of points whose density can be made arbitrarily close to the sum of the passband widths. Applications and a multidimensional extension of this technique will be discussed in a future paper.

I. Introduction and Statement of Result

A set S of real numbers is *bounded* if it is contained in a finite interval. The *content* of a bounded set S is the unique real number, denoted by $cont(S)$, which satisfies the following two conditions:

Condition 1: Whenever $\{[a_i, b_i]: i = 1, \ldots, M\}$ is a finite collection of closed intervals such that

$$S \subset \bigcup_{i=1}^{M} [a_i, b_i] \tag{1}$$

then

$$cont(S) \leqslant \sum_{i=1}^{M} (b_i - a_i) \tag{2}$$

Condition 2: cont(S) is the largest number which satisfies condition 1.

In particular, if S is a finite union of finite intervals, then S is bounded and $cont(S)$ is the sum of the lengths of the intervals.

A function $f(t)$ is called *bandlimited* if its Fourier transform $F(x)$ defined by

$$F(x) = \int_{-\infty}^{\infty} f(t) \exp(-2 \pi i x t)\, dt \tag{3}$$

satisfies the following property:

$$S = \{x \text{ such that } F(x) \neq 0\} \text{ is bounded} \tag{4}$$

The *frequency content* of a bandlimited function f is the content of the set S in (4).

The object of this paper is to prove the following result:

Theorem 1: Given any bounded set S of real numbers and any real number $\epsilon > 0$, there exists an integer P, real numbers T_i for $1 \leqslant i \leqslant P$, a real number $T > 0$, and functions $s_i(t)$ for $1 \leqslant i \leqslant P$ such that, for *every* function $f(t)$ whose Fourier transform vanishes outside of S, the following equation is valid:

$$f(t) = \sum_{i=1}^{P} \sum_{N=-\infty}^{+\infty} f(NT + T_i) s_i(t - NT - T_i) \qquad (5)$$

and furthermore

$$cont(S) + \epsilon > P/T \qquad (6)$$

Equation (5) expresses the fact that $f(t)$ can be determined by its values on (or *sampled* on) a set of points $\{NT + T_i\}$, called a *sampling set*, whose average *density*, which is P/T, can be chosen to be arbitrarily close to the frequency content of $f(t)$. In the proof of theorem 1 the *order P, step size T, phases* T_i, and *sampling functions* $s_i(t)$ will be specified. The following observations place theorem 1 in a historical perspective.

The classical sampling in Refs. 1 and 2 corresponds to theorem 1 where

$$S = [-W, W], P = 1, T_1 = 0, T = 1/2W$$

and

$$s_i(t) = \mathrm{Sin}(2\pi Wt)/(2\pi Wt)$$

In this case, the sampling set has density = $cont(S) = 2W$ and consists of uniformly placed points.

Kohlenberg's sampling theorem in Ref. 3 for narrowband functions corresponds to theorem 1 where

$$W_0 > 0, S = [-W_0 - W, -W_0]\ U\ [W_0, W_0 + W], P = 2,$$

$$T_1 = 0$$

T_2 is subject to weak restrictions, $T = 1/W$ and $s_1(t)$ and $s_2(t)$ are rather complicated. In this case, the sampling set has density = $cont(S) = 2W$ and consists of nonuniformly placed points. Applications of this sampling technique (and of the more restrictive phase quadrature technique) to radar, sonar and radio interferometry are discussed in Refs. 4-6.

Any function $f(t)$ which represents a single channel electronic signal will be real valued; hence its Fourier transform $F(x)$ will satisfy the relation $F(-x)$ = complex conjugate of $F(x)$ and the set S in (4) will be symmetric about the origin. For this reason, the passband of a narrowband signal is usually specified by the set

$$[W_0, W_0 + W]$$

rather than the set

$$S = [-W_0, -W_0 - W]\ U\ [W_0, -W_0 + W].$$

However, this paper treats complex valued functions because the analysis involved is simpler and because complex random processes are of interest in applied analysis (see Ref. 7). Also, it should be noted that theorem 1 can easily be extended to wide sense stationary random processes whose spectral densities satisfy the hypothesis of theorem 1.

Many signals which occur in spread spectrum communication, navigation and remote sensing (radar, sonar, laser scanning) can be modelled as finite sums of narrowband functions whose passbands are widely separated with respect to their widths. A special example, in which each of the passbands is extremely narrow, arises for the Mobil Automated Field Instrument System (MAFIS) navigation system being developed at the Jet Propulsion Laboratory (communicated by Dr. William Hurd). In this system, four nearly pure frequency tones are transmitted simultaneously and the received signal is digitally sampled. The calculated phases associated with each tone are converted to navigation ranging data. This phase estimation problem is a special case of the problem discussed in Ref. 8. However, for the more general case where the passbands are wider, theorem 1 is appropriate.

Applications of theorem 1 and of a multivariable extension of theorem 1 will be discussed in a sequel to this paper. The latter applications include digital sampling strategies for imaging radar and optical systems.

II. Derivation of Main Result

The proof of theorem 1 requires the following formula which was known to Gauss (Ref. 9).

If $F(x)$ is the Fourier transform of $f(t)$ then for every value of T, x, and y:

$$T^{1/2}\exp(\pi i\,x\,y) \sum_{N=-\infty}^{+\infty} f(NT+y)\exp(2\pi i\,NT x) = \tag{7}$$

$$T^{-1/2}\exp(-\pi i\,x\,y) \sum_{N=-\infty}^{+\infty} F(NT^{-1}-x)\exp(2\pi i\,NT^{-1}\,y)$$

(note that the left side of Eq. (7) is transformed to the right side by the substitution $T \to T^{-1}$, $x \to y$, $y \to -x$ and $f \to F$; the beauty of this symmetry is reflected in Gauss' title to Ref. 9). This formula has (mistakenly) acquired the name of the Poisson summation formula under which it is to be found in most serious books on signal processing (Ref. 10) and Fourier analysis (Refs. 11 and 12).

The proof of theorem 1 also requires the Fourier inversion theorem (see Refs. 10-12), which states that a function $f(t)$ can be determined by its Fourier transform $F(x)$ as follows:

$$f(t) = \int_{-\infty}^{+\infty} F(x)\exp(2\pi i\,x\,t)dx \tag{8}$$

Now, let S be a bounded set of real numbers and let $\epsilon > 0$ as in the hypothesis of theorem 1. From the definition of the content of S an integer M and closed intervals $[a_i, b_i]$, $1 \leqslant i \leqslant M$ can be chosen such that;

$$S \subset I = \bigcup_{i=1}^{M} [a_i, b_i] \tag{9}$$

and

$$cont(S) + \frac{\epsilon}{2} > L = \sum_{i=1}^{M} (b_i - a_i) \tag{10}$$

Choose $T > 2M/\epsilon$; therefore,

$$M/T < \epsilon/2 \tag{11}$$

For any real number x define the set $N(x)$ of integers by

$$N(x) = \{N \text{ such that } NT^{-1} - x \in I\} \tag{12}$$

and define

$$n(x) = \text{number of elements in } N(x) \tag{13}$$

and define

$$P = \max\ \{n(x)\text{: } x \text{ is real}\} \tag{14}$$

Now, choose real numbers T_i, $1 \leqslant i \leqslant P$ to be arbitrary but fixed. Sufficient restrictions on the set $\{T_i\text{:}1 \leqslant i \leqslant P\}$ will be formulated which imply the conclusion of theorem 1 (where S, ϵ, P, T_i and T in theorem 1 coincide with the parameters chosen above).

Since $\{NT^{-1} - x \in I; N \text{ an integer}\}$ consists of a set of points pairwise spaced at a distance at least T^{-1} and lying within a union of M intervals whose lengths sum to L, a simple combinatoric argument implies the inequality

$$P/T \leqslant M/T + L \tag{15}$$

which, together with inequality (11) implies

$$P/T \leqslant \epsilon/2 + L \tag{16}$$

Finally, combining inequalities (10) and (16) yields

$$cont(S) + \epsilon > P/T \tag{17}$$

which establishes inequality (6).

In order to determine $f(t)$ from sample values as in Eq. (5), Eq. (8) suggests that first the function $F(x)$ should be determined from sample values of $f(t)$. The relationship between $F(x)$ and sample values of $f(t)$ is exactly described by Eq. (7). Define, for every real number x, the P by $n(x)$ size matrix $B(x)$ as follows:

$$B(x) = [b_{ij}(x)],\ 1 \leqslant i \leqslant P,\ 1 \leqslant j \leqslant n(x) \tag{18}$$

where (of course the non subscript i is $\sqrt{-1}$)

$$b_{ij}(x) = \exp(2\pi i\,N_j(x)\,T^{-1}\,T_i) \tag{19}$$

and $N_j(x)$ is the jth element, in ascending order, of the set $N(x)$ in (12). Define, for every real number x, a P size column vector $V(x)$ by

$$V(x) = [v_i(x)]\,, \; 1 \leqslant i \leqslant P \tag{20}$$

where

$$v_i(x) = T \sum_{N=-\infty}^{+\infty} f(NT + T_i) \exp\,[2\pi i\, x(NT + T_i)] \tag{21}$$

and a $n(x)$ size column vector $W(x)$ by

$$W(x) = [w_j(x)]\,, 1 \leqslant j \leqslant n(x) \tag{22}$$

where (with $N_j(x)$ defined as in (19))

$$w_j(x) = F(N_j(x)T^{-1} - x) \tag{23}$$

Then the set of equations obtained by substituting the value $y = T_i$ for $1 \leqslant i \leqslant P$ into Eq. (7) can be written in matrix form as:

$$V(x) = B(x)\, W(x) \tag{24}$$

Clearly, in order to determine $F(x)$ it suffices to determine $W(x)$ for values of x in the interval $[0, T^{-1}]$. As x ranges from 0 to T^{-1} the set of values $N_j(x)$ $(1 \leqslant j \leqslant n(x))$ form a finite set of integers $\{K_j : 1 \leqslant j \leqslant g\}$ where $g \geqslant P$. Now, the set of exponential functions $\exp\,(2\pi i\, K_j\, u) : 1 \leqslant j \leqslant g\}$ are orthogonal and hence linearly independent functions of the variable u. Therefore, there exists a set of real numbers $\{u_i : 1 \leqslant i \leqslant g\}$ such that the g by g matrix $Q = [\exp(2\pi i\, K_j\, u_i)]$ has rank g. Then choose $T_i = Tu_i$ for $1 \leqslant i \leqslant P$. Since any choice of $n(x) \leqslant P \leqslant g$ columns of Q results in a P by $n(x)$ matrix having rank $n(x)$ and $B(x)$ arises by choosing the first $n(x)$ columns of Q, it follows that such a choice of T_i, $1 \leqslant i \leqslant P$ will imply that for any value of x between 0 and T^{-1} the $n(x)$ by $n(x)$ matrix $B^*(x)\; B(x)$ (where * denotes the transpose of a matrix) has rank $n(x)$ and is therefore invertible. Hence Eq. (24) can be solved for $W(x)$ to yield a (generalized inverse) solution.

$$W(x) = [B^*(x)\, B(x)]^{-1}\, B^*(x)\, V(x) \tag{25}$$

Furthermore, the complicated matrix valued function preceding $V(x)$ in (25) is, by (18), (19), and the definition of $N_j(x)$, constant over each of a finite number of intervals. Therefore, there exists an integer r, real numbers $x_1 < x_2 < \ldots < x_r$, and a set of coefficients $\{c_{ij} : 1 \leqslant i \leqslant P, 1 \leqslant j \leqslant r\}$ such that

$$F(x) = \begin{cases} \displaystyle\sum_{i=1}^{P} c_{ij}\, v_i\,(-x) \text{ for } x_j < x < x_{j+1} \text{ (for } j < r) \\ 0 \text{ for } x > x_r \text{ or } x < x_1 \\ \text{unspecified otherwise} \end{cases} \tag{26}$$

Finally, applying Eq. (8) yields

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} F(x) \exp\,(2\pi i\, x\, t)\, dx \qquad (27) \\ &= T \sum_{j=1}^{r-1} \sum_{i=1}^{P} c_{ij} \sum_{N=-\infty}^{+\infty} f(NT \\ &\quad + T_i) \int_{x_j}^{x_{j+1}} \exp\,[2\pi i\, x\,(t - NT - T_i)]\; dx \\ &= \sum_{i=1}^{P} \sum_{N=-\infty}^{+\infty} f(NT + T_i)\, s_i\,(t - NT - T_i) \end{aligned}$$

where

$$s_i\,(t) = T \sum_{j=1}^{r-1} \frac{\exp\,(2\pi i\, x_{j+1}\, t) - \exp\,(2\pi i\, x_j t)}{2\pi i\; t}\; c_{ij} \tag{28}$$

are the sampling functions in Eq. (5). The proof of theorem 1 is concluded.

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