A Statistical Model of the Photomultiplier Gain Process With Applications to Optical Pulse Detection

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The complete statistical behavior of the random gain of a photomultiplier tube (PMT) has not previously yielded to exact analysis. In this paper a Markov diffusion model is used to determine an approximate probability density for the random gain. This approximate density preserves the correct second-order statistics and appears to be in reasonably good agreement with previously reported experimental data. The receiver operating curve for a pulse counter detector of PMT cathode emission events is analyzed using this density. The error performance of a simple binary direct detection optical communication system is also derived.

I. Introduction

The photomultiplier tube (PMT) is an optical energy detector which has high enough internal gain to provide adequate output signal levels at low light levels. Electrons that are emitted at the cathode of a PMT are directed through a series of dynodes by an applied electric field. A single electron emitted at the cathode causes a number of secondary electrons to be emitted at the first dynode. These secondary electrons from the first dynode are in turn directed to a second dynode where this multiplication process is repeated for each impinging electron. This electron multiplication process is repeated through a series of several dynodes until the electrons from the last dynode are collected at the PMT anode, with the resulting anode current being the PMT output.

The PMT *gain* is defined to be the total number of electrons collected at the anode as a result of a single electron emission event at the cathode. Since the number of secondary electron emissions at a dynode for each primary impinging electron is a random quantity, the overall PMT gain is a random variable. Thus the PMT output current signal resulting from a single electron emission event at the cathode is also random in nature.

In optical communications, direct detection receivers employ photodetectors with internal gain such as PMTs in low light level situations to overcome thermal noise in the amplification stages following the PMT. The probability distribution of the random PMT gain is then required to determine error performance in systems that use a PMT detector. It is well known (Refs. 1, 2) that the random electron multiplication process can be modelled by a Galton-Watson branching process (Ref. 3). Although the mean and variance of the gain can be readily determined using the branching process model, the problem of determining an explicit expression for its probability distribution appears to be intractable. An approach to circumvent this intractable problem is to obtain an accurate approximation of the probability distribution of the PMT gain. This approximate distribution can then be used to evaluate the error performance of the direct detection digital communication system.

Feller (Ref. 4) was the first to suggest the use of a Markov diffusion process approximation to analyze the statistical behavior of a Galton-Watson branching process. As is well known, the Galton-Watson branching process is a discrete-time discrete-state Markov process that can be specified by its conditional state transition probability distribution. On the other hand, the diffusion process is a continuous-time continuous-state Markov process whose incremental state transition statistics are specified by conditional incremental mean and variance parameters called the infinitesimal mean and variance, respectively. Feller's approach is to use a diffusion process approximation whose infinitesimal mean and variance parameters simulate the mean and variance respectively of the branching process conditional state transition distribution. Thus the second-order incremental state transition statistics of the diffusion approximation are similar to those of the original branching process.

In this paper we employ Feller's approach to obtain a Markov diffusion process approximation of the PMT gain branching process. The resulting diffusion process then has a marginal probability density which is obtained by solving a Fokker-Planck partial differential equation. This density can then be regarded as an approximate density for the PMT gain. This approximate density of course yields the true mean and variance of the PMT gain. Moreover, the general shape of the approximate density appears to be in good agreement with experimental PMT gain data reported in the literature (Ref. 5).

The number of electrons emitted at the PMT cathode is also random in nature and can be assumed to be Poisson distributed. Thus the random nature of the number of electrons collected at the anode is a result of both the random electron emission process at the cathode and the random PMT gain. It is the probability distribution of the number of PMT anode electrons that is required for evaluating communication system error performance. In order to determine an explicit expression for this distribution, the distribution of the gain is again required. Hence the problem of determining an explicit expression for the distribution of the number of PMT anode electrons appears to be also intractable. In this paper we obtain an approximate density for the number of anode electrons by using the approximate density for the gain random variable derived from the diffusion approximation.

The number of PMT cathode electron emission events is often monitored by using a pulse counter. It is interesting to determine the effect of the random PMT gain, thermal noise in the PMT output amplifier and the counter response time on successful cathode electron emission event detection. The receiver operating curve of this detection process is determined in this paper using the derived approximate PMT gain distribution.

Finally we analyze the error performance of a direct detection communication system with a PMT detector and an on-off binary signalling scheme. Here the laser transmitter light source is either on or off in a binary symbol time. We consider a simple receiver which integrates the PMT amplifier output over the symbol time period and compares the result to a threshold. Detection of the laser light is then declared if and only if the threshold is exceeded. The error performance of this system is analyzed, taking into account the random PMT gain and the thermal noise in the PMT output amplifier.

This paper is organized as follows. Section II describes the branching process model of the PMT gain and the diffusion approximation. Section III considers the detection performance of pulse counter monitoring of PMT cathode emission events using the approximate PMT gain density obtained in Section II. Section IV derives the approximate density of the number of electrons collected at the PMT anode for Poisson distributed cathode electron emissions. This density is then used to evaluate the error performance of the binary on-off direct detection communication system.

II. Branching Process Model and the Diffusion Approximation

A. Branching Process Model for the PMT Gain

A primary impinging electron at a dynode causes a random number of secondary electrons. In this paper it is assumed that the number of secondary electrons generated per primary electron is Poisson distributed with mean μ, where

$$\mu = \text{average gain per dynode stage.}$$

Moreover, it is assumed that the average gain of each dynode stage is identical. The Poisson assumption can be regarded as being valid when the physical nonuniformities across the dynode surfaces are small (Refs. 6, 7). In the dynode electron multiplication process it can be assumed that the secondary electron emission process operates on each individual primary electron in a statistically independent manner (Refs. 1, 2, 5, 6, 7). Hence the number of secondary electrons resulting from different primary electrons are independent random variables. This independence assumption leads directly to a branching process model for the overall PMT gain. Specifically for $k \geqslant 1$ let

S_k = total number of electrons emitted by the kth dynode.

Moreover, define

$$S_0 = 1$$

$$= \text{number of electrons emitted by the cathode,}$$

and

G = total number of electrons collected at the anode as a result of a single electron emission at the cathode

$$\triangleq \text{PMT gain.}$$

Then for a PMT with ν dynode stages,

$$G = S_\nu . \tag{1}$$

Also, we have, under the above assumptions,

$$S_k = \sum_{i=1}^{S_{k-1}} N_{ki}, \ k \geqslant 1, \tag{2}$$

where $\{N_{ki}, k \geqslant 1, i \geqslant 1\}$ are independent Poisson random variables each with mean μ. The process $\{S_k : k \geqslant 0\}$ is known as a Galton-Watson branching process (Ref. 3). The second-order statistics of S_k can be shown (Ref. 3, p. 6) to be given by

$$E[S_k] = \mu^k \tag{3}$$

$$\mathrm{Var}(S_k) = \frac{\mu^k(\mu^k - 1)}{\mu - 1} \tag{4}$$

where $\mu \neq 1$. Thus from (1) and (3), a PMT with ν dynode stages has average gain

$$\overline{G} \triangleq E[G] = \mu^\nu . \tag{5}$$

A typical PMT such as the RCA C31034 has $\nu = 11$ dynode stages with $\overline{G} = 10^6$. Hence $\mu = 3.51$ here. In this case it can be seen from (4) that the standard deviation of the gain G is roughly half of the average gain $\overline{G}$. Hence the distribution of G can be expected to be quite widely spread about its mean. This is the situation with many PMTs where the standard deviation of the gain is of the same order of magnitude as its mean.

In branching process theory, the probability generating function (pgf) of S_k:

$$f_k(z) = E\left[z^{S_k}\right] \tag{6}$$

can be shown (Ref. 3, p. 5] to be given by

$$f_k(z) = \underbrace{g(g(\cdots g(z)\cdots))}_{k \text{ times}}, \tag{7}$$

where

$$g(z) = E\left[z^{N_{ki}}\right]$$

$$= e^{\mu(z-1)} \tag{8}$$

is the pgf of the Poisson random variable N_{ki} with mean μ. So in view of (1), the pgf of the PMT gain can be explicitly determined using (7). However, it is not possible to invert the pgf analytically to obtain the distribution of the gain because the number of dynode stages is usually sufficiently large to render that problem intractable. Thus the problem of determining an explicit expression for the probability distribution of the gain G appears to be intractable.

B. Diffusion Approximation

A viable alternative is to attempt to obtain an accurate approximation of the PMT gain probability distribution. We propose to accomplish this goal by using a diffusion approximation of the branching process gain model. Diffusion approximations have previously been successfully employed in many stochastic process problems such as in the analysis of queuing systems (Ref. 8). Feller (Ref. 4) was the first to suggest the use of diffusion approximations to analyze branching processes. We apply Feller's approach to obtain an approximate distribution for the PMT gain as follows. Let $S(t)$ be a diffusion process satisfying the Ito differential equation (Ref. 9):

$$dS(t) = \beta(S(t), t)\, dt + \alpha(S(t), t)\, dW(t), \tag{9}$$

where $W(t)$ is a Wiener process with zero mean and variance t. Here β is the infinitesimal mean or drift and α is the infinitesimal variance of $S(t)$. That is, for small Δt,

$$E[S(t + \Delta t) - S(t) | S(t) = x] \cong \beta(x, t)\, \Delta t, \tag{10}$$

$$\mathrm{Var}[S(t + \Delta t) - S(t) | S(t) = x] \cong \alpha(x, t)\, \Delta t. \tag{11}$$

Since the PMT gain G, which is usually of the order of 10^6, is large, we can represent it as a continuous random variable. Let us now approximate the branching processing gain model $\{S_k\}$ given by (2) by the continuous-state continuous-time diffusion process $S(t)$. Moreover, this approximation shall be made so that the infinitesimal parameters (10) and (11) of $S(t)$ possess the same behavior of the corresponding incremental mean and variance of S_k. Note from (2) that

$$E\,[S_{k+1} - S_k | S_k = n] = n(\mu - 1) \tag{12}$$

$$\mathrm{Var}\,[S_{k+1} - S_k | S_k = n] = n\mu \tag{13}$$

are both proportional to the population size $S_k = n$ of electrons emitted by the kth dynode. Thus, in order to preserve this behavior, we should set the infinitesimal parameters

$$\beta(x,t) = \beta x\,, \tag{14}$$

$$\alpha(x,t) = \alpha x\,, \tag{15}$$

to also linearly increase with population size x. We shall not at this time specify the constants α and β in view of the difference in time scales between the processes S_k and $S(t)$. Let

$$p(x,t) = \text{marginal probability density of } S(t)\,.$$

Then it follows from (14) and (15) and a well-known result (Ref. 9) on diffusions that $p(x, t)$ satisfies the following Fokker-Planck equation:

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\,\alpha\,\frac{\partial^2}{\partial x^2}\,[xp(x,t)] - \beta\,\frac{\partial}{\partial x}\,[xp(x,t)]\,, \tag{16}$$

$$p(x,0) = \delta(x-1)\,. \tag{17}$$

The solution $p(x, t)$ of (16) and (17) can be obtained as follows. Instead of solving for $p(x, t)$ directly, consider its moment generating function (mgf)

$$\phi(u,t) = E\,[e^{-uS(t)}] = \int_{-\infty}^{\infty} e^{-ux}\,p(x,t)\,dx\,.$$

A technique (Ref. 10, p. 83) for transforming Fokker-Planck equations then obtains the following equations for $\phi(u, t)$ from (16) and (17):

$$\frac{\partial\phi(u,t)}{\partial t} = \left(\beta u - \frac{1}{2}\alpha u^2\right)\frac{\partial\phi(u,t)}{\partial u}\,, \tag{18}$$

$$\phi(u,0) = e^{-u}\,. \tag{19}$$

The solution of (18) and (19) can be shown to be

$$\phi(u,t) = \exp\left[\frac{-Au}{1+Bu}\right], \tag{20}$$

where

$$A = e^{\beta t}, \tag{21}$$

$$B = \frac{\alpha}{2\beta}(e^{\beta t} - 1)\,. \tag{22}$$

Finally (20) is inverted to recover $p(x, t)$ as follows. First rewrite (20) as

$$\phi(u,t) = e^{-A/B}\exp\left[\frac{A/B}{1+Bu}\right]$$

$$= e^{-A/B}\sum_{n=0}^{\infty}\frac{(A/B)^n}{n!}\,\frac{1}{(1+Bu)^n}\,. \tag{23}$$

Next note that $1/(1 + Bu)^n$ is the mgf of the Gamma density:

$$\frac{1}{(1+Bu)^n} = \int_0^{\infty} e^{-ux}\left[\frac{B^{-n}\,x^{n-1}}{(n-1)!}\,e^{-x/B}\right]dx\,. \tag{24}$$

So (23) and (24) yield

$$p(x,t) = e^{-A/B}\,\delta(x) + e^{-A/B}\sum_{n=1}^{\infty}\frac{(A/B^2)^n\,x^{n-1}\,e^{-x/B}}{n!(n-1)!}$$

$$= e^{-A/B}\left[\delta(x) + \frac{e^{-x/B}\sqrt{A/x}}{B}\,I_1\!\left(\frac{2\sqrt{Ax}}{B}\right)\right], \tag{25}$$

for $x \geqslant 0$, where I_1 is the modified Bessel function of the first kind. Finally, we want to choose the parameters A and B so that $S(t)$ represents the gain $G = S_\nu$ of a ν-stage PMT with average gain $\bar{G}$. In particular, A and B are chosen so that the respective means and variances of G and $S(t)$ are equal. It follows from (1), (4) and (5) that

$$E[G] = \bar{G}\,, \tag{26}$$

$$\mathrm{Var}[G] = \frac{\bar{G}(\bar{G}-1)}{\sqrt[\nu]{\bar{G}} - 1} \tag{27}$$

Also, from (23) we have

$$E[S(t)] = -\left.\frac{\partial \phi(u,t)}{\partial u}\right|_{u=0} = A,$$

$$Var[S(t)] = \left.\frac{\partial^2 \phi(u,t)}{\partial u^2}\right|_{u=0} - A^2 \tag{28}$$

$$= 2AB \tag{29}$$

So in order that G and $S(t)$ have the same second-order statistics, we require

$$A = \bar{G}, \tag{30}$$

$$B = \frac{1}{2}\left(\frac{\bar{G}-1}{\sqrt[\nu]{\bar{G}-1}}\right). \tag{31}$$

So we can conclude that an approximate probability density $p_G(x)$ of the gain G of a PMT with average gain $\bar{G}$ and ν dynode stages is given by

$$p_G(x) = \begin{cases} e^{-A/B}\left[\delta(x) + \dfrac{e^{-x/B}\sqrt{A/x}}{B} I_1\left(\dfrac{2\sqrt{Ax}}{B}\right)\right], & x \geqslant 0 \\ 0 & , x < 0 \end{cases} \tag{32}$$

where A and B are given by (30) and (31)

Let us consider the appropriateness of this approximate density (32). First, it is clear from (26) – (31) that (32) yields the correct mean and variance for G. It is also known (Ref. 3, pp. 13-16) that S_n/μ^n converges to a nonnegative random variable W with probability one as n tends to infinity. Moreover, the distribution of this limiting random variable W has a point mass at 0 (delta function at 0) and is absolutely continuous (has a density) elsewhere. Thus the structural form of $p_G(x)$ appears to be correct for PMTs with a large number of dynodes. As another indication of the appropriateness of (32), we note from (7) that

$$P(G = 0) = P(S_\nu = 0)$$

$$= \underbrace{g(g(\cdots g(0)\cdots))}_{\nu \text{ times}}, \tag{33}$$

where g is given by (8). On the other hand (32) gives

$$P(G = 0) \cong e^{-A/B} \tag{34}$$

with A and B given by (30) and (31). For the RCA 31034 PMT with $\nu = 11$ dynodes, (33) yields 3.36×10^{-2} and 1.4×10^{-2} for $P(G = 0)$ when $\bar{G} = 10^6$ and 10^7 respectively, whereas (34) yields 6.6×10^{-3} and 1.3×10^{-3}. Thus (32) also gives a fairly reasonable approximation of $P(G = 0)$.

The 11-dynode RCA 31034 PMT has usable average gain $\bar{G}$ in a range from 10^5 to over 10^7. Figures 1 and 2 show the approximate density (32) normalized to give a density for $G/\bar{G}$ for $\bar{G} = 10^6$ and 10^7. As can be seen from these figures, the density is not symmetric about the mean and in fact peaks at a point below the mean value. Also there is considerable probability mass below the mean. These properties appear to be in good agreement with experimental PMT gain data reported in the literature (Ref. 5). In particular, the asymmetric nature of the density (32) appears in the experimental results. Also shown in these figures are corresponding density functions which are positive truncations of the Gaussian density with the same mean and variance as G. The truncated Gaussian approximation is sometimes used (Ref. 11) to simplify the analysis of systems using PMTs. However, the simpler truncated Gaussian approximation has a mean larger than $\bar{G}$ and substantially more probability mass above the mean than (32). Hence, using the Gaussian approximation to analyze communication system performance could produce overly optimistic results.

III. Pulse Counter Detection of Cathode Emissions

In many PMT experiments such as determining the dark current, cathode emission event data is required. The cathode emission events are usually recorded by monitoring the PMT output with a pulse counter. In this section the effect of the random PMT gain, thermal noise in the PMT output amplifier and the counter response time on successful emission event detection is determined. We shall derive the probability of successful emission event detection and the probability of false alarm to obtain the receiver operating curve for this detection process.

Suppose a single electron is emitted at the PMT cathode. Denote e = electron charge = 1.6×10^{-19} coulombs, W_p = PMT bandwidth and $T_p = 1/W_p$ = PMT response time. Then a current pulse of GeW_p amps is generated at the PMT anode as a result of this single electron emission. Suppose the anode is terminated with resistance R. Then the voltage pulse signal

$S(t)$ across the resistor can be represented as

$$S(t) = (GeW_p R)\, p(t) \text{ volts,} \tag{35}$$

where $p(t) \geq 0$ can be assumed to be a pulse of duration $T_p = 1/W_p$ seconds satisfying

$$\frac{1}{T_p} \int_0^{T_p} p(t)\, dt = 1. \tag{36}$$

The requirement (36) arises because the total charge in the current pulse $S(t)/R$ due to the single emitted electron must be equal to Ge. Here G is of course the random PMT gain. We assume that the amplified PMT output is monitored using a pulse counter. Let W_c = counter bandwidth and $T_c = 1/W_c$ = counter response time. We assume that the counter bandwidth W_c is less than the PMT bandwidth W_p so that

$$T_c > T_p \tag{37}$$

The only degradation introduced by the PMT output amplifier is assumed to be additive thermal noise. As a first approximation we model the counter as a short-term averager of its input over the counter response time T_c seconds followed by a threshold comparator to determine whether a positive pulse has occurred. So the PMT amplifier – pulse counter combination first introduces an independent additive white Gaussian noise process $V(t)$ to the PMT output and then integrates the PMT output signal plus $V(t)$ process over T_c seconds. The result is first normalized by T_c and then compared to a decision threshold γ. An emission event is declared if and only if this threshold is exceeded. This model is shown in Fig. 3. Let H_1 be the hypothesis of the occurrence of an emission event and H_0 the null hypothesis. If $X(t)$ is the PMT amplifier output, then under

$$H_1:\ X(t) = (e\, W_p R)\, G\, p(t) + V(t)\,, \tag{38}$$

$$H_0:\ X(t) = V(t)\,, \tag{39}$$

where $V(t)$ has spectral density

$$N_0 = k\,\theta\, R\ (\text{volts})^2/\text{Hz}. \tag{40}$$

Here k = Boltzmann constant = 1.38×10^{-23} watts/Hz-°K, θ = amplifier noise equivalent temperature (°K) and R = amplifier equivalent input resistance (assumed to be matched to the PMT anode load resistance). Let X be the normalized integrator output:

$$X = \frac{1}{T_c} \int_0^{T_c} X(t)\, dt, \tag{41}$$

and denote

$$V = \frac{1}{T_c} \int_0^{T_c} V(t)\, dt. \tag{42}$$

So from (35) – (42), it follows that under

$$H_1:\ X = (e\, W_c R)\, G + V\,, \tag{43}$$

$$H_0:\ X = V\,, \tag{44}$$

where V is a $N(0, \sigma^2)$ random variable independent of G with

$$\sigma^2 = N_0/T_c\,. \tag{45}$$

In order to assess the performance of this pulse counter detection of emission events, consider the probabilities:

P_d = probability of correctly detecting an emission event

$$= P\left((e\, W_c R)\, G + V \geq \gamma\right) \tag{46}$$

and

P_f = false alarm probability of declaring an emission event when it actually was absent

$$= P(V \geq \gamma)\,. \tag{47}$$

The statistics of the random PMT gain G are required to determine P_d. We shall use the approximate density (32) for this purpose. So it follows from (46) and (47) that

$$P_f = Q(\gamma/\sigma)\,, \tag{48}$$

and

$$P_d = \int_0^\infty P[V \geq \gamma - (e\, W_c R)\, x]\; p_G(x)\, dx$$

$$= \int_0^\infty Q\left(\frac{\gamma - (e\, W_c R)\, x}{\sigma}\right) p_G(x)\, dx\,, \tag{49}$$

where $p_G(x)$ is given by (32) and

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt. \qquad (50)$$

The plot of P_d versus P_f determines the receiver operating curve for this detection process. Figure 4 gives receiver operating curves for a 11-dynode PMT such as the RCA 31034 for various values of average gain $\overline{G}$ when the counter bandwidth W_c and noise spectral density height N_0 are fixed. The receiver operating curves improve with increasing average gain $\overline{G}$ since the average pulse height increases with $\overline{G}$. Figure 5 shows receiver operating curves for fixed $\overline{G}$ and N_0 for various counter bandwidths W_c. Here the receiver operating curves improve with increasing W_c since the average pulse height increases linearly with W_c while the noise standard deviation σ increases only according to $\sqrt{W_c}$.

IV. Error Performance of a Binary Direct Detection Communication System

In this section we consider a direct detection optical communication system utilizing a PMT as a photodetector with a binary signalling scheme in which the signalling time period is divided into successive time slots of T_s seconds duration. A given slot is either a "noise slot", in which case no incident light from the transmitter light source is received at the PMT cathode, or a "signal slot," when incident light of constant intensity from the transmitter is received. The problem of concern here is to detect whether a given time slot is a signal slot or a noise slot based on the amplified PMT output in that time slot.

We shall consider a receiver which integrates the amplified PMT output during a slot time T_s, normalizes the integrator output by the integration time T_s and compares the result to a threshold γ. A signal slot is declared if and only if the threshold is exceeded. The error performance of this simple binary receiver shall be analyzed here.

A block diagram of this system is shown in Fig. 6. Let $V(t)$ be the additive white Gaussian noise process with spectral density N_0 given by (40) representing the thermal noise in the amplifier. Suppose

N = number of electrons emitted at the photocathode during the slot time $[0, T_s]$

and $\{t_i\}_{i=1}^N$ are the emission times of those N electrons. Also suppose the electron emitted at time t_i undergoes a gain G_i through the PMT dynode chain. Since the PMT electron multiplication process operates independently on each electron emitted at the cathode, the gain random variables G_i are mutually independent. So, similar to (38), the PMT amplifier output process $X(t)$ is given by

$$X(t) = (e\, W_p\, R) \sum_{i=1}^{N} G_i\, p(t - t_i) + V(t), \qquad (51)$$

where $p(t)$ is as before in (35). The number N of cathode emission events in the signal slot time can be assumed to be Poisson with intensity equal to the average number of electron emissions per slot time. In a noise slot, electron emissions are due to the PMT dark current, whereas in the signal slots they are due to both dark current and received light excitation. In particular, if H_1 is the hypothesis of a signal slot and H_2 is the hypothesis of a noise slot, then

$$P(N = n | H_i) = \frac{\alpha_i^n}{n!} e^{-\alpha_i}, n \geq 0, \qquad (52)$$

where

$$\alpha_2 = \overline{N}_n$$

$\triangleq$ average number of dark current cathode emissions per time slot, (53)

$$\alpha_1 = \overline{N}_n + \overline{N}_s, \qquad (54)$$

and

$\overline{N}_s$ = average number of cathode emissions due to received light excitation per signal time slot

$$= \frac{\eta P_s}{hf}. \qquad (55)$$

In (55), η is the photocathode quantum efficiency, P_s is the incident signal light intensity at the cathode surface, h is Planck's constant and f the incident light center frequency.

It is reasonable to assume that the slot time T_s is greater than the PMT response time T_p. Then, since the detector statistic

$$X = \frac{1}{T_s} \int_0^{T_s} X(t)\, dt,$$

it follows from (36) and (51) that

$$X = \left(\frac{eR}{T_s}\right) \sum_{i=1}^{N} G_i + V , \quad (56)$$

where V is a $N(0, \sigma_1^2)$ random variable with

$$\sigma_1^2 = \frac{N_0}{T_s} . \quad (57)$$

In order to determine the receiver's error performance, the distribution of

$$Y = \sum_{i=1}^{N} G_i \quad (58)$$

must be determined under each hypothesis H_i. This requires the PMT gain probability distribution, which is the common distribution of all the random variables G_i in (58). We shall use the approximate density (32) here. First consider the mgf $\phi_Y(u\,|\,H_i)$ of Y under H_i:

$$\phi_Y(u\,|\,H_i) = E\,(e^{-uY}\,|\,H_i)$$

$$= E\,\left[E\,(e^{-uY}|N, H_i)\,|H_i\right] . \quad (59)$$

Let $\phi_G(u)$ be the mgf of each of the G_is. Then

$$E(e^{-uY}|N{=}n, H_i) = E\left[e^{-u\sum_{i=1}^{n} G_i}\right]$$

$$= [\phi_G(u)]^n. \quad (60)$$

So (52), (59) and (60) yield

$$\phi_Y(u\,|\,H_i) = \sum_{n=0}^{\infty} [\phi_G(u)]^n \frac{\alpha_i^n}{n!} e^{-\alpha_i}$$

$$= e^{\alpha_i[\phi_G(u)-1]} . \quad (61)$$

Since we are using the density (32) as the density of each of the G_is, $\phi_G(u)$ is taken to be the mgf (20) of (32). So using (20) in (61) yields

$$\phi_Y(u|H_i) = e^{\alpha_i\left[e^{-A/B}\exp\left(\frac{A/B}{1+Bu}\right)-1\right]} . \quad (62)$$

Finally we shall invert the mgf (62) to obtain the probability density $p_Y(y|H_i)$ of Y under H_i by using a technique similar to (23) - (25). That is, rewrite (62) as

$$\phi_Y(u|H_i) = e^{-\alpha_i} \sum_{n=0}^{\infty} \frac{\left(\alpha_i\, e^{-A/B}\right)^n}{n!} \exp\left(\frac{nA/B}{1+Bu}\right)$$

$$= e^{-\alpha_i} \sum_{n=0}^{\infty} \frac{\left(\alpha_i\, e^{-A/B}\right)^n}{n!} \left[1 + \sum_{k=1}^{\infty} \frac{(nA/B)^k}{k!} \cdot \frac{1}{(1+Bu)^k}\right]$$

$$= e^{\alpha_i(e^{-A/B}-1)} + e^{-\alpha_i} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{\left(\alpha_i\, e^{-A/B}\right)^n}{n!} \cdot \frac{(nA/B)^k}{k!} \frac{1}{(1+Bu)^k} . \quad (63)$$

Next, similar to (25), (63) is inverted by using (24) to yield

$$p_Y(y|H_i) = e^{\alpha_i(e^{-A/B}-1)}\,\delta(y) + e^{-\alpha_i} \sum_{n=0}^{\infty} \frac{\left(\alpha_i\, e^{-A/B}\right)^n}{n!} \cdot \left[\sum_{k=1}^{\infty} \frac{(nA/B^2)^k\, y^{k-1}}{k!\,(k-1)!}\, e^{-y/B}\right]. \quad (64)$$

But, since

$$\sum_{k=1}^{\infty} \frac{(nA/B^2)^k\, y^{k-1}}{k!\,(k-1)!}\, e^{-y/B} = e^{-y/B} \sqrt{\frac{nA}{y\,B^2}} \sum_{k=0}^{\infty} \cdot \frac{\left(\frac{1}{2}\sqrt{4n\,Ay/B^2}\right)^{2k+1}}{k!\,(k+1)!}$$

$$= \frac{e^{-y/B}\sqrt{nA/y}}{B}\, I_1\left(\frac{2\sqrt{nAy}}{B}\right), \quad (65)$$

(64) and (65) together yield the following expression for $p_Y(y|H_i)$:

$$p_Y(y|H_i) = e^{\alpha_i(e^{-A/B}-1)}\,\delta(y) + \frac{\sqrt{A/y}\,e^{-(\alpha_i+y/B)}}{B} \sum_{n=0}^{\infty}$$

$$\cdot \frac{\sqrt{n}\,(\alpha_i\,e^{-A/B})^n}{n!}\, I_1\left(\frac{2\sqrt{nAy}}{B}\right), \tag{66}$$

where $y \geqslant 0$. Further reduction of (66) to a closed form expression does not appear to be possible.

The error performance of this binary communication system is specified by

$$P_{ds} = \text{probability of correctly detecting a signal slot}$$

$$= P(X \geqslant \gamma|H_1), \tag{67}$$

and

$$P_{dn} = \text{probability of correctly detecting a noise slot}$$

$$= 1 - P(X \geqslant \gamma|H_2)\,\cdot \tag{68}$$

So from (56) - (58), (67) and (68) we have

$$P_{ds} = \int_0^\infty P\left(V \geqslant \gamma - \left(\frac{eR}{T_s}\right)y\right)\, p_Y(y|H_1)\,dy$$

$$= \int_0^\infty Q\left(\frac{\gamma - \left(\frac{eR}{T_s}\right)y}{\sigma_1}\right) p_Y(y|H_1)\,dy\,, \tag{69}$$

and similarly

$$P_{dn} = 1 - \int_0^\infty Q\left(\frac{\gamma - \left(\frac{eR}{T_s}\right)y}{\sigma_1}\right) p_Y(y|H_2)\,dy, \tag{70}$$

where σ_1 is given by (57) and $p_Y(y/H_i)$ is given by (66) with α_i given by (53) and (54). Typical receiver operating curves of P_{ds} versus $1 - P_{dn}$ are shown in Fig. 7 for various values of signal counts $\overline{N}_s$ for fixed PMT average gain $\overline{G}$, slot time T_s, thermal noise spectral height N_0 and dark current count $\overline{N}_n$. The receiver operating curve can be seen to improve with increasing $\overline{N}_s$ as expected.

V. Discussion

We have used a Markov diffusion approximation of the PMT electron multiplication process to obtain an approximate density for the PMT random gain. This approximate density was subsequently used to determine the receiver operating curve for a pulse counter detector of PMT cathode emission events. It was also used for analyzing the error performance of a simple on-off binary direct detection optical communication system. These latter results are used elsewhere (Ref. 13) to analyze the error performance of uncoded and coded PPM direct detection optical communication systems employing a PMT detector.

We have assumed here for simplicity that all the dynode stages in the PMT have identical average gain. In some applications, the first dynode stage has a higher average gain than the remaining stages, which have equal average gains. This is achieved by applying a higher interdynode voltage between the first two dynodes of the PMT. The methods in this paper can be extended to obtain an approximate density for the overall PMT gain in the situation as follows. Consider a ν-stage PMT with overall average gain $\overline{G}$ so that the first dynode stage average gain is μ_1. Then the remaining ν-1 dynode stages have overall average gain equal to $\overline{G}/\mu_1$. Based on the assumptions discussed in Section II, it can be seen that the overall PMT gain has the same distribution as the number of collected anode electrons for a (ν-1)-stage PMT with average gain $\overline{G}/\mu_1$ when the number of cathode emissions is Poisson-distributed with mean μ_1. Thus it follows from the results in Section IV that an approximate density for the overall PMT gain in this case is given by the density (66) with parameter $\alpha_i = \mu_1$ and parameters A and B corresponding to the (ν-1)-stage PMT with overall average gain $\overline{G}/\mu_1$. We note that for a PMT operated as such with higher first dynode stage gain, the probability density for the number of anode elections when the cathode emissions are Poisson-distributed is no longer represented by (66). However, the basic technique in Section IV can still be extended to obtain this latter density although yielding a more complex double series solution. It would be interesting to investigate ways of simplifying this series solution.

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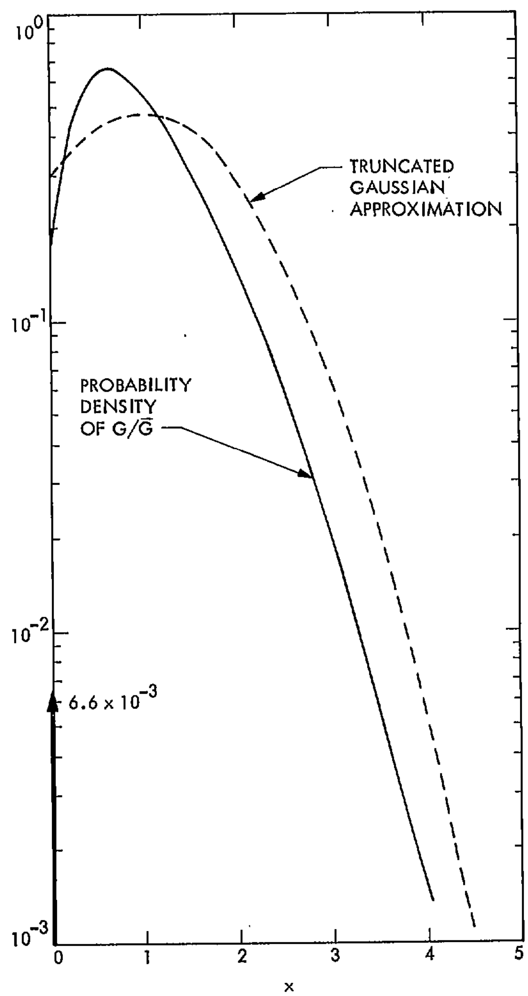

Fig. 1. Approximate probability density of $G/\overline{G}$ for $\overline{G} = 10^6$ and $\upsilon = 11$

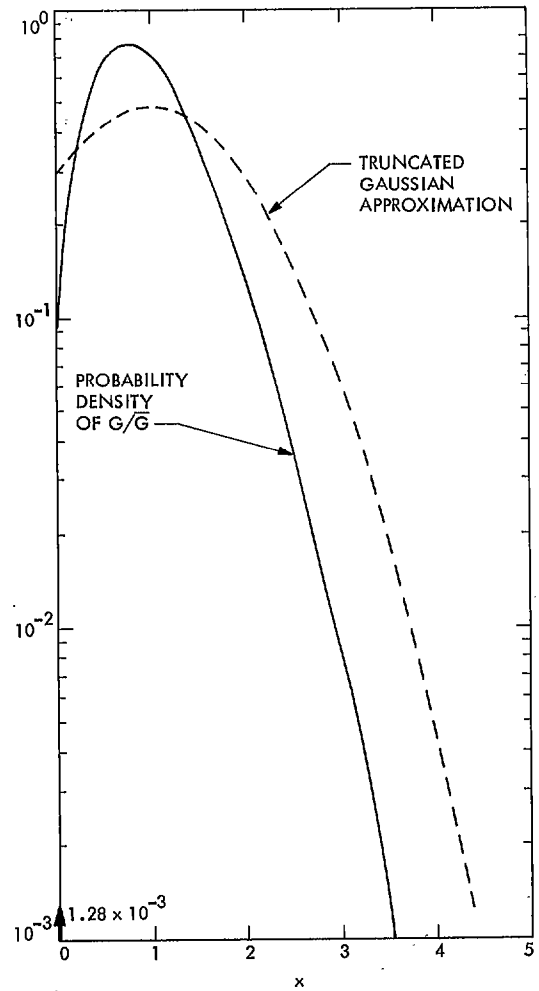

Fig. 2. Approximate probability density of $G/\overline{G}$ for $\overline{G} = 10^7$ and $\upsilon = 11$

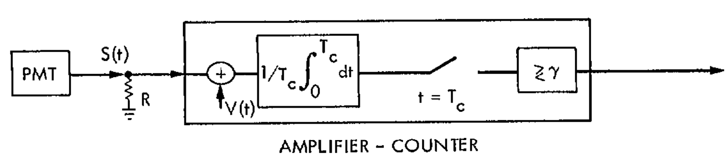

Fig. 3. Model for pulse counter detection of cathode emission events

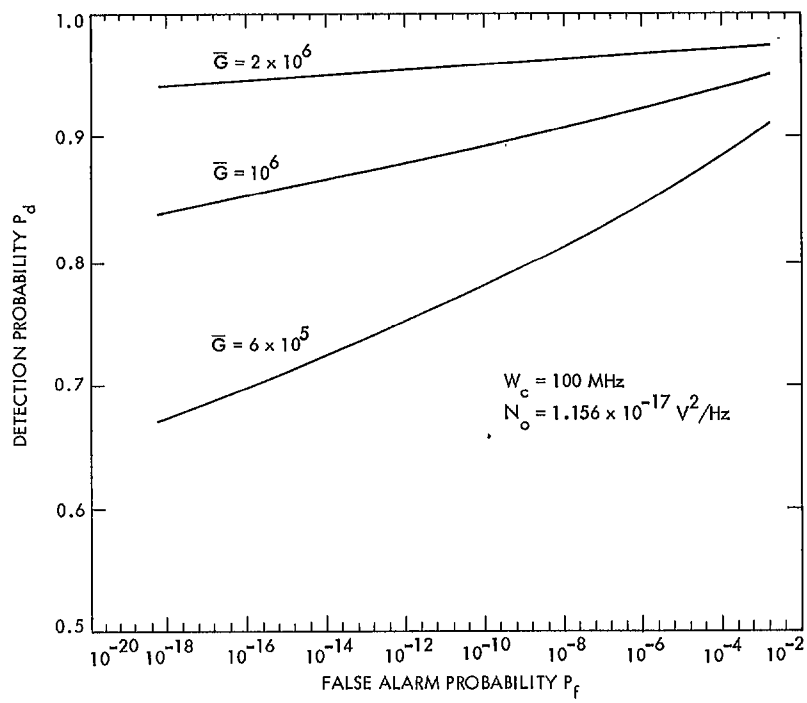

Fig. 4. Cathode emission event detection receiver operating curve as a function of PMT gain

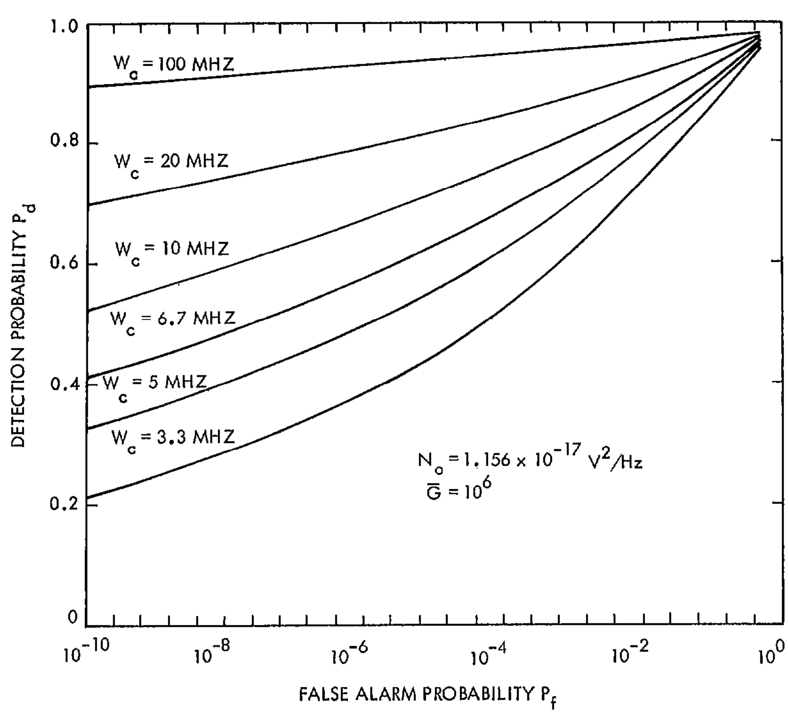

Fig. 5. Cathode emission event detection receiver operating curve as a function of counter bandwidth

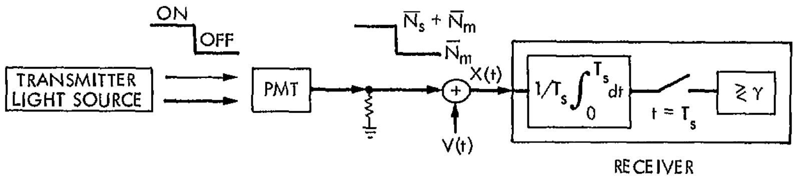

Fig. 6. Direct detection binary communication system

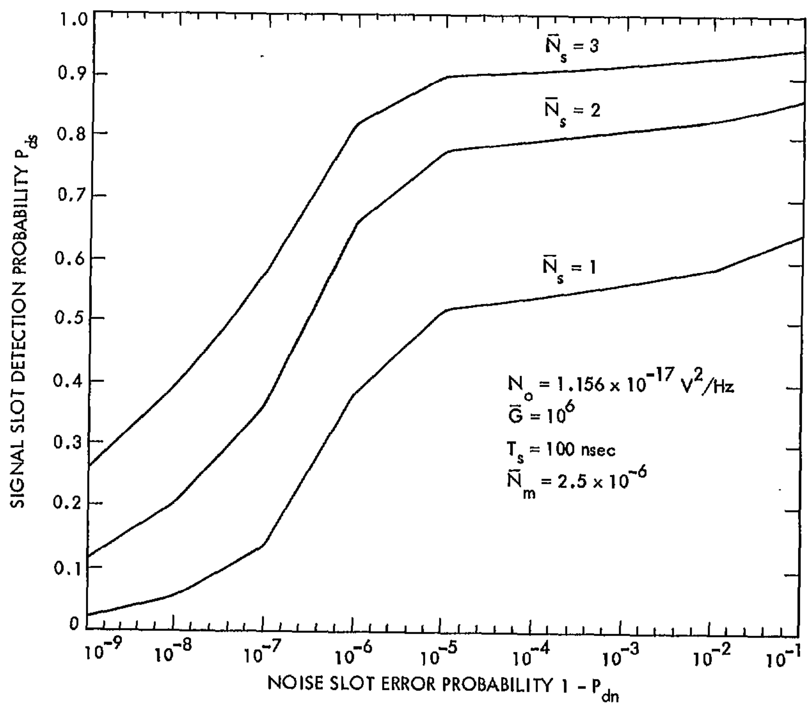

Fig. 7. Binary communication receiver operating curve as a function of $\bar{N}_s$