Error Probabilities in Optical PPM Receivers With Gaussian Mixture Densities

R. M. Gagliardi
University of Southern California

Analysis is presented for evaluating PPM error probabilities in optical photodetector receivers governed by statistics due to Gaussian mixture densities. A Gaussian mixture density arises when a discrete variable (e.g., a photodetector count variable) is added to a continuous Gaussian variable (e.g., thermal noise). Making use of some properties of photomultiplier Gaussian mixture distributions, some approximate error probability formulas can be derived. These appear as averages of M-ary orthogonal Gaussian error probabilities, of which the latter are well documented in the literature. The use of a pure Gaussian assumption is considered, and when properly defined, appears as an accurate upper bound to performance.

I. Introduction

In optical PPM communications, digital data are transmitted by placing an optical light pulse in one of a set of designated pulse slots, the latter constituting a PPM frame (Ref. 1). Thus each pulse represents a data word, or symbol, depending on its pulse slot location. At the PPM optical receiver, the photodetected output is integrated over each slot time to generate a slot voltage. These voltages are then compared over a PPM frame, and the largest is used to decode the PPM symbol. In past studies, symbol error probabilities for this optical PPM link have been analyzed for Gaussian, Poisson, and erasure voltage models (Refs. 1, 2, 3). These various models arise from different assumptions made on the receiver model. Discrete count statistics arise from use of ideal, high-gain photomultipliers, for which receiver thermal noise is negligible, while erasure models occur if in addition background noise is neglected. Gaussian models occur if high receiver power levels are assumed and receiver thermal noise is included. In this report we extend the study to general Gaussian mixture densities. A Gaussian mixture density is the probability density of the sum of a discrete and a continuous Gaussian random variable and will occur if both photodetector count statistics and thermal noise are included.

Let the PPM slot integrations generate the sequence of voltage variables

$$y_j = ak_j + n_j, \quad j = 1, 2, \ldots M \tag{1}$$

where $\{k_j\}$ is the discrete (count) sequence, M is the number of slots, a is a scalar, and $\{n_j\}$ is a sequence of independent

Gaussian zero mean variables. The voltage variable y_j corresponds to that which would be generated from the integration of a photodetector output containing additive thermal noise. For this model the scalar corresponds to (Ref. 3)

$$a = eR_L/T \tag{2}$$

where

e = electron charge

R_L = photodetector load resistance

T = slot integration time

The sequence k_j are independent count variables, in which the signaling slot has discrete count probabilities.

$$Prob(k_j = k) \stackrel{\Delta}{=} P_1(k) \tag{3}$$

while all other (nonsignaling) slots have

$$Prob(k_j = k) \stackrel{\Delta}{=} P_0(k) \tag{4}$$

The slots therefore generate a voltage y given by the mixture density

$$p_i(y) = \sum_{k=0}^{\infty} P_i(k)\,\Psi(y - ak), \quad i = 0, 1 \tag{5}$$

where

$$\Psi(y) = \frac{1}{\sqrt{2\pi}\sigma_n} e^{-y^2/2\sigma_n^2}$$

Here σ_n^2 is the variance of the integrated Gaussian thermal noise variates. For the typical optical receiver at noise temperature T°

$$\sigma_n^2 = 4\kappa T^\circ R_L/T \tag{6}$$

where κ is Boltzmann's constant. We point out that, based on a true slot voltage comparison test among all slots, the PPM channel with mixture densities *cannot* be an erasure channel, since the probability of equal slot voltages (an erasure event) is always zero with continuous densities as in (5).

Although primary photoelectrons released from photo emissive surfaces are usually modeled as obeying a Poisson process, secondary electrons generated via multi-anode secondary emissions as in photomultiplier vacuum tubes, or by avalanche mechanisms, as in avalanche photodetectors (APD), generally produce more symmetrical distributions. These photomultiplied electron distributions are usually modeled (Refs. 4, 5) with Gaussian-shaped discrete probabilities of the form

$$P_i(k) = \frac{C_i}{\sqrt{2\pi\sigma_{di}^2}} e^{-(k-Gm_i)^2/2\sigma_{di}^2}, \quad i = 0, 1 \tag{7}$$

where m_i is the mean primary count, G is the mean photodetector gain, C is a proportionately constant and σ_{di} is the standard deviation of the output counts. The parameter σ_{di} is often called the photodetector "spread," and typically

$$\sigma_{di}^2 = G^\delta m_i \tag{8}$$

where $2 \leqslant \delta \leqslant 3$. (For a photomultiplier tube $\delta \cong 2$, while for an APD, $\delta \cong 3$). The coefficient C_i in (7) is defined by

$$C_i^{-1} = \frac{1}{\sqrt{2\pi\sigma_{di}^2}} \sum_{k=0}^{\infty} e^{-(k-Gm_i)^2/2\sigma_{di}^2} \tag{9}$$

For $Gm_i \geqslant 10$, C is almost identically unity. The PPM symbol error probability, based on a comparison test among the M slot voltage, is then given by

$$PSE = 1 - \int_{-\infty}^{\infty} p_1(y) \left[\int_{-\infty}^{y} p_0(x)\,dx \right]^{M-1} dy \tag{10}$$

The bracketed term corresponds to the distribution function of the mixture density $p_0(x)$. This distribution function can be written as

$$\int_{-\infty}^{y} p_0(x)\,dx = F\left(\frac{y - aGm_0}{\sigma_0}\right) [1 + \epsilon(y)] \tag{11}$$

where

$$F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt \tag{12}$$

$$\sigma_0^2 = a^2\sigma_{d0}^2 + \sigma_n^2 \tag{13}$$

and $\epsilon(y)$ is the fractional error in representing the left side of (11) by a Gaussian distribution function. Extensive studies

(Refs. 3, 6) have shown that $|\epsilon(y)|$ is extremely small. Figure 1a, extracted from Ref. 4, shows how the magnitude $|\epsilon(y)|$ behaves under the condition $\sigma_{d0} = \sigma_n = \beta$ for $y \geqslant 0$. The magnitude function is monotonic in either σ_{d0} or σ_n, and is essentially constant for all $y \geqslant 0$. Figure 1b replots $|\epsilon(y)|$ as a function of β for several values of y. These results simply state that if the variances of the discrete count and the continuous added noise are each large enough, the Gaussian mixture density loses its "discreteness" in integration. Furthermore the resulting integral is within $|\epsilon|$ of integrating an equivalent Gaussian density with the same mean and combined variance. In essence, the Gaussian mixture density behaves as a continuous Gaussian density, as far as integration is concerned.

With (11), the PPM mixture error probability becomes

$$PSE = 1 - \int_{-\infty}^{\infty} p_1(y)\left[F\left(\frac{y - aGm_0}{\sigma_0}\right)\right]^{M-1}\left[1 + \epsilon(y)\right]^{M-1} dy \tag{14}$$

where $\epsilon(y)$ can be either positive or negative at each y, with its magnitude plotted in Fig. 1. We can immediately write

$$[1 + \epsilon(y)]^{M-1} \geqslant [1 - |\epsilon_\beta|]^{M-1} \tag{15}$$

where $|\epsilon_\beta| = |\epsilon(\infty)|$, and we use the subscript β to indicate that we have assumed $\sigma_{d0} = \beta$ and $\sigma_n/a = \beta$ in evaluating $\epsilon(\infty)$. (For all purposes ϵ_β can be taken as the variable in the plot in Fig. 1b at $y = 10$). We can therefore bound the integral in (14) as

$$\int_{-\infty}^{\infty} p_1(y)\, F^{M-1}\left(\frac{y - aGm_0}{\sigma_0}\right)\left[1 + \epsilon(y)\right]^{M-1} dy$$

$$\geqslant [1 - |\epsilon_\beta|]^{M-1} P_c \tag{16}$$

where

$$P_c = \int_{-\infty}^{\infty} p_1(y)\left[F\left(\frac{y - aGm_0}{\sigma_0}\right)\right]^{M-1} dy \tag{17}$$

This also means

$$PSE \leqslant 1 - [1 - |\epsilon_\beta|]^{M-1} P_c \tag{18}$$

Hence the right side serves as an upper bound to PSE. To evaluate this bound, we can write

$$P_c = \int_{-\infty}^{\infty} p_1(y)\left[F\left(\frac{y - aGm_0}{\sigma_0}\right)\right]^{M-1} dy$$

$$= \sum_{k=0}^{\infty} P_1(k)\, \Phi(M, k) \tag{19}$$

with

$$\Phi(M, k) \triangleq \int_{-\infty}^{\infty} \Psi\left(\frac{y - ak}{\sigma_n}\right)\left[F\left(\frac{y - aGm_0}{\sigma_0}\right)\right]^{M-1} dy \tag{20}$$

We recognize $\Phi(M, k)$ as the probability that a Gaussian variable with mean ak and variance σ_n^2 exceeds $M - 1$ independent Gaussian variable with mean aGm_0 and variance σ_0^2. This Φ is simply the detection probability associated with M Gaussian orthogonal variable, with the correct one having mean $(ak - aGm_0)$, all incorrect having mean zero, and all have variance $(\sigma_n^2 + \sigma_0^2)$. These detection probabilities are known (Ref. 7) to depend only on the number of variables M and the signal-to-noise ratio ρ, the latter defined by

$$\rho = \frac{(ak - aGm_0)^2}{\sigma_0^2 + \sigma_n^2} = \frac{(k - Gm_0)^2}{(\sigma_0^2 + \sigma_n^2)/a^2} \tag{21}$$

Thus we can rewrite (20) as simply

$$\Phi(M, k) = PD(M, \rho(k)) \tag{22}$$

where $PD(M, \rho)$ is the M-ary Gaussian word detection probability at an E_b/N_0 of ρ. This forms (19) as the average, over k, of a Gaussian M-ary word detection probability whose bit energy is $(k - Gm_0)^2$.

Some useful approximations to this PSE bound can be derived. For example, since $(1 - \epsilon)^{M-1} \geqslant 1 - M\epsilon$ we can write

$$PSE \leqslant 1 - (1 - M|\epsilon_\beta|)P_c = (1 - P_c) + M|\epsilon_\beta| P_c \tag{23}$$

The first term is now an average word error probability, while the second appears as a correction term. Clearly, if $M|\epsilon_\beta| << 1 - P_c$, the correction term can be neglected. This simply requires the discrete integration error plotted in Fig. 1 to be significantly less than $1/M$ times the desired word error probability. In this case, the bound $1 - P_c$ can be evaluated by simply averaging the Gaussian word error probabilities instead of evaluating (22). This means

$$PSE \leqslant \sum_{k=0}^{\infty} P_1(k)\, PWE(M, \rho(k)) \tag{24}$$

where $PWE(M, \rho)$ is the Gaussian word error probabilities (Ref. 7) of an M-ary test with an E_b/N_0 of ρ. Equation (24) can be further evaluated by introducing the union bound to the Gaussian PWE:

$$PWE(M, \rho) \leqslant \frac{M-1}{2}\, e^{-\rho/2} \tag{25}$$

Equation (24) becomes

$$PSE \leqslant \frac{M-1}{2} \sum_{k=0}^{\infty} P_1(k)\, e^{-(k-Gm_0)^2/2\sigma^2} \tag{26}$$

where we have used

$$\sigma^2 \stackrel{\Delta}{=} \sigma_{d0}^2 + (2\,\sigma_n^2/a^2) \tag{27}$$

For the discrete count distribution of (7) we have

$$PSE \leqslant \left(\frac{M-1}{2}\right)\left(\frac{C_1}{\sqrt{2\pi\sigma_{d1}^2}}\right) \sum_{k=0}^{\infty} \exp - \left[\frac{(k-aGm_1)^2}{2\sigma_{d1}^2} + \frac{(k-Gm_0)^2}{2\sigma^2}\right] \tag{28}$$

Algebraically combining inside the summand yields

$$PSE \leqslant \left\{\left(\frac{M-1}{2}\right) \exp\left[\frac{-G^2(m_1-m_0)^2}{2(\sigma_{d1}^2 + \sigma_{d0}^2 + 2\sigma_n^2/a^2)}\right]\right\} g(m_1, m_0) \tag{29}$$

where

$$g(m_1, m_0) = \sum_{k=0}^{\infty} \frac{C_1}{\sqrt{2\pi}B} e^{-(k-A)^2/2B^2}$$

$$A = \frac{\sigma^2 Gm_1 + \sigma_{d1}^2\, Gm_0}{\sigma_{d1}^2 + (\sigma^2/a^2)}$$

$$B^2 = \frac{\sigma^2 \sigma_{d1}^2/a^2}{\sigma_{d1}^2 + (\sigma^2/a^2)} \tag{30}$$

The term in braces is simply the M-ary union bound based on the mean primary counts m_1 and m_0. The g parameter represents a modification of this bound caused by the mixture density models. Again, if Gm_1 and Gm_0 are large, $g(m_1, m_0)$ will be almost identically one. In this case the bracket accurately estimates performance and corresponds to the result that would be computed from a purely Gaussian assumption on all receiver statistics.

If we assume $g = 1$, substitute from (8) and divide through by G^2 in the exponent of (29) we have

$$PSE \leqslant \frac{M-1}{2} \exp\left[-\frac{(m_1-m_0)^2/2}{F(m_1+m_0) + 2\sigma_n^2/a^2G^2}\right] \tag{31}$$

where $F = G^\delta/G^2$ is referred to as the excess noise factor of the photomultiplier. Note that the denominator represents the usual shot noise-plus-thermal noise contributions obtained in optical receiver analysis.

II. Conclusion

An analysis is presented for determining the symbol error probability of an optical, direct detection, PPM communication system when background noise, nonideal photomultipliers, and postdetection thermal noise are included. This study extends earlier studies based on pure count statistics and simple Gaussian noise models. The effect of the nonideal photomultiplication is to redistribute the count statistics into more symmetrical discrete distribution from those used earlier. The additive thermal noise adds to this density, providing a combined continuous density for the PPM slot integration variables. This combined discrete and Gaussian variable, referred to here as a mixture density, interconnects the earlier discrete count and Gaussian models. When the photomultiplier is high gain and ideal, the count statistics prevail. However, nonideal and low gain devices redistribute the counts, and additive noise can no longer be neglected.

Earlier published results have contended that the redistributed counts appear to have a discrete Gaussian envelope model. It is shown here that under this model the mixture densities begin to behave as a continuous Gaussian density as far as computing error probabilities is concerned. This allows Gaussian M-ary word error probabilities to be used to estimate performance, with signal-to-noise ratio obtained from the usual squared mean to variance ratios. Thus, even though detector statistics are not Gaussian, performance appears to be adequately obtained from Gaussian performance curves.

References

1. Gagliardi, R., and Karp, S., *Optical Communications*, Wiley, 1976, Chapter 8.
2. Massey, J. L., "Capacity, Cut-off Rate, and Coding for a Direction Detection Optical Channel," *IEEE Trans. Comm.*, Vol. COM-26, Nov. 1981.
3. Gagliardi, R., and Prati, G., "On Gaussian Error Probabilities in Optical Receivers," *IEEE Trans Comm.*, pp. 1742-1747, Sept. 1980.
4. McIntyre, R., "Distribution of Gains in Avalanche Photodiodes: Theory," *IEEE Trans. Electron Devices*, pp. 703-712, June 1972.
5. Conradi, J., "Distribution of Gains in Avalanche Photodiodes: Experimental," *IEEE Trans. Electron Devices*, pp. 713-718, June 1972.
6. Prati, G., and Gagliardi, R., "Encoding and Decoding for Optical PPM Channel," *IEEE Trans. IT*, pp. 88-92, Jan. 1982.
7. Lindsey, W. and Simon, M., *Telecommunication System Engineering*, Prentice-Hall, Chapter 6.

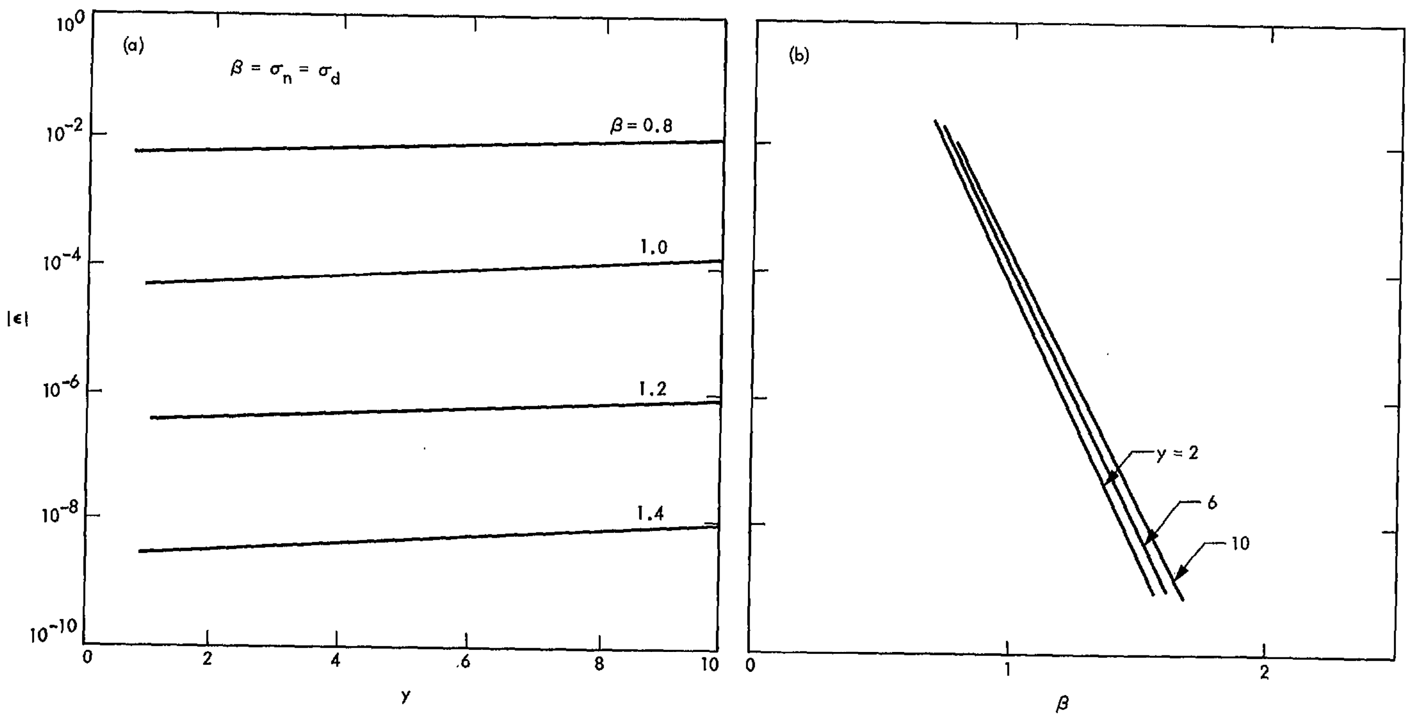

Fig. 1. Fractional error in replacing mixture distribution in y by Gaussian distribution with same mean and variance, σ_n^2 = thermal noise variance; σ_d = photon detector spread