Thermal Analysis of Antenna Backup Structure
Part I — Methodology Development

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An analytic method is devised for predicting the temperature distribution in typical antenna structural back-up members. The results are in good agreement with those obtained by a numerical shooting method. The analytic method developed in this work has shown a good potential in greatly simplifying the thermal analysis process for complex back-up antenna structures.

I. Introduction

Future requirements for deep space navigation and telemetry link communications point to the need for higher frequency bands. At these higher frequencies, data rates are better but antenna surface accuracy requirements are more severe than those of the presently used S- or X-bands. Guided by the accuracy needs, studies are now in progress on the feasibility of constructing large Ka-band antennas which will use frequencies at 32 GHz or higher (Ref. 1).

The construction requirements for these Ka-band antennas are much more demanding than those for presently used devices. For example, the reflective surfaces must be positioned and aligned much more accurately and must be maintained during the life of the antenna against varying gravity, wind, and thermal loadings. Environmental changes such as temperature variations can have deleterious effects on the antenna performance. Field thermal measurements (Ref. 2) and analytic investigations are being conducted to study the environmental effects on the structural members of the large 64-m antenna. In the analytical investigation, which is the subject of this article, we are interested in simulating the temperature distribution throughout the complex back-up

structural members given their physical properties, geometric arrangement, and environmental conditions, such as air temperature, wind velocity, and solar irradiation (insolation).

The simulation of the temperature pattern of a structural member can be done by "conventional" numerical finite difference methods in heat transfer which divide the member under consideration into nodes and then apply heat balance equations to each node. Rather than solving for an excessively large network of nodes for a complex antenna back-up structure, a new method is developed in this paper to save effort and time. The method is analytical in nature and relies on deriving a universal relationship for the temperature variations and heat fluxes within each member. The methodology is described only in this article and will be followed by additional applications in subsequent TDA reports.

II. Methodology Development

Consider a simple bar, of length L, as sketched in Fig. 1 subjected to solar radiation, conduction, convection, and radiation heat exchange with the ambient air. The cross section of the bar, A, and its material properties are assumed

constant along the bar's length. For an element of incremental length dx along the bar axis, whose temperature is $T(X)$, the convection losses are given by:

$$dq_c = h P \, dx \, (T - T_a) \tag{1}$$

the radiative losses by:

$$dq_r = \epsilon \, \sigma \, \mathscr{F} P \, dx \, (T^4 - T_a^4) \tag{2}$$

the absorbed portion of the solar irradiation by:

$$dq_i = \alpha I P \, dx \tag{3}$$

and the change in conductive heat transfer stored in the element by:

$$dq_k = \frac{d}{dx} \left(kA \frac{dT}{dx} \right) dx \tag{4}$$

The complete heat balance equation at steady-state conditions can be written as:

$$\frac{d^2 T}{dx^2} = \frac{hP}{kA} (T - T_a) + \frac{\epsilon \sigma P \mathscr{F}}{kA} (T^4 - T_a^4) - \frac{\alpha I P}{kA} \tag{5}$$

Equation (5), together with the boundary conditions

$$\text{at} \quad x = 0 \quad , \quad T = T(0) \tag{6}$$

$$\text{at} \quad x = L \quad , \quad T = T(L) \tag{7}$$

forms a nonlinear boundary value problem for $T(X)$. In general, such problems cannot be solved analytically and numerical methods must be employed (Refs. 3 and 4). Several of these methods are presented in the paragraphs which follow.

A. The "Shooting" Method

An outline of the "shooting" method and its use is given in Ref. 4. Using the subroutine described in that reference, Eq. (5) can be written as a system of equations in the form:

$$Y_1' = Y_2 \tag{8}$$

$$Y_2' = C_1 Y_1^4 + C_2 Y_1 + C_3 \tag{9}$$

where Y_1 stands for the temperature, T, while Y_2 and Y_2' for dT/dx and d^2T/dx^2 respectively, and the superscript (') for d/dx. The constants C_1, C_2 and C_3 are defined as:

$$C_1 \equiv \frac{\epsilon \sigma P \mathscr{F}}{kA} \tag{10}$$

$$C_2 \equiv \frac{hP}{kA} \tag{11}$$

$$C_3 \equiv -\frac{hP}{kA} T_a - \frac{\epsilon \sigma P \mathscr{F}}{kA} T_a^4 - \frac{\alpha I P}{kA} \tag{12}$$

This numerical method will serve as a useful check on the analytical results to be developed later.

B. Perturbation Method

This analytic method of solving Eq. (5) using perturbation requires that the equation must become linear in the limit of some selected small parameter. The parameter C_1 in Eq. (10) is small because of the Stefan-Boltzman constant σ of the order 10^{-10}, while C_2 and C_3 from Eqs. (11) and (12) are of the order 1 and greater. The perturbation method assumes that the solution can be written in the form:

$$T(x) = T_0(x) + C_1 T_1(x) + C_1^2 T_2(x) + \cdots \tag{13}$$

Then, by inserting (13) into (5), one obtains a set of linear equations such as:

$$\frac{d^2 T_0}{dx^2} - C_2 T_0 - C_3 = 0 \tag{14}$$

$$\frac{d^2 T_1}{dx^2} - C_2 T_1 - C_3 = T_0^4 \tag{15}$$

$$\frac{d^2 T_2}{dx^2} - C_2 T_2 - C_3 = T_1^4 \tag{16}$$

$$\vdots$$

which must be solved sequentially.

The disadvantage of applying this method for our case is that once an expression for T_0 is found by solving Eq. (14), subsequent equations will become very complicated due to the presence of terms such as T_0^4, T_1^4, etc. For this reason, the perturbation method was not examined further.

C. Linearization Method

Equation (5) will be reduced to a closed form solution for $T(X)$ by linearizing the radiation term in Eq. (2) by writing it as:

$$dq_r = h_r P\, dx\, (T - T_a) \quad (17)$$

By comparing Eq. (17) with Eq. (2) one obtains:

$$h_r = \epsilon\sigma\,\mathscr{F}(T^2 + T_a^2)(T + T_a) \quad (18)$$

The initial value of h_r can be obtained by making the approximation $T \cong T_a$. This assumption is valid in cases where one would not expect too much difference between the temperature of the structure and that of the ambient air. Note, however, that such an assumption cannot be used for a structure in space where the temperature of the structure is much different from the surrounding space temperature, ($T_a = 0$ K). As the temperature pattern is known, a modified value of h_r can be obtained at the average link temperature.

By using Eq. (17) in Eq. (5), one obtains the linearized differential equation:

$$\frac{d^2T}{dx^2} = \left[\frac{(h_r + h_c)P}{kA}\right]T - \left[\frac{(h_r + h_c)P}{kA}T_a + \frac{\alpha IP}{kA}\right] \quad (19)$$

and by making use of the abbreviations

$$\psi \equiv \frac{(h_r + h_c)P}{kA} \quad (20)$$

$$\xi \equiv \frac{(h_r + h_c)P}{kA}T_a + \frac{\alpha IP}{kA} \quad (21)$$

one obtains the solution of Eq. (19) as:

$$T(x) = \frac{\left[T(L) - \frac{\xi}{\psi}\right] - \left[T(0) - \frac{\xi}{\psi}\right]\cosh\sqrt{\psi}L}{\sinh\sqrt{\psi}L}\sinh\sqrt{\psi}x$$

$$+ \left[T(0) - \frac{\xi}{\psi}\right]\cosh\sqrt{\psi}x + \frac{\xi}{\psi} \quad (22)$$

The ratio ξ/ψ from Eqs. (20) and (21) is:

$$\frac{\xi}{\psi} = T_a + \frac{\alpha I}{h_r + h_c} \quad (23)$$

Hence, the physical meaning of ξ/ψ represents a balance between the incoming insolation and the rate of radiative and convective losses, all with respect to the ambient temperature, T. Therefore, ξ/ψ is an "equilibrium" temperature which will be denoted by T_e. In terms of T_e, Eq. (22) becomes

$$T(x) = \frac{[T(L) - T_e] - [T(0) - T_e]\cosh\sqrt{\psi}L}{\sinh\sqrt{\psi}L}\sinh\sqrt{\psi}x$$

$$+ [T(0) - T_e]\cosh\sqrt{\psi}x + T_e \quad (24)$$

III. Numerical Example

To check the validity of Eq. (24) versus the more accurate shooting method, let us consider a numerical example. In this example we assume a steel bar having the following geometry:

$$\text{Perimeter}, P = 0.305\text{m (1 ft)}$$

$$\text{Length}, L = 0.91\text{m (3 ft)}$$

$$\text{Cross sectional area}, A = 5.81 \times 10^{-3}\text{ m}^2\ (0.0625\text{ ft}^2)$$

The two end temperatures are kept at:

$$T(0) = 311\text{ K (560°R)}$$

$$T(L) = 322\text{ K (580°R)}$$

while the ambient temperature is:

$$T_a = 294\text{ K (530°R)}$$

For these temperatures, and assuming typical Goldstone values for wind speed [16.1 m/hr (10 mph)] and insolation (800 W/m^2), we will use the following heat transfer coefficients:

conductivity, $k = 45.0$ W/mK (26.0 BTU/hr ft°R)

radiation heat transfer coefficient $h_r = 3.1$ W/m^2 K (0.55 BTU/hr ft^{2}R)

convective heat transfer coefficient, $h = 22.7$ W/m^2 K (4.0 BTU/hr ft^2°R)

emissivity, $\epsilon = 0.5$

view factor, $\mathscr{F} = 1.0$

absorptivity, $\alpha = 0.4$

insolation, $I = 800$ W/m^2 (254 BTU/hr ft^2)

For these values, $\psi = 30.1$ m^{-2} (2.8 ft.$^{-2}$) and $T = 308$ K (552°R). The results of the shooting method and the linearization method in Eq. (24) are shown in Tables 1 and 2. A plot of the results of Table 2 is made in Fig. 2. The excellent agreement between the analytical and numerical results validates the use of Eq. (24). Note that the temperature distribution within the bar need not be monotonic; in this particular case, the temperature within the bar reaches a point where it is lower than either of the end temperatures.

IV. The Multi-link Problem

Consider now a system of 3 bars linked together as illustrated in Fig. 3. The physical and geometric properties of each bar can be different and so can their exposure to solar radiation. The temperatures at the junctions are assumed to be unknown. The problem is to find a simple way of determining these temperatures and then, by using Eq. (24), to obtain the temperature distribution throughout the links.

The solution relies on the fact that, at each junction of two or more members, the heat flux balance at steady state is written as:

$$\Sigma q_{i,j,k, \ldots} = 0$$

Since the only heat transfer at the junction points is through conduction, for the case we are considering, the following equations are valid:

$$q_{1,2} = \left[-k_1 A_1 \left(\frac{dT}{dx} \right)_0 \right]_1 - \left[k_2 A_2 \left(\frac{dT}{dx} \right)_L \right]_2 = 0 \quad (25)$$

$$q_{2,3} = \left[-k_2 A_2 \left(\frac{dT}{dx} \right)_0 \right]_2 - \left[k_3 A_3 \left(\frac{dT}{dx} \right)_L \right]_3 = 0 \quad (26)$$

$$q_{3,1} = \left[-k_3 A_3 \left(\frac{dT}{dx} \right)_0 \right]_3 - \left[k_1 A_1 \left(\frac{dT}{dx} \right)_L \right]_1 = 0 \quad (27)$$

In Eqs. (25) through (27), the square brackets are subscripted by the link number, while the derivitive dT/dx is to be taken at the "beginning" (0) or the "end" (L) of the respective link. The assignment of (0) and (L) to each link is completely arbitrary. Note that, based on Eq. (24):

$$\left(\frac{dT}{dx} \right)_L = \frac{-\sqrt{\psi}}{\sinh \sqrt{\psi} L} [(T(0) - T_e) - (T(L) - T_e) \cosh \sqrt{\psi} L] \quad (28)$$

$$\left(\frac{dT}{dx} \right)_0 = \frac{\sqrt{\psi}}{\sinh \sqrt{\psi} L} [(T(L) - T_e) - (T(0) - T_e) \cosh \sqrt{\psi} L] \quad (29)$$

By substituting Eq. (28) and (29) into Eqs. (25) through (27), one gets three equations and six unknowns, the unknowns being the temperatures $T(L)_i$ and $T(0)_i$ ($i = 1, 2, 3$). However, three of these temperatures are redundant because of the junction conditions:

$$T(0)_1 = T(L)_2 \quad (30)$$

$$T(0)_3 = T(L)_1 \quad (31)$$

$$T(L)_3 = T(0)_2 \quad (32)$$

We choose to solve for $T(0)_i$ ($i = 1, 2, 3$) and to use the following abbreviations:

$$\Theta_i \equiv k_i A_i \frac{\sqrt{\psi_i}}{\sin h \sqrt{\psi_i} L_i} \quad (33)$$

$$\beta_i \equiv \Theta_i T_{e_i} (\cosh \sqrt{\psi_i} L_i - 1) \quad (34)$$

$$\delta_i \equiv \Theta_i \cosh \sqrt{\psi_i} L_i \quad (35)$$

Then, the system Eqs. (25) through (27), by using Eqs. (30) through (32) and (24), can be written in matrix form as:

$$
\begin{pmatrix}
\delta_1 + \delta_2 & \theta_2 & -\theta_1 \\
-\theta_2 & \delta_2 + \delta_3 & -\theta_3 \\
-\theta_1 & -\theta_3 & \delta_1 + \delta_3
\end{pmatrix}
\begin{pmatrix}
T(0)_1 \\
T(0)_2 \\
T(0)_3
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
\beta_1 + \beta_2 \\
\beta_2 + \beta_3 \\
\beta_1 + \beta_3
\end{pmatrix}
\tag{36}
$$

The solutions of Eq. (36) are given by:

$$
T(0)_1 = \frac{
\begin{bmatrix}
(\beta_1 + \beta_2) & \theta_2 & -\theta_1 \\
(\beta_2 + \beta_3) & \delta_2 + \delta_3 & -\theta_3 \\
(\beta_1 + \beta_3) & -\theta_3 & \delta_1 + \delta_3
\end{bmatrix}
}{\mathscr{D}}
\tag{37}
$$

$$
T(0)_2 = \frac{
\begin{bmatrix}
\delta_1 + \delta_2 & (\beta_1 + \beta_2) & -\theta_1 \\
-\theta_2 & (\beta_2 + \beta_3) & -\theta_3 \\
-\theta_1 & (\beta_1 + \beta_3) & \delta_1 + \delta_3
\end{bmatrix}
}{\mathscr{D}}
\tag{38}
$$

$$
T(0)_3 = \frac{
\begin{bmatrix}
\delta_1 + \delta_2 & \theta_2 & (\beta_1 + \beta_2) \\
-\theta_2 & \delta_2 + \delta_3 & (\beta_2 + \beta_3) \\
\theta_1 & -\theta_3 & (\beta_1 + \beta_3)
\end{bmatrix}
}{\mathscr{D}}
\tag{39}
$$

where:

$$
\mathscr{D} =
\begin{bmatrix}
\delta_1 + \delta_2 & \theta_2 & -\theta_1 \\
-\theta_2 & \delta_2 + \delta_3 & -\theta_3 \\
-\theta_1 & -\theta_3 & \delta_1 + \delta_3
\end{bmatrix}
\tag{40}
$$

V. Conclusions

In this first report we have outlined a simple, analytic method for obtaining the temperature distribution in a typical arrangement of antenna structural back-up member. The results were verified by the numerical shooting method. Good agreement between the two methods was obtained. Further work will deal with setting up the matrices for more complex structures, with radiation exchange, and with comparison between simulations and actual field measurements.

References

1. Baars, J. W., "Technology of Large Radio Telescopes for Millimeter and Submillimeter Wavelengths", *Reviews in Infrared and Millimeter Waves*, Vol. 2, K. J. Button ed., Plenum Publishing Corp., N.Y.

2. Glazer, S., and Gayle, G., "Thermal Measurement Technique of Rib Elements on DSN Antenna Structure", *The Telecommunications and Data Acquisition Progress Report 42-66*, Jet Propulsion Laboratory, Pasadena, Calif., December 15, 1981.

3. Smith, J. P., *SINDA User's Manual*, RO-71-810, TRW Systems Group, April 1971.

4. Schonfeld, D., "Analysis of Capacitive Heat Exchangers, Part II", *The Telecommunications and Data Acquisition Progress Report 42-69*, Jet Propulsion Laboratory, Pasadena, Calif., June 15, 1982.

Table 1. Example of temperature distribution obtained by shooting method (B)	
x	T
0.00	0.560+03
0.10	0.559+03
0.20	0.558+03
0.30	0.557+03
0.40	0.557+03
0.50	0.556+03
0.60	0.556+03
0.70	0.555+03
0.80	0.555+03
0.90	0.555+03
1.00	0.555+03
1.10	0.555+03
1.20	0.555+03
1.30	0.555+03
1.40	0.555+03
1.50	0.555+03
1.60	0.555+03
1.70	0.556+03
1.80	0.556+03
1.90	0.557+03
2.00	0.558+03
2.10	0.559+03
2.20	0.560+03
2.30	0.561+03
2.40	0.562+03
2.50	0.564+03
2.60	0.566+03
2.70	0.569+03
2.80	0.572+03
2.90	0.576+03
3.00	0.580+03

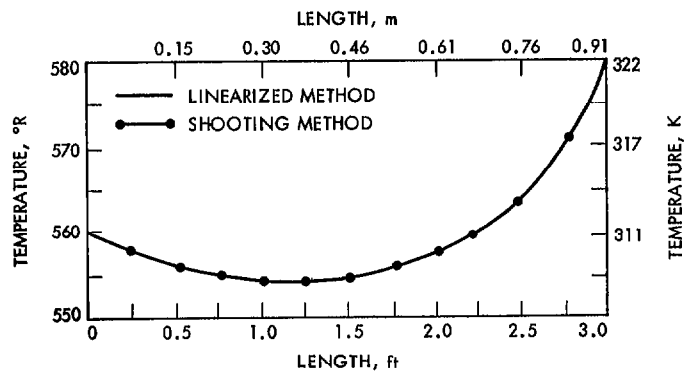| Table 2. Example of temperature distribution using the linearization method (C)	
x	T
0.00	0.560+03
0.10	0.559+03
0.20	0.558+03
0.30	0.557+03
0.40	0.556+03
0.50	0.556+03
0.60	0.555+03
0.70	0.555+03
0.80	0.555+03
0.90	0.555+03
1.00	0.554+03
1.10	0.554+03
1.20	0.554+03
1.30	0.555+03
1.40	0.555+03
1.50	0.555+03
1.60	0.555+03
1.70	0.556+03
1.80	0.556+03
1.90	0.557+03
2.00	0.558+03
2.10	0.558+03
2.20	0.560+03
2.30	0.561+03
2.40	0.562+03
2.50	0.564+03
2.60	0.566+03
2.70	0.569+03
2.80	0.572+03
2.90	0.576+03
3.00	0.580+03

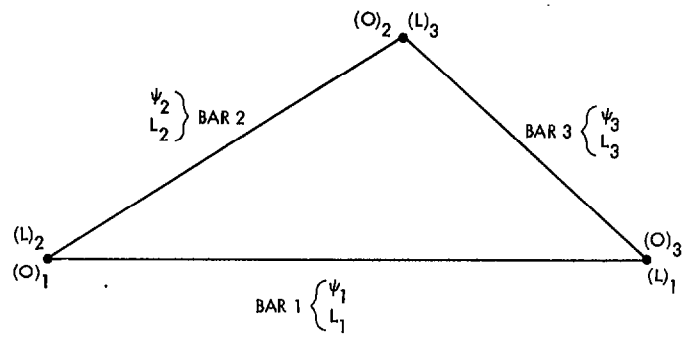Fig. 2. Temperature distribution in a simple bar using linearization method
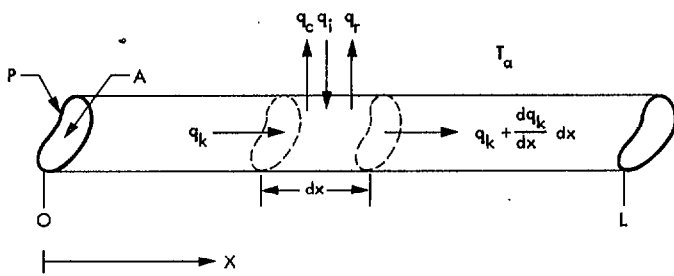
Fig. 1. Physical system for a simple bar

Fig. 3. Three-bar linkage