Rates of Solar Angles for Two-Axis Concentrators

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Determination of the sun's position by the azimuth and elevation angles and its rate of change at any time of day are essential for designing 2-axis tracking mechanisms of solar concentrators. A study of the sun's angles and their rates is presented at four selected months of the year (March, June, September and December) and for seven selected latitudes (0, ±30, ±60, ±90) covering both the northern and southern hemispheres.

I. Introduction

Solar thermal-electric power systems are being studied for the Deep Space Network Energy Conservation Project to evaluate their technical and economical feasibility. The engineering design of high temperature, point-focusing two-axis tracking solar concentrators in an unattended mode of operation requires, among the many design specifications, a knowledge of the sun angles and angle rates to be incorporated in the automatic control loop parameters. To satisfy this need for different network facilities throughout the world, this study was initiated to determine the sun's position vector, rate of movement in the sky at any given location and at any time of year. Both northern and southern latitudes were investigated for their possible differences at different seasons.

II. Solar Angles

Because the earth's equatorial plane is tilted at an angle of 23.5 degrees to the orbital plane as shown in Fig. 1, the solar declination angle δ, which is the angle between the earth-sun line and the equatorial plane, varies throughout the year. This variation causes the changing seasons with their unequal periods of daylight and darkness. The geometric configurations of the position vector of the sun at any time of day relative to the earth center and relative to a horizontal plane passing by point P on earth, are presented in Figs. 2 and 3. From Fig. 2, the latitude angle of location in radians L is defined as the angle between the location position vector OP and the equatorial plane (north latitudes are taken positive). The solar-azimuth angle in radians ϕ as shown in Fig. 3 is defined as the angle between the true south direction and the projection of the sun's position vector on the horizontal plane. The south direction is taken as the reference line for both northern and southern latitudes. The sign convention for the angle ϕ is that east of the south is taken positive, and west of the south is taken negative. The hour angle H, which is an indication of the local solar time, is changing with a rate of 15 deg/hr (360 deg/24 hours). The sign for H is taken positive in the morning, and negative in the afternoon, and zero at solar noon. The solar elevation, sometimes called the altitude angle β, and the solar zenith angle z are also shown in Fig. 3 as the angles the solar position vector makes with the horizontal plane and the vertical line, respectively.

Since the declination angle δ does not vary significantly from one day to the next within each month (Refs. 1, 2), a one-day representation of a given month was considered satisfactory. The variation of solar declination throughout the year from a maximum of about 23.5 deg on June 21 to a minimum of –23.5 deg on December 21, gave us a reason to select the twenty-first day of each month to be the representative day of the month. By treating the solar declination tabular data (in Ref. 2) as a periodic function with one complete cycle per year, the following analytic expression was obtained by least squares curve fitting:

$$\delta = (\pi/180)(0.2833 - 23.188 \cos P - 0.15 \cos 2P - 0.211 \sin P + 0.1155 \sin 2P) \quad (1)$$

where

$$P = \pi M/6 \quad (2)$$

and M is a month index ($M = 1, 2, \ldots, 12$).

The time of day, described by the hour angle H in radians, is also written as

$$H = \pi(12 - T)/12 \quad (3)$$

where T is the hourly time index in hours ($T = 1, 2, 3, \ldots, 24$). Note $T = 12$ means a solar noon time.

The direction cosines of a unit vector along the location position vector are found from the geometry of OP in Fig. 2 as: $\cos L \cos H$, $-\cos L \sin H$ and $\sin L$, with the X, Y and Z axes, respectively. Note that the equatorial plane lies on the X - Y plane, and point P' is the projection of point P (representing the location on earth) on the equatorial plane. The direction cosines of the sun's radiation unit vector can be also obtained from Fig. 2 as: $\cos \delta$, 0 and $\sin \delta$ with the X, Y and Z axes, respectively. The scalar product of the above two unit vectors will determine the cosine of the zenith angle z (or the sine of the elevation angle β):

$$\cos z = \sin \beta = \cos L \cos \delta \cos H + \sin L \sin \delta \quad (4)$$

On the other hand, a unit vector along the direction of true south ($\overline{PS}$ in Fig. 2) will have the direction cosines: $\sin L \cos H$, $-\sin L \sin H$ and $-\cos L$ with the X, Y and Z axes, respectively. Accordingly, the angle ω, from Fig. 3, which is the angle between the sun's vector and the south direction is obtained by scalar product as

$$\cos \omega = \cos \beta \cos \phi = \cos \delta \sin L \cos H - \cos L \sin \delta$$

or

$$\cos \phi = (\cos \delta \sin L \cos H - \cos L \sin \delta)/\cos \beta \quad (5)$$

An additional relationship for the azimuth angle ϕ, can be obtained also by multiplying scalarly the true east vector and the sun's position vector as shown in Fig. 3, to yield

$$\sin \phi = \cos \delta \sin H/\cos \beta \quad (5a)$$

At solar noon (i.e., $H = 0$), the azimuth angle ϕ is always zero at all locations throughout the year. Also, the angle β is always equal to $(90 - L + \delta)$ deg for both northern and southern latitudes (from Eq. 4). Equations (4) or (5) are subject to one constraint: the argument of the right-hand side should be within ±1.

The hour angle limit H^*, which determines the hour angle at either sunrise or sunset as measured from solar noon, is given by equating the elevation angle β to zero. Using Eq. (4), the angle H^* (which is half the solar day) is written as:

$$H^* = \cos^{-1}(-\tan L \tan \delta) \quad (6)$$

where the term $(-\tan L \tan \delta)$ in Eq. (6) should not exceed 1, and the minimum value should not be less than -1. Hence, H^* (in radians) can vary between zero and π. If H^* equals to zero, the location on earth will be in complete darkness for the entire day. If H^* equals to π, the solar day will be 24 hours and the location will receive continuous sunlight for 24 hours. The above limiting conditions can only occur at higher latitude angles in the northern hemisphere (or lower angles in the southern hemisphere) than + 66.5 deg, which is a marginal latitude given by Eq. (6) at $\delta = 23.5$ deg. For example, the city of Bettles, Alaska, USA (with 66.5 deg north latitude) has twenty-four hours of sunlight in June and receives no sunlight in December. This fact could be explained by using Eq. (6) since the declination angle is +23.5 deg on June 21 and -23.5 deg on December 21 for all latitudes.

The azimuth angle at sunrise or sunset ϕ^* can be obtained, also, from combining Eqs. (5) and (6) at any latitude or declination angles where β is set equal to zero.

$$\cos \phi^* = -\sin \delta/\cos L \quad (7)$$

Special cases arise for Eqs. (4) through (6). First is the case during the equinox (March 21 and September 21) when the declination angle is zero for all latitudes. The elevation and azimuth angles, for these two months, become:

$$\left.\begin{aligned} \sin \beta &= \cos L \cos H \\ \cos \phi &= \sin L \cos H/\cos \beta \\ \sin \phi &= \sin H/\cos \beta \\ H^* &= \pi/2 \end{aligned}\right\} \text{ at } \delta = 0 \quad (8)$$

The above indicates that the day length (from sunrise to sunset) is always 12 hours during the equinox.

Second is the case of a location at the equator. The solar angles at different declination and hour angles are reduced to:

$$\left.\begin{aligned} \sin\beta &= \cos\delta\cos H \\ \cos\phi &= -\sin\delta/\cos\beta \\ H^* &= \pi/2 \end{aligned}\right\} \quad \text{at } L = 0 \qquad (9)$$

Eq. (9) shows that the day length at the equator is always 12 hours.

Third is the case at the north pole (L = +90 deg). The solar angles at different declination and hour angles are reduced to:

$$\left.\begin{array}{lll} \beta = \delta & \text{if} & \delta \geqslant 0 \\ \phi = H & \text{if} & \delta \geqslant 0 \\ H^* = \pi & \text{if} & \delta \geqslant 0 \\ \text{No sun} & \text{if} & \delta < 0 \end{array}\right\} \quad \text{at } L = +90 \text{ deg} \qquad (10)$$

The fourth special case is at the south pole (L = -90 deg) where the solar angles become:

$$\left.\begin{array}{lll} \beta = -\delta & \text{if} & \delta \leqslant 0 \\ \phi = (\pi - H) & \text{if} & \delta \leqslant 0 \\ H^* = \pi & \text{if} & \delta \leqslant 0 \\ \text{No sun} & \text{if} & \delta > 0 \end{array}\right\} \quad \text{at } L = -90 \text{ deg} \qquad (11)$$

III. Angle Rates

The time rate of change of solar angles can be determined by differentiating Eqs. (3), (4) and (5) with respect to time for the hour angle, elevation angle and azimuth angle, respectively. The units of the angle rate are expressed in degrees per second. The elevation angle rate $\dot{\beta}$ (in deg/sec), at a given δ, L, can be expressed from Eqs. (3) and (4) as

$$\dot{\beta} = \frac{d\beta}{dt} = \frac{0.00417 \cos L \cos\delta \sin H}{\cos\beta} \qquad (\beta \neq 0 \text{ des}) \quad (12)$$

Similarly, the azimuth angle rate $\dot{\phi}$ (in deg/sec), at given δ and L, can be expressed as

$$\dot{\phi} = \frac{d\phi}{dt} = 0.00417 \left\{ \frac{\sin^2 H \cos L \sin\beta \cos^2\delta - \cos H \cos^2\beta \cos\delta}{\cos^3\beta \cos\phi} \right\} \qquad (13)$$

where β and $|\phi|$ must not equal to 90 deg.

Equations (12) and (13) are valid for any H, L, or δ angles. However, they are reduced to other forms for some special cases presented as follows:

A. Noon Time

At noon time (H = 0), the rate $\dot{\beta}$ will be zero for any latitude and any month; which means that β either reaches a maximum point or remains constant. The rate $\dot{\phi}$ is also obtained from Eq. (13) at noon time where:

$$\left.\begin{aligned} \beta &= 90 - (L - \delta) \\ \phi &= 0 \\ \dot{\beta} &= 0 \\ \dot{\phi} &= -0.00417 \cos\delta/\sin(L - \delta) \end{aligned}\right\} \quad H = 0 \qquad (14)$$

B. Sunrise or Sunset Time

The angle rates at sunrise or sunset times (denoted by an asterisk) are obtained from Eqs. (6), (7), (12) and (13) at any latitude or declination angles where

$$\left.\begin{aligned} \beta^* &= 0 \\ \phi^* &= \cos^{-1}(-\sin\delta/\cos L) \\ \dot{\beta}^* &= 0.00417\sqrt{\cos(L+\delta)\cos(L-\delta)} \\ \dot{\phi}^* &= -0.00417 \sin L \end{aligned}\right\} \quad \text{at } H^* \qquad (15)$$

C. At the Equator

At the equator (where L = 0), the angles are obtained from Eq. (9). The angle rates are obtained from Eqs. (12) and (13) at any hour angle H as:

$$\left.\begin{aligned}\dot\beta &= 0.00417\cos\delta\sin H/\sqrt{1-\cos^2\delta\cos^2 H}\\ \dot\phi &= 0.00417\sin\delta\cos\delta\cos H/(1-\cos^2\delta\cos^2 H)\end{aligned}\right\} L=0 \qquad (16)$$

At noon time, for a location at the equator, the rates $\dot\beta$ and $\dot\phi$ from Eq. (16) are reduced to:

$$\left.\begin{aligned}\dot\beta &= 0\\ \dot\phi &= 0.00417\cos\delta\end{aligned}\right\} \begin{aligned}L&=0\\ H&=0\end{aligned} \qquad (17)$$

Eq. (17) is identical to Eq. (14) when setting $H = 0$.

Also, at sunrise (or sunset), for a location at the equator ($H^* = \pi/2$), the rates $\dot\beta$ and $\dot\phi$ from Eq. (15) or Eq. (16) are

$$\left.\begin{aligned}\dot\beta &= 0.00417\cos\delta\\ \dot\phi &= 0\end{aligned}\right\} \begin{aligned}L&=0\\ H&=H^*\end{aligned} \qquad (18)$$

Equation (16) for the equator is plotted as shown in Fig. (6) at different H and δ angles. Peak values for $\dot\beta$ are evidenced at sunrise or sunset time in March and September ($\delta = 0$) as 0.00417 deg/sec. On the other hand, the peak values for $\dot\phi$ can reach infinity as $\delta \to 0$. As δ changes between ±23.5 deg, the rate $\dot\phi$ changes between ±0.0096 deg/sec.

D. At the North Pole

The angles β, ϕ, H^* at the north pole, from Eq. (10), are substituted in Eqs. (12) and (13) where $L = 90$ and $\delta \geqslant 0$, hence,

$$\left.\begin{aligned}\dot\beta &= 0\\ \dot\phi &= -0.00417\end{aligned}\right\} L = +90 \text{ deg} \qquad (19)$$

Eq. (19) is applicable for any hour-of-day variation at any month where $\delta \geqslant 0$.

E. At the South Pole

The angles β, ϕ, H^* at the south pole, from Eq. (11), are substituted in Eqs. (12) and (13), where $L = -90$, and $\delta \leqslant 0$. Hence,

$$\left.\begin{aligned}\dot\beta &= 0\\ \dot\phi &= +0.00417\end{aligned}\right\} L = -90 \text{ deg} \qquad (20)$$

Eq. (20) is applicable for any hour-of-day variation at any month where $\delta \leqslant 0$.

IV. Discussion of Results

A. Solar Angles

The computations of the solar angles at some selected hours of day, months of year and latitudes were compared against the tabulated data given in Ref. 2. A good agreement was found in all cases. Further, the rates of solar angle variations were determined at seven different latitudes covering northern and southern hemispheres including the equator. In the northern hemisphere, latitudes +30, +60, and +90 deg were selected. In the southern hemisphere, latitudes -30, -60, and -90 were selected.

For four selected months throughout the year (March, June, September and December), the solar angles were plotted as shown in Figs. 4 and 5. Figure 4 shows the solar elevation angles. If $\delta = 0$, the solar elevation angle reaches its peak (90 deg) ar solar noon, and the sun's position vector travels in a vertical plane containing the east-west directions. Figure 4 also gives the solar elevation angles for the remaining six latitudes: ±30, ±60 and ±90 deg. Figure 5 shows the comparison of the solar azimuth angle for the seven selected latitudes. The profile of the solar altitude angles, however, is similar in both northern and southern locations of equal latitude. The two hemispheres experience an opposite climatic effect; the "summer" season, including the months of June, July and August in the northern hemisphere, is the "winter" season in the southern hemisphere.

At the north pole, the sun is barely visible at the horizon on March 21 and September 21 as evidenced from Eq. (10). In June, the sun maintains a constant +23.5 deg with the horizontal plane all day long. On the contrary, the south pole experiences 24 hours of daylight in December ($\delta = -23.5$) as evidenced from Eq. (11).

The azimuth angle is always zero (or 180°) at solar noon, and at sunset it is obtained from Eq. (7). In northern latitudes, the trace of the azimuth angle is mostly located in the southern quadrants of the horizontal plane; hence solar concentrators should be oriented facing south. The opposite situation exists in the southern hemisphere; i.e., solar concentrators in the southern hemisphere should be oriented facing north.

B. Angle Rates

Figures 6 through 10 show the time rate of change of solar angles ($\dot\beta$ and $\dot\phi$) at the selected latitudes. Figure 6 shows the angle rates at the equator as obtained from Eq. (16). The rate of elevation angle in March and September ($\delta = 0$) is constant at 15 deg per hour) throughout the day except at solar noon where the rate is zero. The maximum rate of azimuth angle $\dot{\phi}$ is 0.0096 deg/sec at solar noon in June.

Figures 7 and 8 present the angle rates in the northern hemisphere where L = 30 and 60, respectively. In general, the angle rates in the northern hemisphere decrease as the latitudes increase. In the north pole, the rate of elevation angle is zero, while the azimuth angle travels at a constant rate of -15 deg/hr (-0.00417 deg/sec).

Figures 9 and 10 present the angle rates for the southern hemisphere where L = -30 and -60 degrees respectively. In all cases studied, with the exception of the north and south poles, the rate of elevation angle $\dot{\beta}$ decreases to zero at noon time from its maximum value at sunrise (or sunset) while the rate of azimuth angle $\dot{\phi}$ increases from its low sunset value to a peak value at solar noon.

The development of the above angle and angle rate analytical expressions for any month, hour of day, and latitude represents an initial study which provides the solar concentrator designer with a quantitative determination of the limiting sun's position and angle rates for an accurate automatic tracking mechanism.

References

1. Threlkeld, J. L., *Thermal Environmental Engineering*, Prentice-Hall, 1962.
2. ASHRAE, *Handbook of Fundamentals*, American Society of Heating, Refrigeration and Air Conditioning Engineers, 1972.

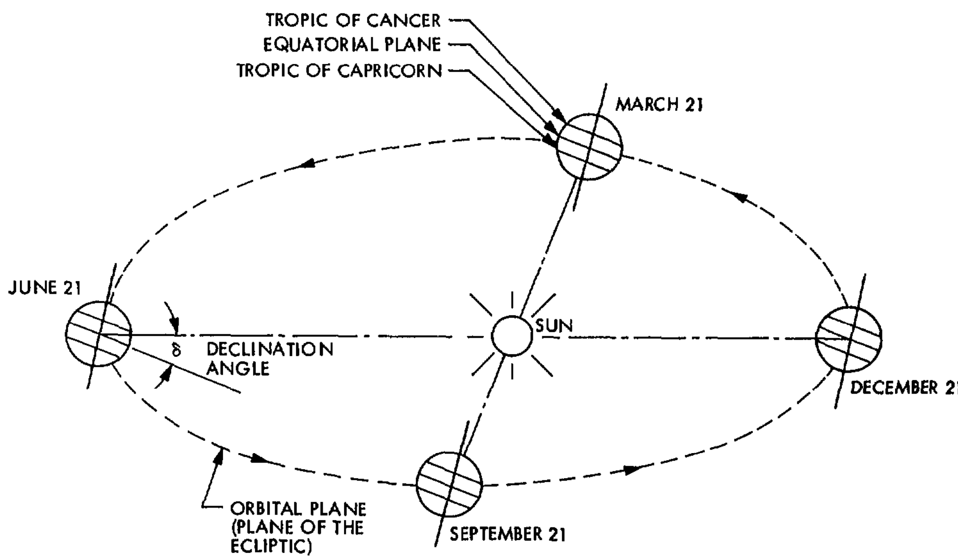

Fig. 1. The earth's motion around the sun

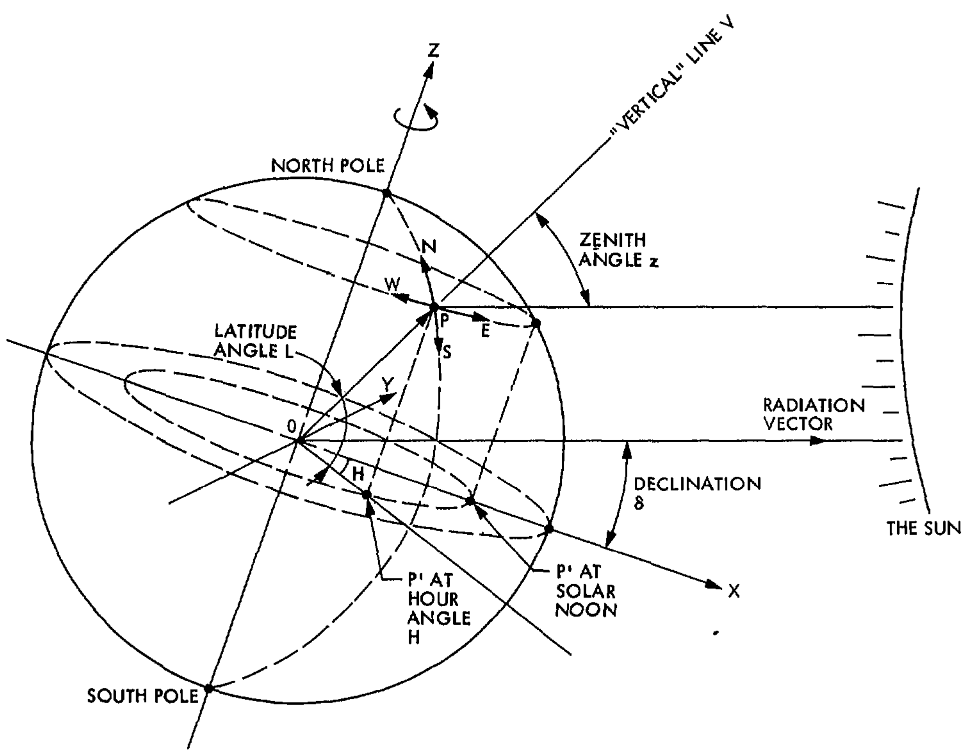

Fig. 2. Earth-sun angles

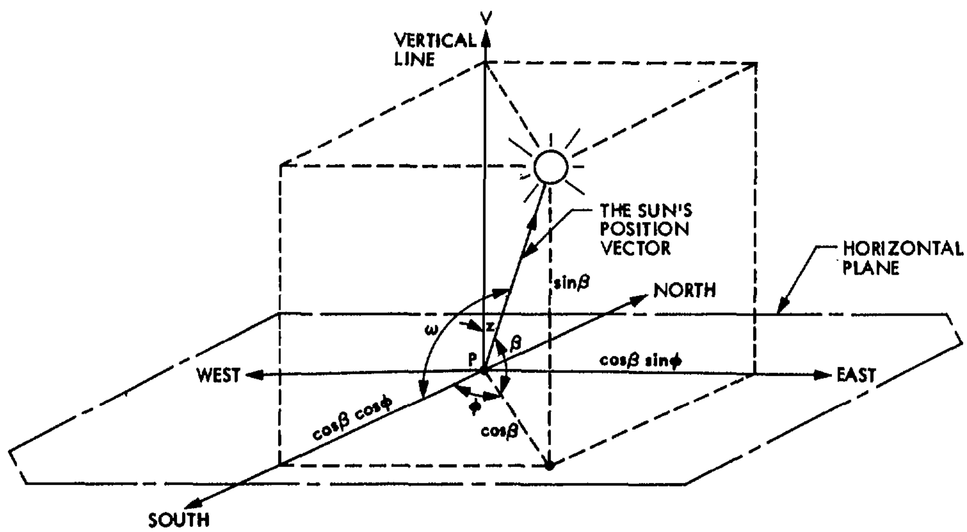

Fig. 3. Direction cosines of the sun's position vector viewed from a point on earth

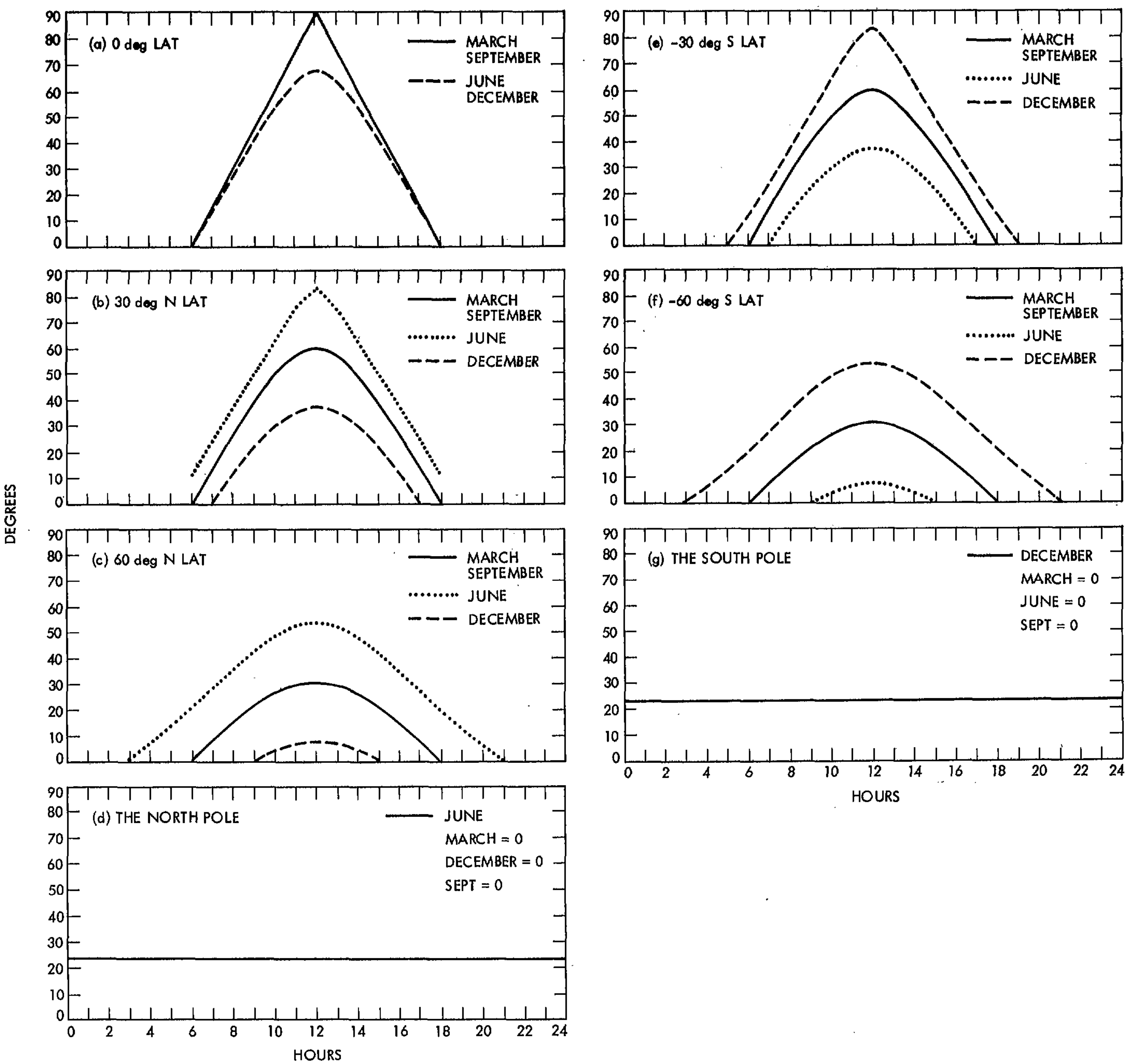

Fig. 4. Comparison of solar elevation angles for seven different latitudes

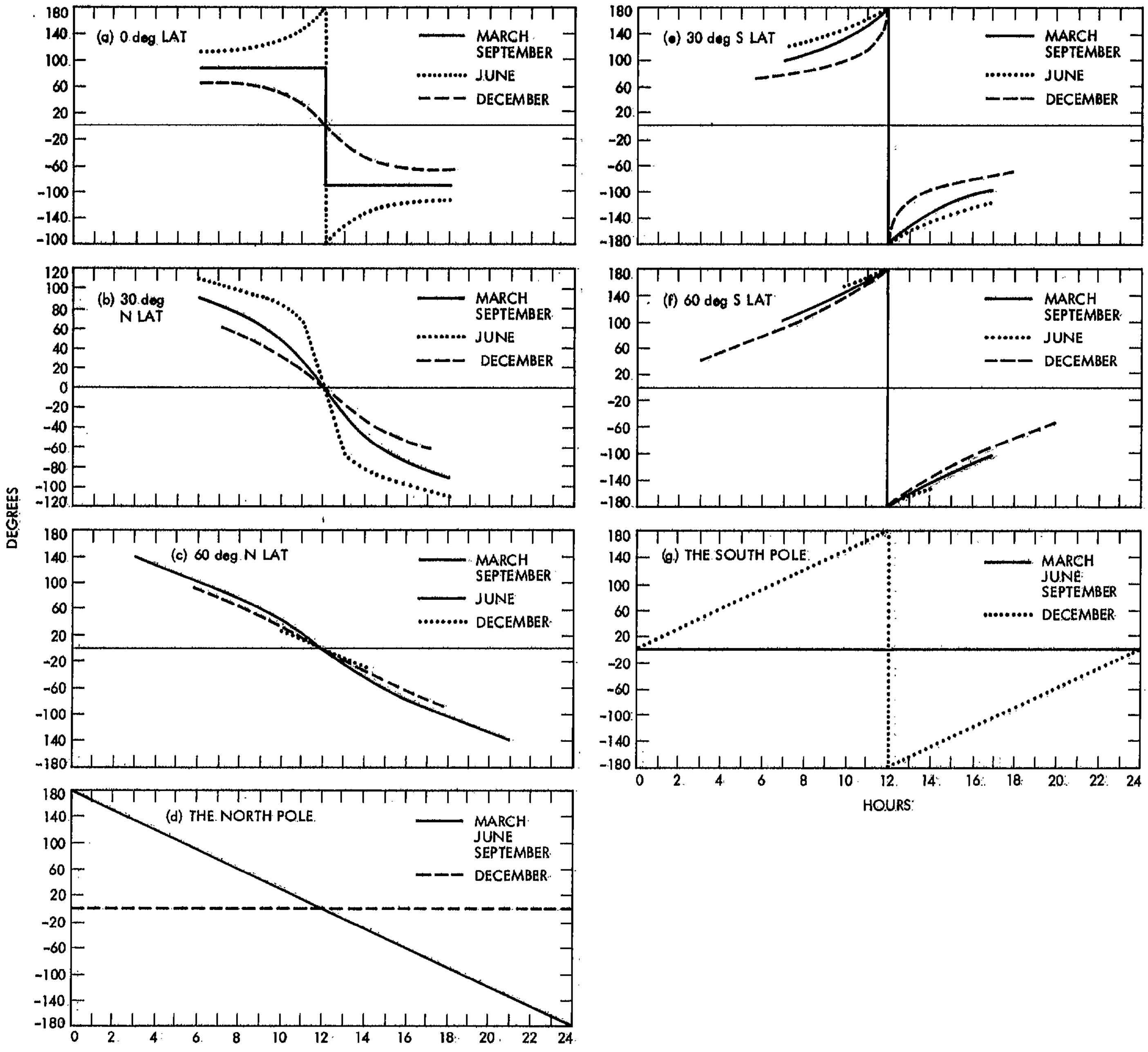

Fig. 5. Comparison of solar azimuth angles for seven different latitudes

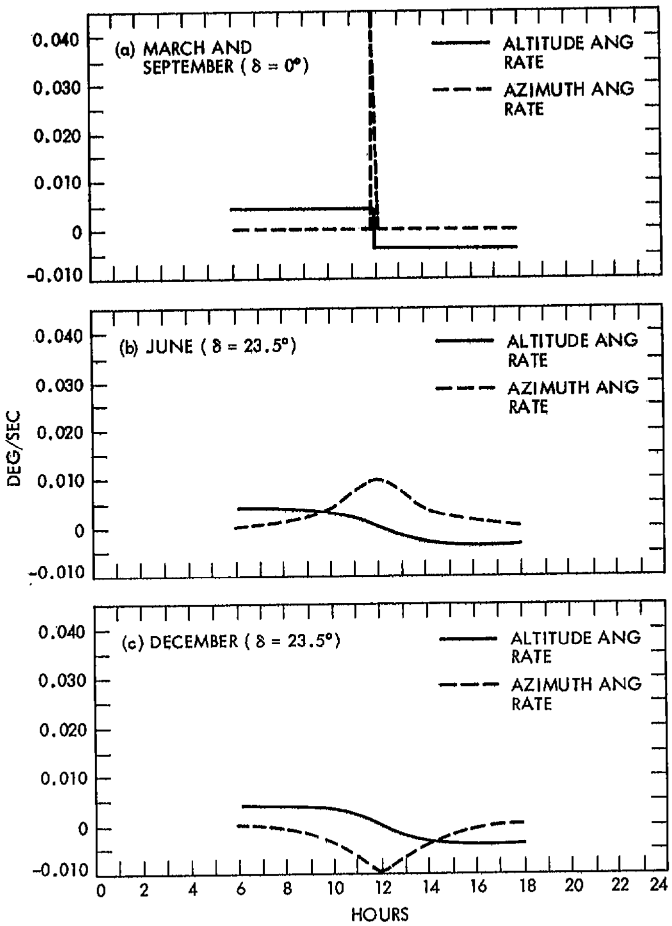

Fig. 6. Rate of solar angles for the equator

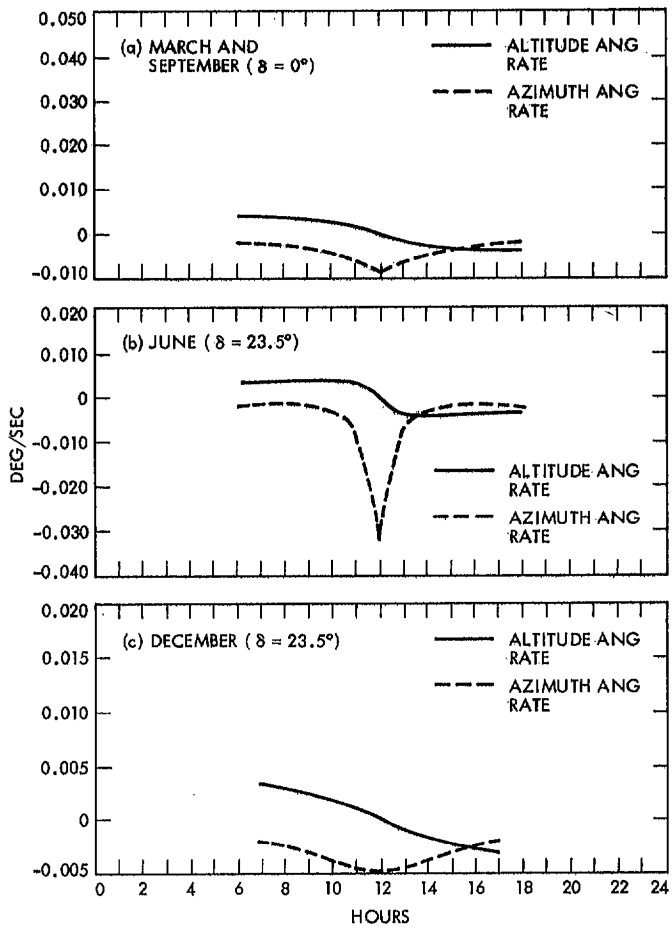

Fig. 7. Rates of solar angles for 30.00 deg north latitude

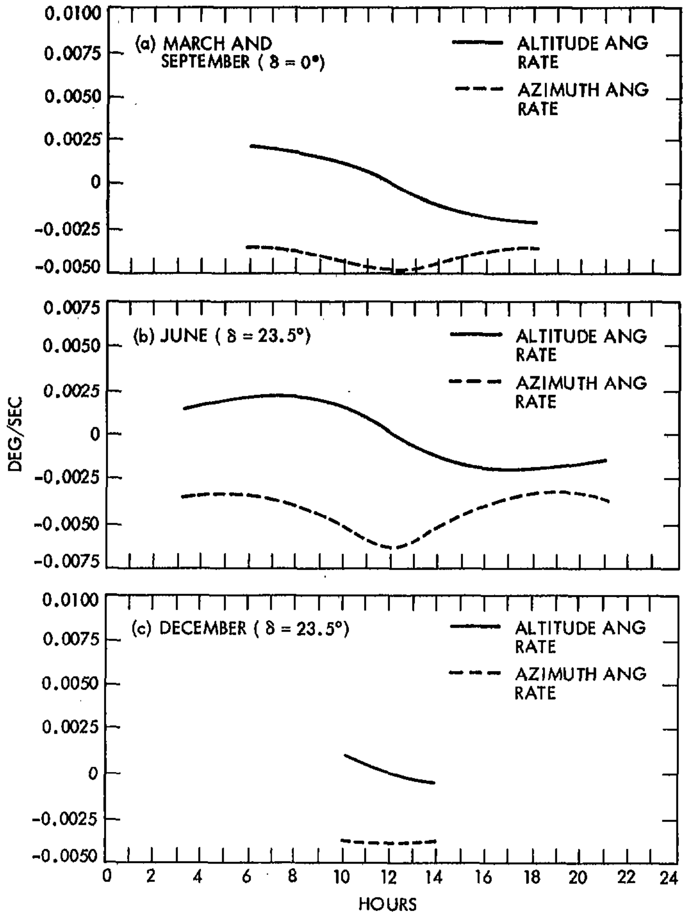

Fig. 8. Rates of solar angles for 60.00 deg north latitude

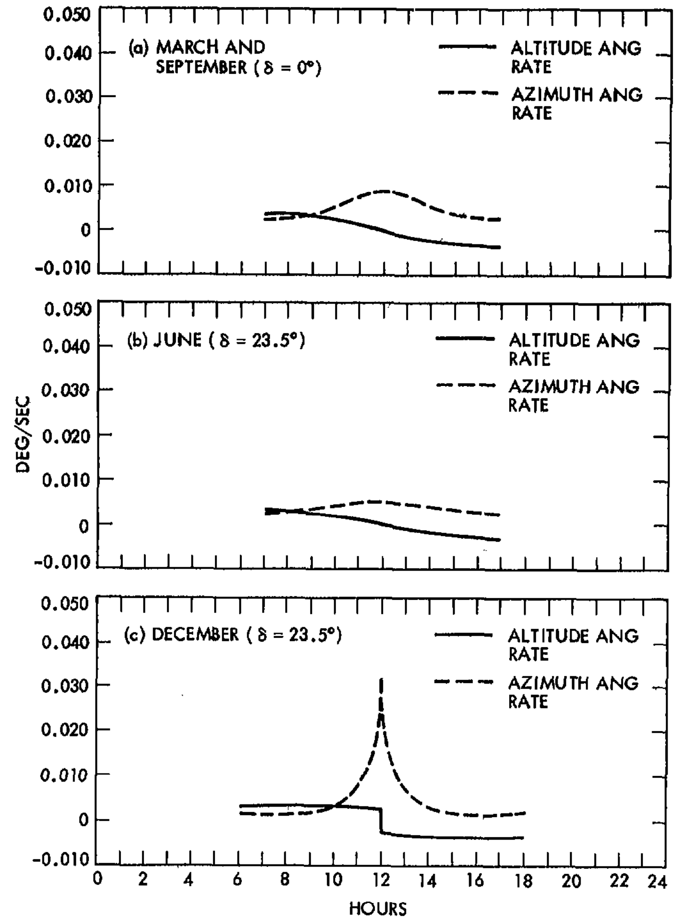

Fig. 9. Rates of solar angles for 30.00 deg south latitude

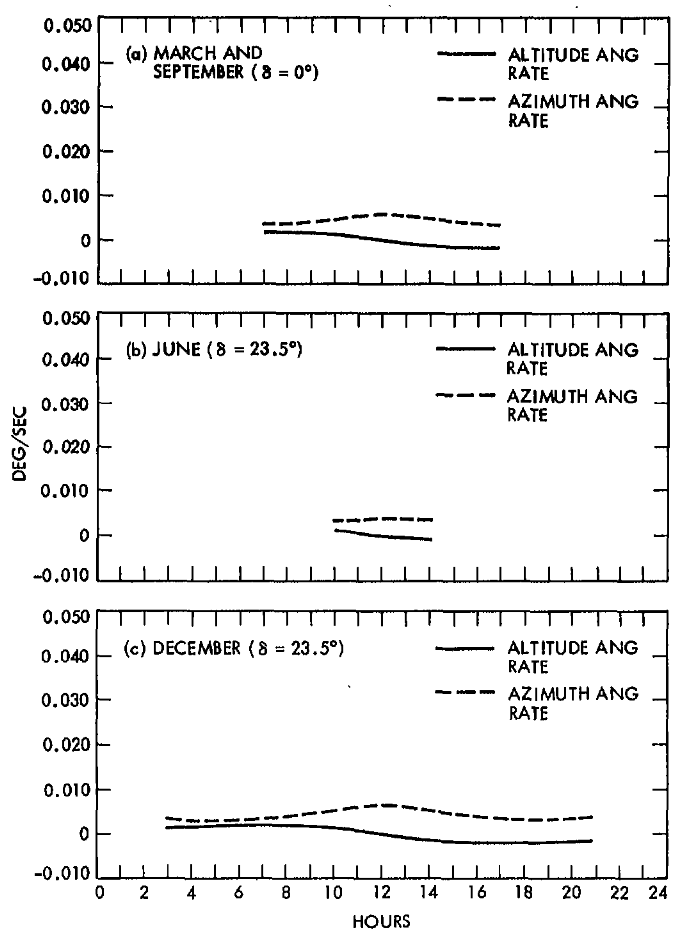

Fig. 10. Rates of solar angles for 60.00 deg south latitude