Performance of Concatenated Reed-Solomon/Viterbi Channel Coding

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This paper briefly reviews the concatenated Reed-Solomon (RS)/Viterbi coding system. Then the performance of the system is analyzed and results are derived with a new simple approach. We present a functional model for the input RS symbol error probability π. Based on this new functional model, we compute the performance of a concatenated system in terms of RS word error probability, output RS symbol error probability, bit error probability due to decoding failure, and bit error probability due to decoding error. Finally we analyze the effects of the noisy carrier reference and the slow fading on the system performance.

I. Introduction

The Voyager 2 spacecraft, which was launched in 1977 and has encountered Jupiter and Saturn systems in 1979 and 1981 respectively, will fly by Uranus in 1986 and Neptune in 1989. Voyager delivered 115.2 kbps at Jupiter (5 AU from earth), and 44.4 kbps at Saturn (10 AU from earth), both with 5×10^{-3} bit error probability. In order to enhance Voyager's communications capability, NASA's Jet Propulsion Laboratory is expected to exercise a planned option by switching on its Reed-Solomon (RS) encoder on the Voyager spacecraft. This results in a RS/Viterbi concatenated coded communication link to increase the achievable data rate. Reed-Solomon decoders will be installed in NASA's Deep Space Network (DSN) ground stations in time for the 1986 Uranus encounter.

The concatenated RS/Viterbi channel coding and its performance has been considered in the past; for example, see Refs. 1-5. This paper gives a brief review of this topic and shows the derivation of the results with a new simple approach that will be useful to system design engineers.

In concatenated RS/Viterbi channel coding, the key parameter for evaluating the performances of the system is the input RS symbol error probability π. Based on previous simulation and measurement results, we give a simple functional model for RS decoder input symbol error probability π. Using this model, we evaluate the RS word error probability, output RS symbol error probability and bit error probability. Finally we consider the effects of phase jitter and slow fading channels:

namely, the effects of Rayleigh, Rician, and log-normal fading channels on the performance of this concatenated RS/Viterbi channel. We use the outer code RS (255, 223), convolutional rate 1/2 and constraint length 7 code to illustrate these results.

II. System Model

The block diagram of the concatenated coding system is given in Fig. 1. The Reed-Solomon/Viterbi concatenated code consists of a Reed-Solomon (RS) outer code and a convolutional inner code which is Viterbi decoded.

The binary input data sequence is divided into J bit sequences to form symbols over a $M = 2^J$-ary alphabet; i.e., there are M possible RS symbols. The RS coder then encodes the symbols such that any combination of t or fewer symbol errors per RS word ($2^J - 1$ symbols per word) can be corrected. A very simple block diagram of an RS block coder is shown in Fig. 2. All coding and decoding operations involve RS symbols, not individual bits. Here $K = M - 1 - 2t$ information symbols (or $J[M - 1 - 2t]$ information bits) from some data source enter the RS coder to the left. The result of coding operations is a *codeword* of length $N = 2^J - 1 = M - 1$ symbols, of which the first $K = M - 1 - 2t$ are the same symbols as those entering to the left. This makes the code *systematic*. The remainder of the codeword is filled in with $2t$ parity symbols; $t = (N - K)/2$ represents the number of correctable RS symbol errors in an RS codeword. That is, if t or less RS symbols are in error in any way, the decoder will be capable of correcting them. An RS symbol is in error if any of the J bits making up the symbol are in error. The minimum distance of RS code is $2t + 1$ symbols.

The interleaving buffers are required because the inner Viterbi decoder errors tend to occur in bursts, which occasionally are as long as several constraint lengths. Without interleaving, Viterbi decoder burst error events would tend to occur within one RS codeword. Thus over a period of time there would be a tendency for some codewords to have "too many" errors to be corrected. The performance of the RS decoder is severely degraded by highly correlated errors among several successive symbols. The purpose of interleaving and de-interleaving is to make the RS symbol errors, at the input of the RS decoder, independent of each other and disperse the RS symbol errors, in other words, to break the burst errors out of the Viterbi decoder among several code words.

The level of interleaving I corresponds to the number of RS code words involved in the interleaving and de-interleaving operation. Interleaving and de-interleaving operations over a Viterbi channel can be explained simply by considering two $I \times 2^J - 1$ matrices, one at the input of the channel and one at the output. For interleaving, put the code words with length $2^J - 1$ in rows $1, 2, \cdots, I$ of the matrix, then transmit the symbols of columns $1, 2, \cdots, 2^J - 1$ through the channel. For de-interleaving, do the reverse operation. For the RS code (255, 223), simulation results (Ref. 3) have shown that the interleaving level of $I = 16$ is sufficient to make the RS symbol errors independent of each other for the ($K = 7$, $r = 1/2$) Viterbi channel. Thus we can assume henceforth that the combination of interleaving, convolutional code, additive white Gaussian noise (AWGN) channel, Viterbi decoder and de-interleaving creates an equivalent M-ary discrete memoryless channel (DMC) with transition error probability π, shown in Fig. 3.

III. A Functional Model for the RS Decoder Input Symbol Error Probability

Based on simulation results of Linkabit for π given in Ref. 3 and then using a least-squares curve fit, we proposed a simple model for π as

$$\pi\left(\frac{E_b}{N_0}\right) = \begin{cases} \exp\left(\beta_0 - \beta_1 \frac{E_b}{N_0}\right) ; & \frac{E_b}{N_0} \geq T^* \\ 1 - \frac{1}{M} ; & \frac{E_b}{N_0} \leq T^* \end{cases} \quad (1)$$

where $\beta_0 = 4.9551$, $\beta_1 = 5.2275$, $T^* = \max(\beta_0/\beta_1, T)$, T is Viterbi decoder node synchronization threshold (for $E_b/N_0 < T$ Viterbi decoder produces random output), and E_b/N_0 is the bit SNR of the Viterbi channel (inner code). For perfect node synchronization $T^* = \beta_0/\beta_1$. Figure 4 shows the baseline symbol error probability π for the (255, 223) RS code vs E_b/N_0 of the Viterbi channel (inner code), using (1).

IV. Error Events in RS Decoding

Consider an (N, K) Reed-Solomon code. We know that t-error correcting RS code can decode a code word correctly if the number of symbol errors are t or less where $t = (N - K)/2$. Consider the space of all received vectors y of length N and spheres of radius t around code words in this N-dimensional vector space, as is shown in Fig. 5. These spheres do not overlap because the minimum distance between codewords is $2t + 1$. Since we have a linear code, without loss of generality, we assume the all zero codeword $\mathbf{x}_0$ is sent. In decoding the RS code words, depending on what region in the observation space of Fig. 5 the received vector y falls, three disjoint events may occur. The first event occurs if there are t or less input RS

symbol errors in a received codeword. This occurs if the received vector y falls in the double-shaded sphere around $\mathbf{x}_0$, $\mathbf{y} \epsilon S_C$. In this case the decoder successfully corrects the errors, decodes $\mathbf{x}_0$ as the codeword and outputs the correct information block $\hat{\mathbf{u}}_0 = \mathbf{u}_0$. Then, the "correct decoding" event occurs. The second event happens if there are more than t input RS symbol errors in a received codeword and the corrupted codeword is not within a distance of t symbols to any other codeword. In other words, y does not fall in any sphere of radius t around the codewords, $\mathbf{y} \epsilon S_F$. In this case the RS decoder fails to decode and we say that a "decoding failure" event occurs. Here, the decoder just outputs the first $N - 2t$ undecoded Viterbi channel symbols (input RS symbols) that may contain symbol errors. The third event happens if there are more than t input RS symbol errors in a received codeword and the corrupted codeword is within a distance of t symbols to some other codeword than the correct codeword $\mathbf{x}_0$. This happens if y falls in any one of the single-shaded spheres, $\mathbf{y} \epsilon S_I$. Here the decoder decodes incorrectly and outputs an incorrect information block. Then, a "decoding error" event occurs.

V. Error Performance Analysis

In this section we derive expressions for RS codeword, RS symbol, RS bit, and RS information block error probabilities.

A. RS Codeword Error Probability

Denote the word error probability for an (N, K) RS code by $P_W(N, t)$. An RS codeword is in error when there are more than t channel symbol errors in a received codeword y; i.e., $\mathbf{y} \epsilon \{S_F \cup S_I\}$. Therefore, we should consider all combinations of k channel symbol errors within the N symbols, for all $k > t$. Thus

$$P_W(N, t) = \sum_{k=t+1}^{N} \binom{N}{k} \pi^k (1 - \pi)^{N-k} \tag{2}$$

where π is the RS decoder input symbol error probability leaving the Viterbi decoder (groups of J bits).

B. RS Symbol Error Probability

Denote the RS symbol error probability by P_S. This error may result when the received codeword is either $\mathbf{y} \epsilon S_F$ or $\mathbf{y} \epsilon S_I$. Denote the RS symbol error probabilities when $\mathbf{y} \epsilon S_F$ and $\mathbf{y} \epsilon S_I$ by $P_{S,F}$ and $P_{S,I}$, respectively. Since these events are disjoint, we have

$$P_S = P_{S,F} + P_{S,I} \tag{3}$$

1. Derivation of $P_{S,F}$. Recall that when the RS decoder fails to decode, $\mathbf{y} \epsilon S_F$, the decoder simply outputs the first $M - 1 - 2t$ Viterbi channel symbols as the "decoded" information block $\mathbf{u}_0$. Hence the output RS symbol error pattern is exactly the same as the first $M - 1 - 2t$ undecoded input symbols to the RS decoder. An output RS symbol at position say k, $k = 1, 2, \cdots, M - 1 - 2t$ is in error, when the input RS symbol at the same position k is in error and there are at least t input RS symbol errors in the remaining $N - 1$ symbols of received codeword (i.e., other than the position k). The probability of the input RS symbol at position k being in error is π. The probability that there will be t or more input RS symbol errors in the remaining positions, i.e., the $N - 1$ places of the received codeword, is

$$P_W(N - 1, t - 1) = \sum_{m=t}^{N-1} \binom{N-1}{m} \pi^m (1 - \pi)^{N-m-1} \tag{4}$$

Thus

$$P_{S,F} = \pi \cdot P_W(N - 1, t - 1) \tag{5}$$

2. Derivation of $P_{S,I}$. As we have seen, a decoding error occurs if the received codeword y belongs to one of the spheres in S_I. Consider a particular codeword $\mathbf{x}_n$ which has exactly n particular nonzero symbols. This codeword has a distance n from the all zero codeword $\mathbf{x}_0$ and hence has weight n. Let

$$W_1, W_2, W_3, \cdots, W_n$$

be n independent identically distributed (iid) random variables corresponding to the n nonzero symbols of $\mathbf{x}_n$, and

$$Z_1, Z_2, Z_3, \cdots, Z_{N-n}$$

be $N - n$ iid random variables corresponding to the $N - n$ zero symbols of $\mathbf{x}_n$. Each of these random variables W_i; $i = 1, 2, \cdots, n$, and Z_j; $j = 1, 2, \cdots, N - n$, can be either 0 or 1. Let us compute the probability that $\mathbf{x}_0$ is sent and the decoder decodes it as $\mathbf{x}_n$, i.e., the decoder has made an incorrect decision. Denote this probability by $Pr\{\mathbf{x}_0 \rightarrow \mathbf{x}_n\}$. This event occurs if y falls in a sphere with radius t around the codeword $\mathbf{x}_n$. The probability that a zero symbol is sent and a nonzero symbol of a *particular* value is received is $\pi/(M - 1)$. But the probability that a zero symbol is sent and a nonzero symbol of any value is received is π, since there are $M - 1$ nonzero symbols. For computation convenience, without loss of generality, we can assume the first n symbols in $\mathbf{x}_n$ are the nonzero symbols. Now let $\mathbf{y} = (y_1 \cdots, y_N)$ and $\mathbf{x}_n = (x_{1,n}, x_{2,n}, \cdots, x_{N,n})$, and define

$$W_i = \begin{cases} 0\ ;\ y_i = x_{i,n} \\ 1\ ;\ y_i \neq x_{i,n} \end{cases} \quad i = 1, 2, \cdots, n \tag{6}$$

then given $\mathbf{x}_0$ is sent

$$Pr\{W_i = 0\} = \frac{\pi}{M-1}$$
$$Pr\{W_i = 1\} = 1 - \frac{\pi}{M-1} \tag{7}$$

Similarly, if we define

$$Z_i = \begin{cases} 0\ ;\ y_{n+i} = x_{n+i,n} \\ 1\ ;\ y_{n+i} \neq x_{n+i,n} \end{cases} \quad i = 1, 2, \cdots, N-n \tag{8}$$

then given $\mathbf{x}_0$ is sent

$$Pr\{Z_i = 0\} = 1 - \pi$$
$$Pr\{Z_i = 1\} = \pi \tag{9}$$

Let

$$W = \sum_{i=1}^{n} W_i \text{ and } Z = \sum_{i=1}^{N-n} Z_i$$

Then denoting the weight between two vectors by $\omega(\cdot,\cdot)$ we have

$$Pr\{\mathbf{x}_0 \to \mathbf{x}_n\} = Pr\{\omega(\mathbf{x}_n, \mathbf{y}) \leqslant t | \mathbf{x}_0\}$$
$$= Pr\{W + Z \leqslant t\} \tag{10}$$

But

$$Pr\{W + Z \leqslant t\} = \sum_{i=0}^{t} Pr\{W = i\}\, Pr\{Z \leqslant t - i\}$$
$$= \sum_{i=0}^{t} Pr\{W = i\} \sum_{j=0}^{min(t-i, N-n)} Pr\{Z = j\} \tag{11}$$

where

$$Pr\{W = i\} = \binom{n}{i} \left(1 - \frac{\pi}{M-1}\right)^i \left(\frac{\pi}{M-1}\right)^{n-i} \tag{12}$$

and

$$Pr\{Z = j\} = \binom{N-n}{j} \pi^j (1-\pi)^{N-n-j} \tag{13}$$

Therefore

$$Pr\{\mathbf{x}_0 \to \mathbf{x}_n\} = \sum_{i=0}^{t} \sum_{j=0}^{min(t-i, N-n)} Pr\{W = i\}\, Pr\{Z = j\} \tag{14}$$

This is an exact analytic result. An upper bound using the Chernhoff bound technique (Ref. 6) is derived in Appendix A. This gives a simpler expression than the exact result in (14).

Let $q(n)$ denote the number of codewords of distance n from $\mathbf{x}_0$; then the symbol error probability $P_{S,I}$ due to decoding error is

$$P_{S,I} = \sum_{n=2t+1}^{N} \frac{n}{N} q(n)\, Pr\{\mathbf{x}_0 \to \mathbf{x}_n\} \tag{15}$$

RS code is a maximum distance separable (MDS) code. Therefore we have (Ref. 7)

$$q(n) = (M-1)\binom{N}{n} \sum_{m=0}^{n-2t-1} \binom{n-1}{m} (-1)^m M^{n-2t-m-1} \tag{16}$$

C. RS Bit Error Probability

Denote the RS bit error probability by $P_b(RS)$. This bit error is either due to decoding failure or decoding error. Denote the RS bit error probability when $y \in S_F$ and $y \in S_I$ by $P_{b,F}(RS)$ and $P_{b,I}(RS)$, respectively. Since these events are disjoint, we have

$$P_b(RS) = P_{b,F}(RS) + P_{b,I}(RS) \tag{17}$$

1. Derivation of $P_{b,F}(RS)$. Recall that when the RS decoder fails to decode, it outputs the first $M-1-2t$ input RS symbols as "decoded" information block $\mathbf{u}_0$. A bit at

position say m, $m = 1, 2, \cdots, J$ within the RS output symbol at position say k; $k = 1, 2, \cdots, M - 1 - 2t$ of a codeword is in error if the bit at position m within the input RS symbol at position k of an undecoded codeword is in error and there are t or more RS symbol errors in the received codeword out of the Viterbi decoder in other positions than position k. But the probability that a bit would be in error in an undecoded RS codeword is the bit error rate of the Viterbi decoder denoted by P_b (Viterbi). The probability that there are t or more RS symbol errors in an undecoded codeword in positions other than position k is given by (4). Thus the RS bit error rate $P_{b,F}(RS)$ is

$$P_{b,F}(RS) = P_b\,(Viterbi) \cdot P_W(N - 1, t - 1) \qquad (18)$$

2. **Derivation of $P_{b,I}(RS)$.** We have derived the symbol error probability $P_{S,I}$ in (15). Now the bit error probability $P_{b,I}$ is

$$P_{bI}(RS) = \frac{M}{2\,(M - 1)}\,P_{S,I} \qquad (19)$$

Based on our functional model for π, $P_{b,I}(RS)$ is shown in Fig. 6 for (255, 223) RS code using (19) with (14) and (15). In Fig. 7 we have shown $P_{b,F}(RS)$, $P_{S,F}$ and P_w for concatenated coding with (255, 223) RS code and (7, 1/2) convolutional code vs the E_b/N_0 of the concatenated channel. For $E_b/N_0 < 2$dB, $P_W(N - 1, t - 1) \cong 1$; hence (18) becomes $P_{b,F}(RS) = P_b$(Viterbi). Note that the code rate of the RS code is

$$RS\ code\ rate = (N - 2t)/N \qquad (20)$$

As is seen from Figs. 6 and 7

$$P_{S,I} << P_{S,F}$$

and

$$P_{b,I}(RS) << P_{b,F}(RS) \qquad (21)$$

Therefore

$$P_S \cong P_{S,F}$$

and

$$P_b(RS) \cong P_{b,F}(RS) \qquad (22)$$

At this point, since the probability of a decoding error event is very small, it can be ignored for further computations. Using this fact, we determine the RS of information block error probability.

D. RS Information Block Error Probability

Note that if a codeword is in error it is not necessary that the corresponding information block be in error. This error probability is important for source coding (data compression). Denote the information block error probability by $P_I(N, t)$. Ignoring the probability of decoding error event to find $P_I(N, t)$, we should subtract the probability of all possible patterns of symbol error that happen only in the parity check symbols, from P_W. This implies there is no error in the information block. Thus we have

$$P_I(N, t) = P_W(N, t) - \sum_{k=t+1}^{2t} \binom{2t}{k} \pi^k\,(1 - \pi)^{N-k} \qquad (23)$$

Obviously, in the practical range of error probability the last term can be ignored and we can say approximately

$$P_I(N, t) \cong P_W(N, t) \qquad (24)$$

VI. Effects of Phase Jitter

If the data rate is sufficiently large with respect to the PLL loop bandwidth so that the phase error does not vary significantly during the Viterbi decoder error bursts, then the phase error can be assumed to be constant during one RS symbol. In addition, since we have $I = 16$ level of interleaving, the phase errors affect the RS input symbols approximately independently. Noting the approach we have taken for the derivations of P_W, P_S and $P_b(RS)$, we have

$$P_W = E\{P_W(\phi_1, \phi_2, \cdots \phi_N)\} = \sum_{k=t+1}^{N} \binom{N}{k} \overline{\pi(\phi)}^k\ [1 - \overline{\pi(\phi)}]^{N-k} \qquad (25)$$

and

$$P_b(RS) = \overline{P_b(Viterbi\,|\,\phi)} \sum_{k=t}^{N-1} \binom{N-1}{k} \overline{\pi(\phi)}^k\ [1 - \overline{\pi(\phi)}]^{N-k-1} \qquad (26)$$

where

$$\overline{\pi(\phi)} = \int_{-\pi}^{\pi} \pi\left(\frac{E_b}{N_0} \cos^2 \phi\right) p(\phi)\, d\phi \qquad (27)$$

and $p(\phi)$ is the probability density function of phase jitter given by (Ref. 8)

$$p(\phi) = \frac{\exp(\rho \cos \phi)}{2\pi I_0(\rho)}; \quad -\pi \leqslant \phi \leqslant \pi \tag{28}$$

where ρ is the SNR in the tracking loop bandwidth and $\overline{\pi(\phi)}$ is given by (1), replacing E_b/N_0 with $(E_b/N_0)\cos^2\phi$. Figure 8 shows $\overline{\pi(\phi)}$ for various loop SNR ρ. The effect of a noisy carrier reference on Reed-Solomon Viterbi bit error rate performance, given by (26), is shown in Fig. 9. Note that in (26) $\overline{P_b\,(Viterbi|\phi)}$ can be computed from

$$\overline{P_b\,(Viterbi|\phi)} = \int_{-\pi}^{\pi} f\left(\frac{E_b}{N_0}\cos^2\phi\right) p(\phi)\, d\phi \tag{29}$$

where

$$f(x) = \begin{cases} \exp(\alpha_0 - \alpha_1 x); & x \geqslant T \\ \frac{1}{2}; & x < T \end{cases} \tag{30}$$

and

$$\alpha_0 = 4.4514 \qquad \alpha_1 = 5.7230\,.$$

T is the Viterbi decoder node synchronization threshold, where

$$T \geqslant \frac{\ln 2 + \alpha_0}{\alpha_1}$$

and with equality for perfect node synchronization. In this paper we have assumed perfect node synchronization. The effect of node sync on RS decoding can be found using (25) through (30) with the model for π given in (1); results are illustrated in Fig. 9. Further detail is given in Ref. 9.

For the effect of a noisy carrier reference on RS/Viterbi, we also recommend Ref. 5 to the interested reader.

VII. Effects of Slow Fading on the RS/Viterbi Decoder Performance

Usually the RF telemetry signal can be expressed as

$$S(t) = \sqrt{2}A \sin[\omega_c t + m(t)] \tag{31}$$

where $m(t)$ contains the telemetry information which can be a subcarrier, biphase modulated with a binary data stream. Now if this signal is passed through a fading channel, at the output we have

$$r(t) = \sqrt{2}\,a(t) \sin[\omega_c t + m(t) + \phi(t)] + n(t) \tag{32}$$

where $a(t)$ is random amplitude process and $\theta(t)$ is random phase process. If $a(t)$ and $\theta(t)$ change slowly with time, and the spectral bandwidths of $a(t)$ and $\theta(t)$ are narrow with respect to average loop bandwidth of PLL at the receiver, then $\theta(t)$, together with the carrier phase, can be tracked by PLL.

Here we consider a slow varying fading channel with perfect tracking. Depending on the channel, we can have Rayleigh, Rician, or log-normal channels. Consider first the Rician channel, since it is Rayleigh with an added specular component. If we pass transmitted signal (31) through a Rician channel we get

$$r(t) = \sqrt{2}A \sin[\omega_c t + m(t)] + \sqrt{2}\,n_c(t)\cos[\omega_c t + m(t)] + \sqrt{2}\,n_s(t)\sin[\omega_c t + m(t)] + n(t) \tag{33}$$

where n_c and n_s are zero mean Gaussian random processes each with variance σ^2. Then the received signal amplitude $a(t)$ has the Rician probability density function

$$p(a) = \frac{a}{\sigma^2}\exp\left\{-(a^2 + A^2)/2\sigma^2\right\} I_0\left(\frac{aA}{\sigma^2}\right) \quad a \geqslant 0 \tag{34}$$

Define

$$\gamma^2 \triangleq \frac{A^2}{2\sigma^2} \tag{35}$$

as the ratio of the specular power to the fading power. Suppose the received power is P where

$$P = A^2 + 2\sigma^2 \tag{36}$$

Let's normalize $a(t)$ as

$$y(t) = \frac{a(t)}{\sqrt{P}} \tag{37}$$

Then

$$p(y) = 2y(1+\gamma^2)\exp\left\{-(1+\gamma^2)y^2 - \gamma^2\right\} I_0\left[2y\sqrt{\gamma^2(1+\gamma^2)}\right] \quad y \geq 0 \tag{38}$$

with

$$\overline{y^2} = 1 \tag{39}$$

For the Rayleigh channel we don't have specular components, which means $A = 0$ or $\gamma^2 = 0$. Then pdf for y is

$$p(y) = 2y\exp(-y^2) \qquad y \geq 0 \tag{40}$$

Figure 10 shows $p(y)$ given in (38) and (40). For Log-normal channels the received signal amplitude is of the form

$$a(t) = A\,e^{\chi(t)} \tag{41}$$

where $\chi(t)$ is Gaussian random process with variance σ_χ^2. Pdf of normalized $a(t)$ is

$$p(y) = \frac{1}{y\sqrt{2\pi}\,\sigma_\chi}\exp\left\{-(\ln y + \sigma_\chi^2)^2/2\sigma_\chi^2\right\} \quad y > 0 \tag{42}$$

Therefore in all cases we can assume the received signal is

$$r(t) = \sqrt{2P}\,y\cos\left[\omega_c t + m(t) + \theta(t)\right] + n(t) \tag{43}$$

where pdf of y is given by (38), (40) or (42).

If the signal amplitude changes very slowly in comparison to the bit rate, y will remain constant over a large number of bits. On the other hand, 16 levels of interleaving makes y affect RS symbols independently. Therefore, we have

$$\pi = \overline{\pi(y)} = \int_0^\infty \pi\left(\frac{E_b}{N_0}y^2\right)p(y)\,dy \tag{44}$$

and

$$P_b\,(Viterbi) = \overline{P_b\,(Viterbi|y^2)} = \int_0^\infty f\left(\frac{E_b}{N_0}y^2\right)p(y)\,dy \tag{45}$$

Using these averages in (2), (5), and (18) we can get performance of concatenated coded system in the presence of a slow fading channel. The corresponding performance curves are shown in Figs. 11 and 12.

VIII. Combined Effect of Phase Jitter and Slow Fading

With a similar argument as before, we can find the combined effect of phase jitter and slow fading by finding the average of π and P_b (*Viterbi*) over the phase error ϕ and amplitude fading factor y as

$$\pi = \overline{\pi(y,\phi)} = \int_0^\infty\int_{-\pi}^{\pi}\pi\left(\frac{E_b}{N_0}y^2\cos^2\phi\right)p(\phi|y)\,d\phi\,dy \tag{46}$$

where (for details of effects of a fading channel on PLL see Ref. 10).

$$p(\phi|y) = \frac{e^{\rho(y^2)\cos\phi}}{2\pi I_0(\rho(y^2))} \tag{47}$$

$$\rho(y^2) = \frac{y^2 P_c}{N_0 B_L(y^2)\Gamma(y^2)} \tag{48}$$

and similarly

$$P_b\,(Viterbi) = \int_0^\infty\int_{-\pi}^{\pi} f\left(\frac{E_b}{N_0}y^2\cos^2\phi\right)p(\phi|y)\,p(y)\,d\phi\,dy \tag{49}$$

Using these in our performance formulas, we get the desired results.

IX. Conclusion

Key parameters for characterizing the performance of concatenated Reed-Solomon/Viterbi coding have been considered. Simple derivations of close form expressions for a number of error probabilities are presented; these include RS codeword, RS information block, RS symbol, and RS bit error probabilities. A functional model for the RS decoder input symbol error probability is found which enables us to carry out numerical computations of the above-mentioned error probabilities. In addition, the effects of noisy carrier reference and slow fading on the RS/Viterbi decoding performance are determined.

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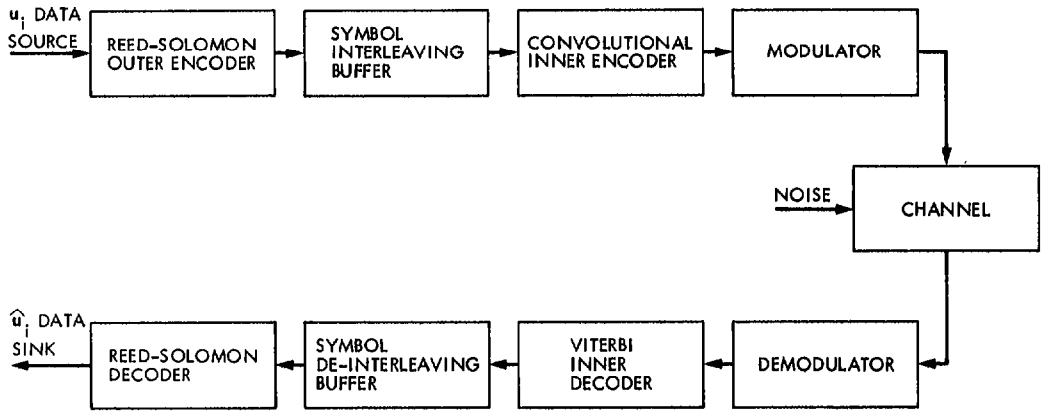

Fig. 1. Concatenated Reed-Solomon/Viterbi coding system diagram

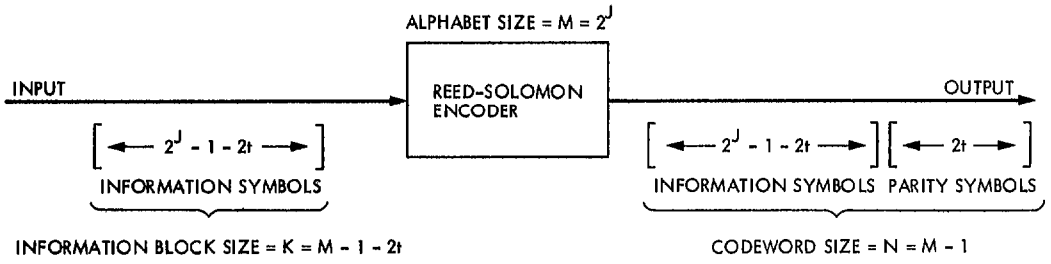

Fig. 2. Basic Reed-Solomon code structure

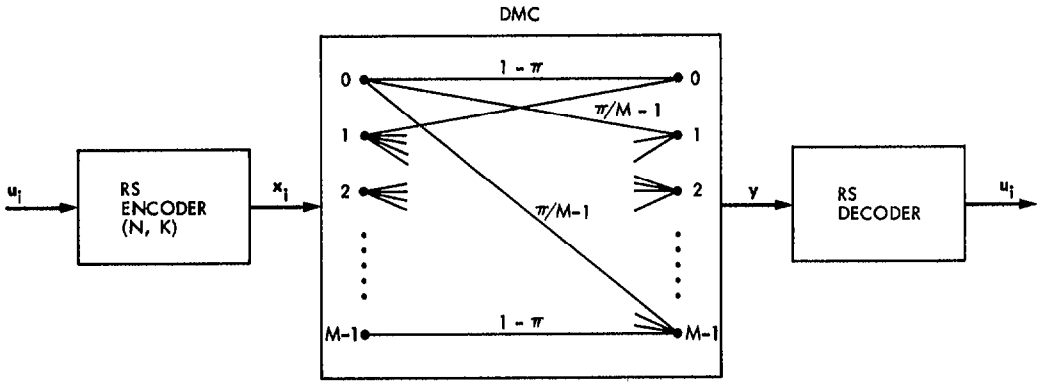

Fig. 3. Equivalent *M*-ary discrete memoryless channel (DMC) of Fig. 1

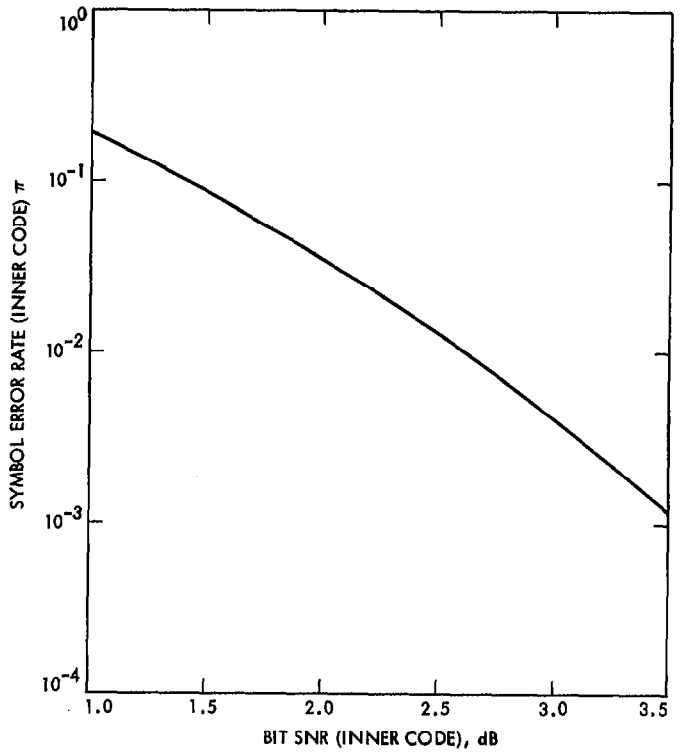

Fig. 4. Symbol error probability π computed from Eq. (1)

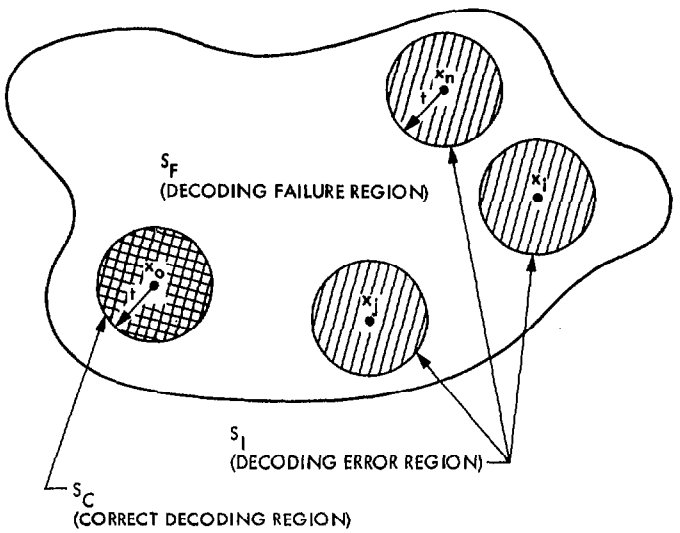

Fig. 5. Received codeword space of all vectors y with length N

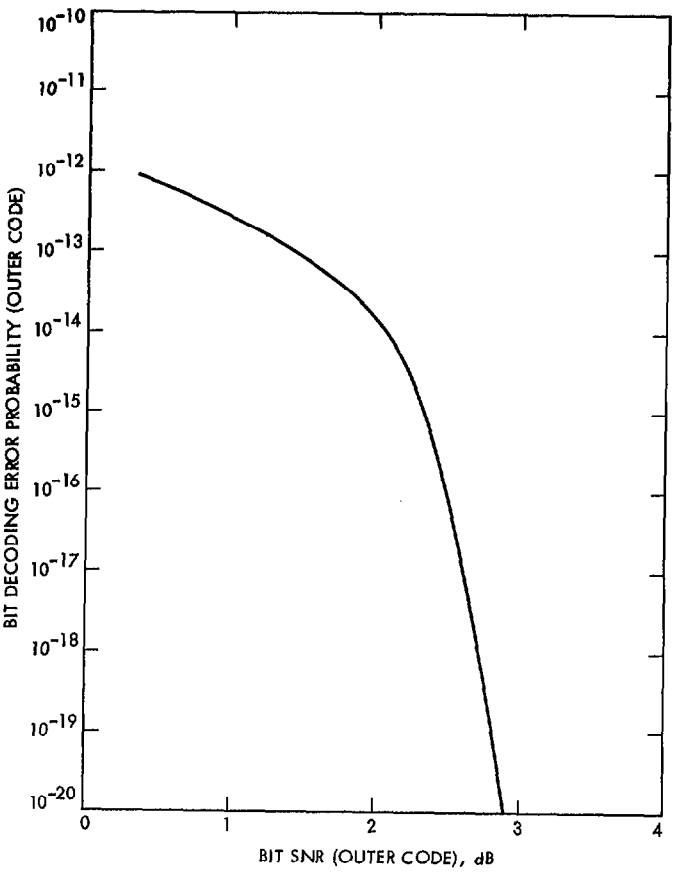

Fig. 6. $P_{b,I}(RS)$ vs bit SNR for (255,223) RS outer code and ($K = 7$, $r = 1/2$) convolutional inner code

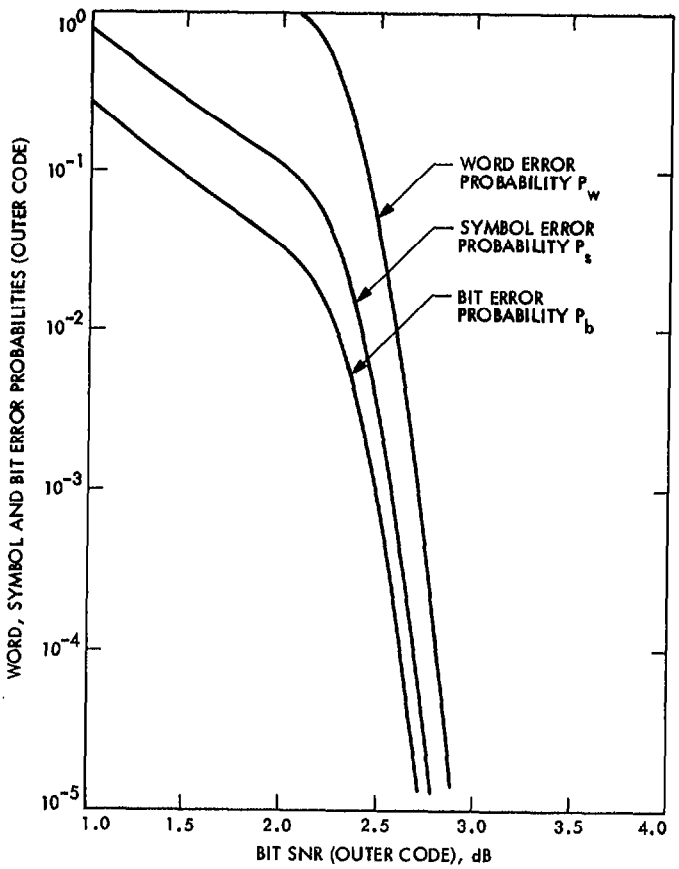

Fig. 7. $P_{b,F}(RS)$, $P_{S,F}$ and P_w performance curves

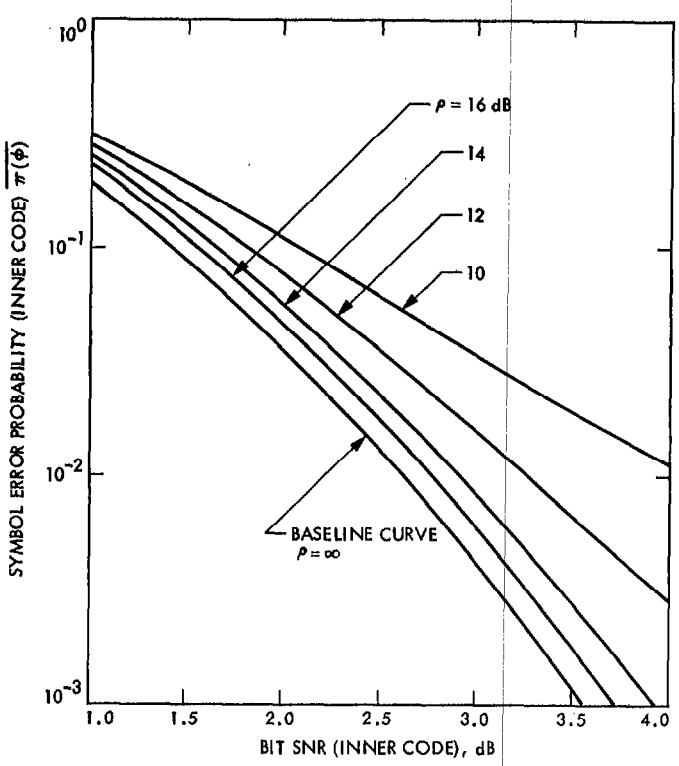

Fig. 8. Effects of noisy carrier reference on symbol error probability π

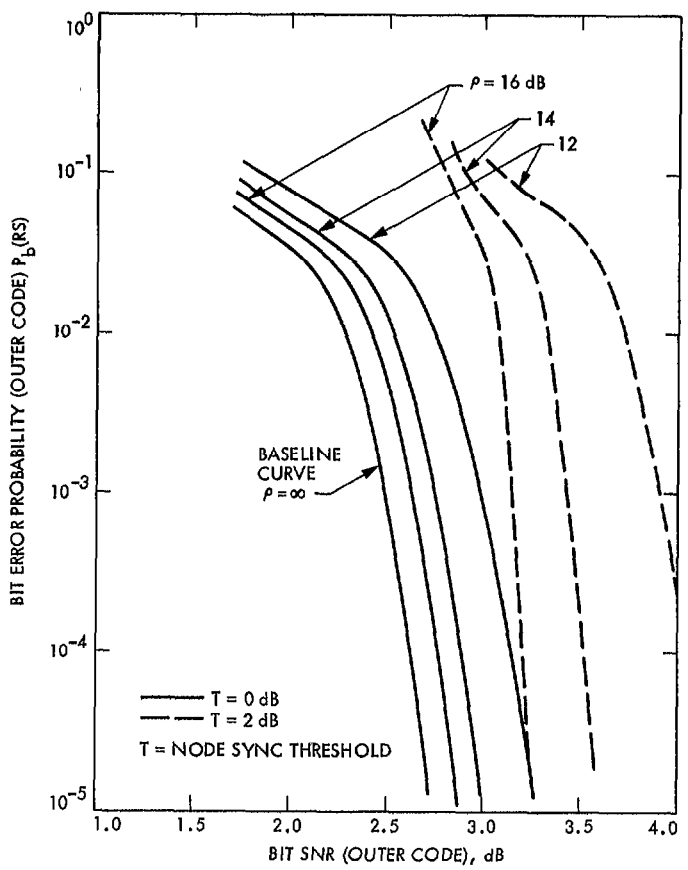

Fig. 9. Effects of noisy carrier reference on concatenated RS/Viterbi bit error rate P_b (RS)

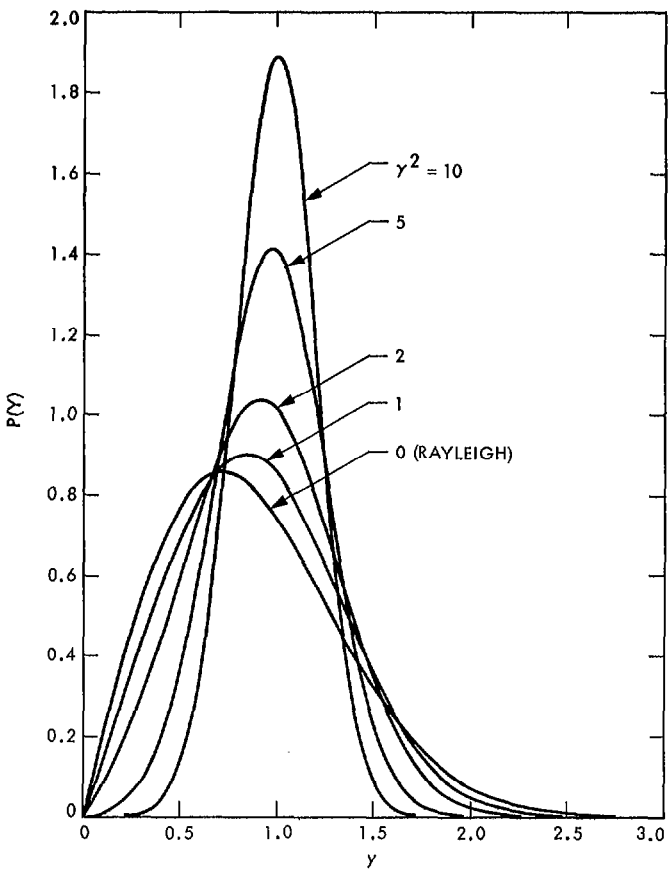

Fig. 10. Rician ($\gamma^2 > 0$) and Rayleigh ($\gamma^2 = 0$) probability density functions

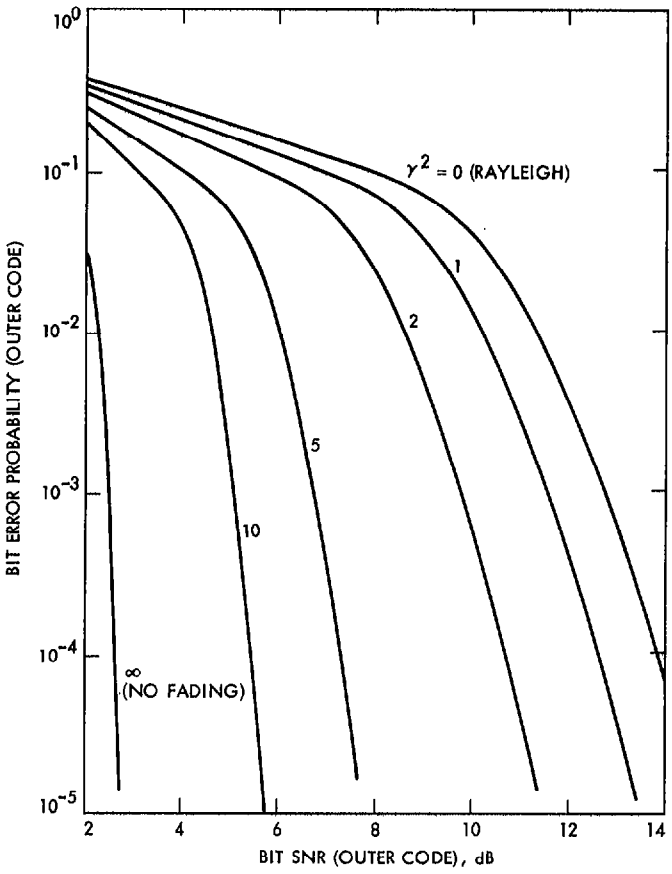

Fig. 11. $P_b(RS)$ vs bit SNR for Rician ($\gamma^2 > 0$) and Rayleigh ($\gamma^2 = 0$) fading channels

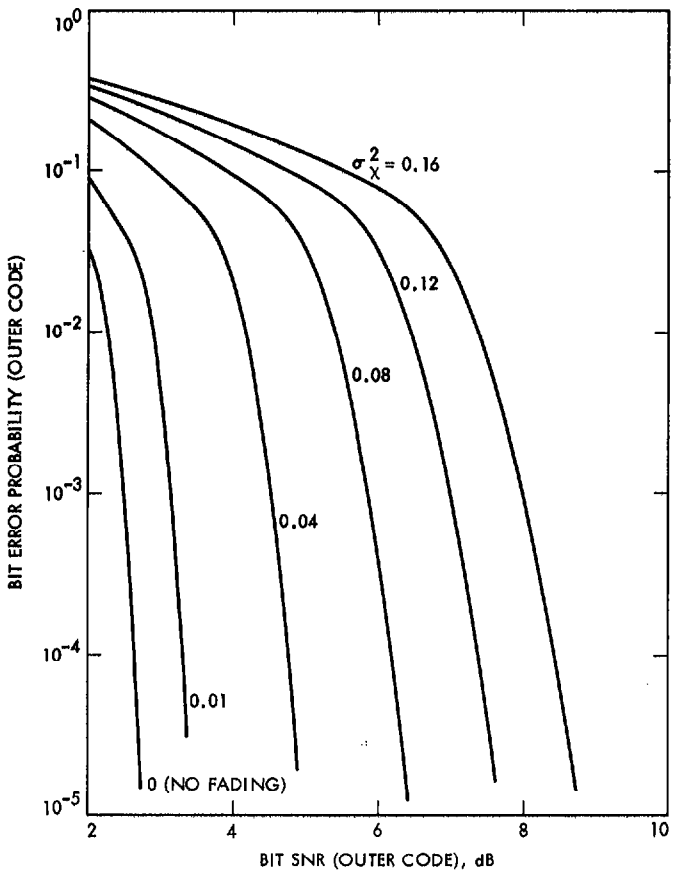

Fig. 12. P_b (RS) vs bit SNR for log-normal fading channels

Appendix
Chernoff Bound on $Pr\left\{\mathbf{x}_0 \rightarrow \mathbf{x}_n\right\}$

Here we would like to find a simple upperbound on $P_r(\mathbf{x}_0 \rightarrow \mathbf{x}_n)$ using the Chernoff bound technique. Note for $\lambda > 0$ we have

$$Pr\{\mathbf{x}_0 \rightarrow \mathbf{x}_n\} = Pr\{W + Z \leqslant t\} = Pr\{t - W - Z \geqslant 0\} \leqslant \{e^{\lambda(t-W-Z)}\}$$

$$= e^{\lambda t} E\{e^{-\lambda W}\} E\{e^{-\lambda Z}\} \tag{A-1}$$

But

$$E\{e^{-\lambda W}\} = E\left\{e^{-\lambda \sum_{i=1}^{n} W_i}\right\} = \prod_{i=1}^{n} E\{e^{-\lambda W_i}\} = \left[\frac{\pi}{m-1} - \left(1 - \frac{\pi}{m-1}\right) e^{-\lambda}\right]^n \tag{A-2}$$

Similarly

$$E\{e^{-\lambda Z}\} = E\left\{e^{-\lambda \sum_{i=1}^{N-n} Z_i}\right\} = [(1-\pi) + \pi e^{-\lambda}]^{N-n} \tag{A-3}$$

Therefore,

$$Pr\{\mathbf{x}_0 \rightarrow \mathbf{x}_n\} \leqslant \min_{\lambda > 0} e^{\lambda t} \left[\frac{\pi}{m-1} + \left(1 - \frac{\pi}{m-1}\right) e^{-\lambda}\right]^n [(1-\pi) + \pi e^{-\lambda}]^{N-n} \tag{A-4}$$