Capacity and Cutoff Rate of (*M*+1)-ary Decision Rules for Noisy *M*-ary Optical PPM Channel

P. J. Lee
Communications Systems Research Section

The channel capacity C *and the cutoff rate* R_0 *of two (*M *+ 1)-ary decision rules for noisy* M *slots/symbol optical pulse position modulation (PPM) with ideal photon counting are computed and compared. Also the values of the optimum thresholds needed to minimize the signal energy requirements are given. With a minor increase in hardware complexity, the symbol-by-symbol threshold decision rule is shown to be superior to the slot-by-slot threshold detection-and-decision rule in two aspects: First, it saves more than 0.5 dB in signal energy for the very noisy cases of more than one noise photon per slot (for low noise cases it also saves signal energy, but a negligibly small amount). Second, it is more robust to variations in the noise level.*

I. Introduction

In this article, the theoretical performance of coded systems for the noisy optical PPM channel with two different $(M + 1)$-ary decision rules will be studied. The system model is depicted in Fig. 1. From the information bits the encoder selects channel input symbols x from an M-ary alphabet $\{1, \cdots, m, \cdots, M\}$. When $x = m$, the optical PPM modulator emits an optical pulse only at the mth time slot during a symbol time period which consists of M time slots. Depending upon the numbers of the received noise and/or signal photons, the optical detector produces output $y_{m'}$ at the end of each m'th time slot, $m' = 1, 2, \cdots, M$. Ordinarily, hard decision M-ary symbols would be produced by the PPM demodulator, and passed to the decoder, which uses M-ary symbols as inputs. However, if we provide, by an $(M + 1)$-ary decision rule, an additional symbol called "erasure" which denotes the fact that no confident decision can be made, and if we use a decoder which can use $(M + 1)$-ary symbols as inputs, then an additional gain can be obtained. Hence, $(M + 1)$-ary decision (or decision with erasure) rules have been investigated.

An $(M + 1)$-ary decision rule based on the slot-by-slot threshold detection, which requires very little complexity, has been proposed and used (Ref. 1). In this case, each y_m, $m = 1, \cdots, M$, is compared to a given threshold γ, $\gamma \geqslant 0$, and the corresponding slot is declared active ($z_m = 1$) if $y_m > \gamma$, or inactive ($z_m = 0$) if $y_m \leqslant \gamma$. Then a nonerased decision is made if and only if there is only one active slot in a PPM symbol time, and the erasure decision is made for all the other cases. This decision rule is shown in Fig. 2(a).

More recently, the optimum and a near optimum $(M + 1)$-ary decision rules for the M-ary orthogonal input channel have been developed (Ref. 2). The near optimum $(M + 1)$-ary decision rule, which uses symbol-by-symbol threshold decision, can be applied to the case of the noisy M-ary input optical PPM channel. The decision rule produces a nonerased symbol if and only if the maximum of y_m is greater than all the other $y_{m'}$ by a given threshold δ, $\delta \geqslant 0$. In all other cases, the erasure decision is made. This decision rule is shown in Fig. 2(b). We can see that the decision rule of Fig. 2(b) is only slightly more complex than that of Fig. 2(a).

The same $(M + 1)$-ary decision rule with symbol-by-symbol threshold decision was also considered, independently, but without showing the near optimality, by Divsalar, et al. (Ref. 3) under the different name of "delta-max demodulation." They compared this decision rule to the slot-by-slot threshold detection-and-decision rule by calculating the decoded bit error rate with a Reed Solomon code in each case. Their results, however, apply to the specific set of codes studied and cannot be easily extended to other situations.

Notice that with any $(M + 1)$-ary decision rule, the inside of the encoder-decoder pair (inside of the dashed line in Fig. 1) becomes an M-ary input, $(M + 1)$-ary output discrete memoryless coding channel. The channel capacity C and the cutoff rate R_0 of a given coding channel have long been recognized as valuable performance indicators of the coding channel. Also, these quantities are independent of the actual codes to be used over the coding channel. Furthermore, their evaluation is usually much easier than computing the decoded bit error rates with specific complex codes.

Therefore, in this article we will compute C's and R_0's with more variety of situations (noise levels and/or channel input alphabet sizes) than Ref. 3 for more detailed comparisons of the two decision rules. We will show that the performance gain of the symbol-by-symbol threshold decision rule, over the slot-by-slot threshold detection-and-decision rule, is larger than 0.5 dB under strong background noise conditions. As a by-product of the evaluation we will, as in Ref. 3, determine the optimum threshold settings. We will also show that under weak background noise conditions (such as in Ref. 1), the improvement is negligibly small.

II. Evaluation of C and R_0

Let $G(a) = Pr\,\{y_m \leqslant a \mid x = m\}$ and $F(a) = Pr\,\{y_{m'} \leqslant a \mid x = m \text{ and } m' \neq m\}$ represent the cumulative probability distribution functions of the receiver output value for the slot with and without signal pulse, respectively. Let $P_c(\epsilon)$, $P_s(\epsilon)$, and $P_e(\epsilon)$ be the probabilities of correct decision, decision error, and erasure decision, respectively, with a given threshold ϵ, which is either γ or δ depending on the decision rule. Recall that γ is the threshold for the slot-by-slot detection-and-decision rule and δ corresponds to the symbol-by-symbol decision rule. For the slot-by-slot threshold detection-and-decision, these probabilities are given by (Ref. 1).

$$P_c(\gamma) = P_{ds}(\gamma) \cdot P_{dn}^{M-1}(\gamma),$$

$$P_s(\gamma) = (M-1) \cdot (1 - P_{ds}(\gamma)) \cdot (1 - P_{dn}(\gamma)) \cdot P_{dn}^{M-2}(\gamma),$$

and

$$P_e(\gamma) = 1 - P_c(\gamma) - P_s(\gamma),$$

where

$$P_{ds}(\gamma) = 1 - G(\gamma) \quad \text{and} \quad P_{dn}(\gamma) = F(\gamma).$$

For the symbol-by-symbol threshold decision (Refs. 2, 3),

$$P_c(\delta) = \int F^{M-1}(t) \cdot dG(t+\delta),$$

$$P_s(\delta) = (M-1) \cdot \int G(t) \cdot F^{M-2}(t) \cdot dF(t+\delta),$$

if y_m takes on continuous values, or

$$P_c(\delta) = \sum_k F^{M-1}(k) \cdot g(k+\delta+1),$$

$$P_s(\delta) = (M-1) \cdot \sum_k G(k) \cdot F^{M-2}(k) \cdot f(k+\delta+1),$$

if y_m takes on integer values (in which case the threshold δ should also be an integer), with

$$g(i) = G(i) - G(i-1) \quad \text{and} \quad f(i) = F(i) - F(i-1)$$

Finally,

$$P_e(\delta) = 1 - P_c(\delta) - P_s(\delta).$$

The capacity C and cutoff rate R_0 in [infor. bits/channel bit] for the resulting M-ary input, $(M + 1)$-ary output coding channel, which can be described with $P_s(\epsilon)$, $P_e(\epsilon)$, and $P_c(\epsilon)$, are given by (Ref. 2),

$$C = \max_{\epsilon \geqslant 0} C(\epsilon)$$

and

$$R_0 = \max_{\epsilon \geqslant 0} R_0(\epsilon)$$

where

$$C(\epsilon) = P_c(\epsilon) + P_s(\epsilon) + (P_c(\epsilon) \cdot \log [P_c(\epsilon)] + P_s(\epsilon)$$
$$\cdot \log [P_s(\epsilon)] - [P_c(\epsilon) + P_s(\epsilon)] \cdot \log [P_c(\epsilon) + P_s(\epsilon)]$$
$$- P_s(\epsilon) \cdot \log [M-1])/\log [M],$$

and

$$R_0(\epsilon) = 1 - \log [1 + (M-1) \cdot P_e(\epsilon) + (M-2) \cdot P_s(\epsilon)$$
$$+ \sqrt{4 \cdot (M-1) \cdot P_c(\epsilon) \cdot P_s(\epsilon)} \;] / \log [M],$$

and ϵ is either γ or δ, depending on the decision rule. (Note that the values of the optimum threshold need not be the same for each different criterion.)

We assume here an ideal photon counting receiver. That is, we neglect thermal noise and assume a constant gain PMT (photomultiplier tube). In this simple case, $f(i)$ and $g(i)$ are given by (Ref. 3)

$$f(i) = \exp(-\bar{N}_n) \cdot \bar{N}_n^i)/i!$$

$$g(i) = \exp(-\bar{N}_n - \bar{N}_s) \cdot (\bar{N}_n + \bar{N}_s)^i/i! \qquad i = 0, 1, 2, \cdots,$$

where $\bar{N}_n$ is the average number of the received background noise photons per PPM slot and $\bar{N}_s$ is the average number of the received signal photons per PPM symbol.

When M and $\bar{N}_n$ are given, we can calculate C's and R_0's as functions of a single parameter of $\bar{N}_s$. Since C and R_0 are monotonically increasing functions of $\bar{N}_s$, they are invertable. The coding theorem (Ref. 4) says that there exists a code with code rate r [info bits/channel bit] which gives arbitrary small decoded error rate provided that

$$r < C(\bar{N}_s).$$

Or equivalently, since C is invertable, the condition can be rewritten as

$$\bar{N}_s > C^{-1}(r).$$

Since the parameter of interest is usually the average number of received signal photons per information bit $\bar{N}_b$ [photons/info bit], we can obtain it from $\bar{N}_s$ by the simple relation: $\bar{N}_b = \bar{N}_s/(r\text{-}\log_2 M)$. Therefore, the condition can again be rewritten as

$$\bar{N}_b > C^{-1}(r)/(r \cdot \log_2 M).$$

Hence, we call $C^{-1}(r)/(r \cdot \log_2 M)$ the "required average number of signal photons per information bit, or simply photons per information bit, to achieve capacity C with a given code rate r." Similarily, $R_0^{-1}(r)/(r \cdot \log_2 M)$ is called the "required average number of photons per info bit to achieve a cutoff rate R_0 with a given code rate r."

For $M = 64$, 256, and 1024 (or $\log_2 M = 6$, 8, and 10), the required N_b [signal photons/info bit] to achieve (a) the channel capacity C and (b) the cutoff rate R_0 vs code rate r plots are given in Figs. 3, 4, and 5 for various noise levels of N_n [noise photons/slot] for the two different $(M+1)$-ary decision rules. The curve marked $N_n = 0$ corresponds to the noiseless channel use.

III. Discussions and Conclusions

First, we notice the nonsmoothness of the curves in the plots. This is due to the discreteness of the observation values and threshold values. Also the discreteness of the observation and threshold values gives large gaps in performance between the two decision rules near the nonsmooth points. This phenomenon is more noticible for the low noise level cases, due to the small threshold values. (Note that there is a much bigger difference when using threshold values of 0 or 1 than the difference corresponding to thresholds of 22 or 23.) If, instead of using an ideal photon counting receiver assumption, we had considered a more realistic random gain PMT with additive thermal noise model (Ref. 1), then the nonsmoothness and the large gaps for some rates would disappear and hence the differences in performance between the two decision strategies would be almost the same for all values of r. (We can see this for very noisy cases where the curves tend to be much smoother, as well as for $\bar{N}_n = 10^{-2}$ of Fig. 4(a) where accidently the optimum threshold levels do not vary for almost all values of r.)

As was concluded in Ref. 3, the symbol-by-symbol threshold decision rule saves more signal energy than the slot-by-slot threshold detection-and-decision rule as the noise level increases. For the noiseless case there is no gain at all. For the 2.5 bit/photon demonstration program (Ref. 1), 256-ary PPM with rate 3/4 or 7/8 coding is used. The typical value of background noise is 10^{-5} [noise photons/slot]. In this case also, we calculated C's and R_0's. The gain in signal energy of using the symbol-by-symbol threshold decision over using the slot-by-slot threshold detection-and-decision is less than 0.03 dB for almost all code rate, hence totally negligible. With this noise level, the performance is about 0.2 dB poorer than the noiseless case. However, for the very noisy case of more than one noise photon per slot, the gain is more than 0.5 dB for all values of M considered. We also observed as in Ref. 3 that the symbol-by-symbol threshold decision rule is more robust to variations in the noise level.

In conclusion, we have found that the symbol-by-symbol decision strategy always requires less signal energy than the simpler slot-by-slot detection-and-decision strategy, with greater benefits both in energy savings and in robustness to noise level variations for the high background noise cases. In the weak background noise case, the differences in performance are negligibly small, and hence the symbol-by-symbol decision will not be attractive since it requires additional complexity.

References

1. Lesh, J. R., Katz, J., Tan, H. H., and Zwillinger, D., "2.5 Bit/Detected Photon Demonstration Program: Analysis and Phase I Results," *TDA Progress Report 42-66*, Jet Propulsion Laboratory, Pasadena, Calif., pp. 115-132, Dec. 1981.

2. Lee, P. J., and Lesh, J. R., "On the Decisions with Erasure for *M*-ary Orthogonal Input Channel," in preparation.

3. Divsalar, D., Gagliardi, R. M., and Yuen, J. H., "PPM Demodulation for Reed-Solomon Decoding for the Optical Space Channel," *TDA Progress Report 42-70*, Jet Propulsion Laboratory, Pasadena, Calif. pp. 47-59, Aug. 1982.

4. Gallager, R. G., *Information Theory and Reliable Communication*, John Wiley and Sons, New York, 1968.

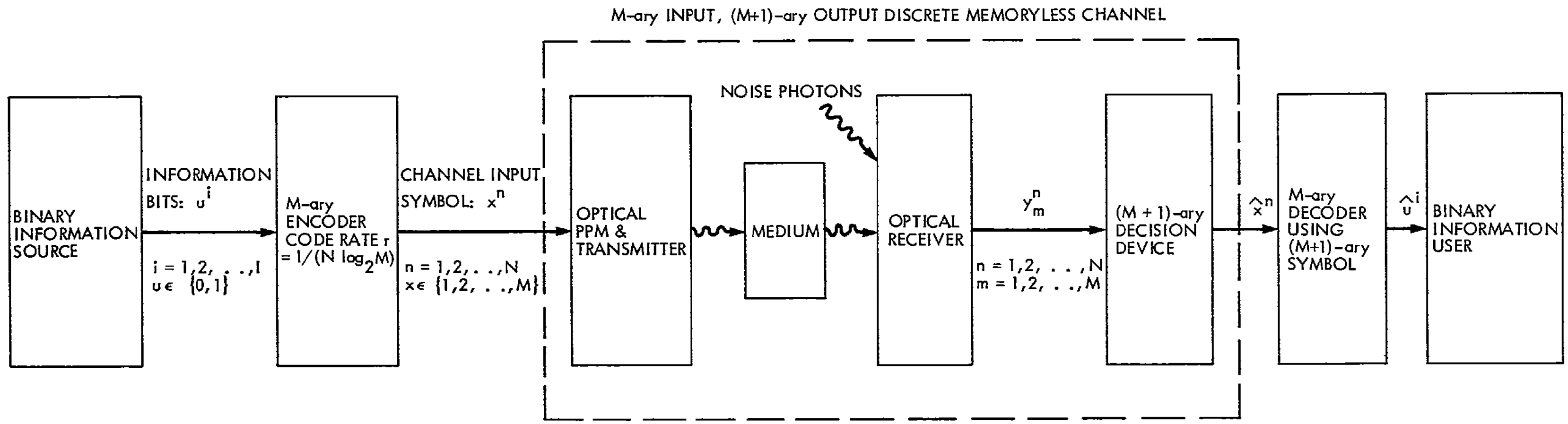

Fig. 1. Coding system for *M*-ary optical PPM channel with (*M*+1)-ary decision rule

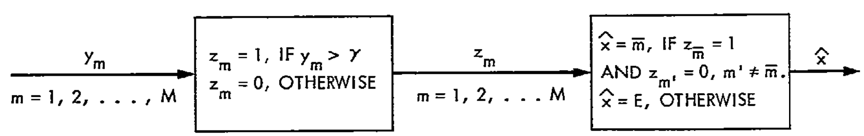

Fig. 2(a). Slot-by-slot threshold detection-and-decision device

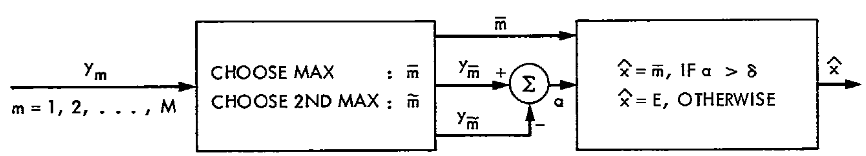

Fig. 2(b). Symbol-by-symbol threshold decision device

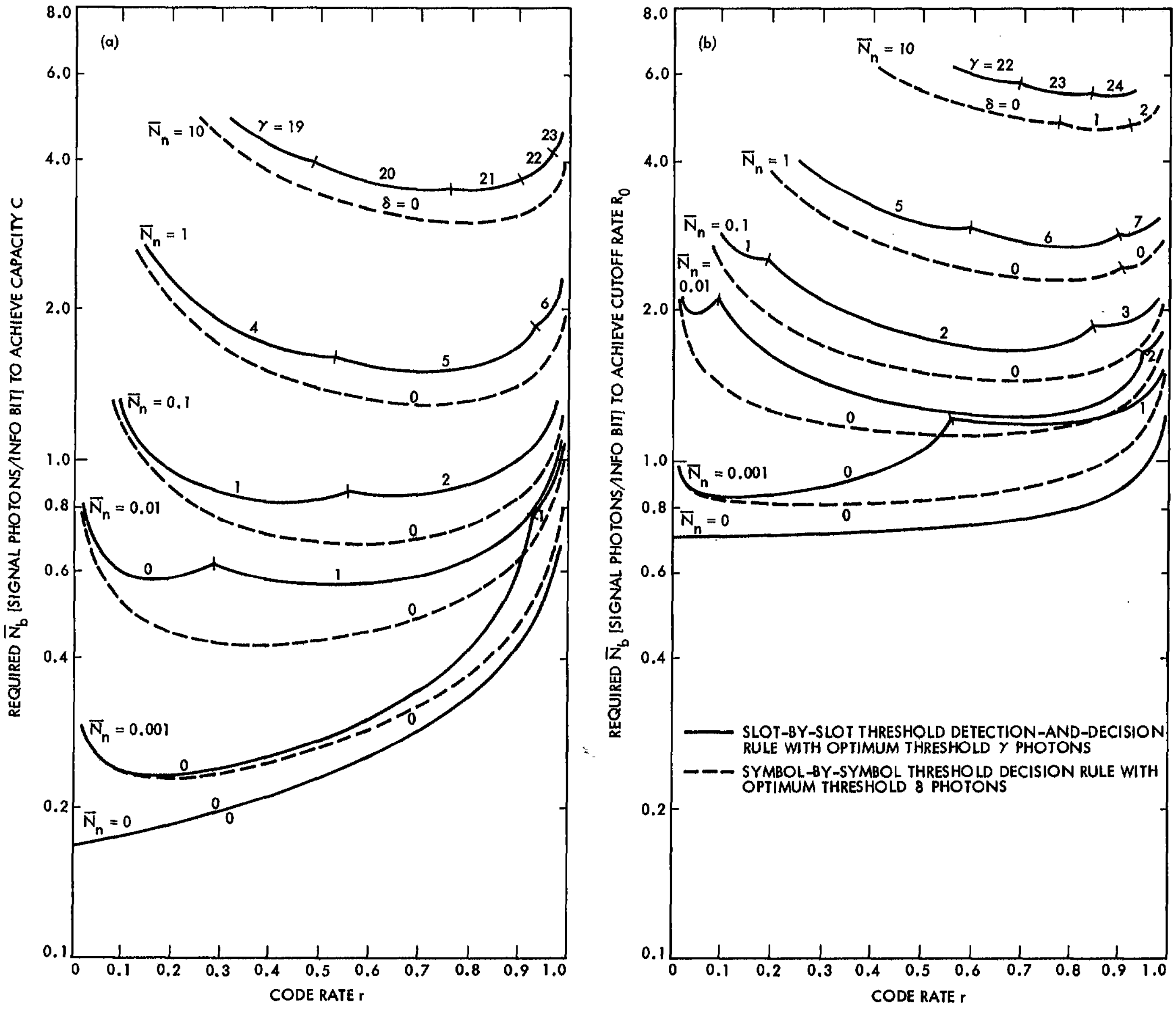

Fig. 3. Required N_b [signal photons/info. bit] to achieve (a) channel capacity C and (b) cutoff rate R_0 vs code rate r for two different $(M+1)$-ary decision rules on ideally direct-detected $M = 64$-ary optical PPM channel with several background noise levels N_n [noise photons/PPM slot]

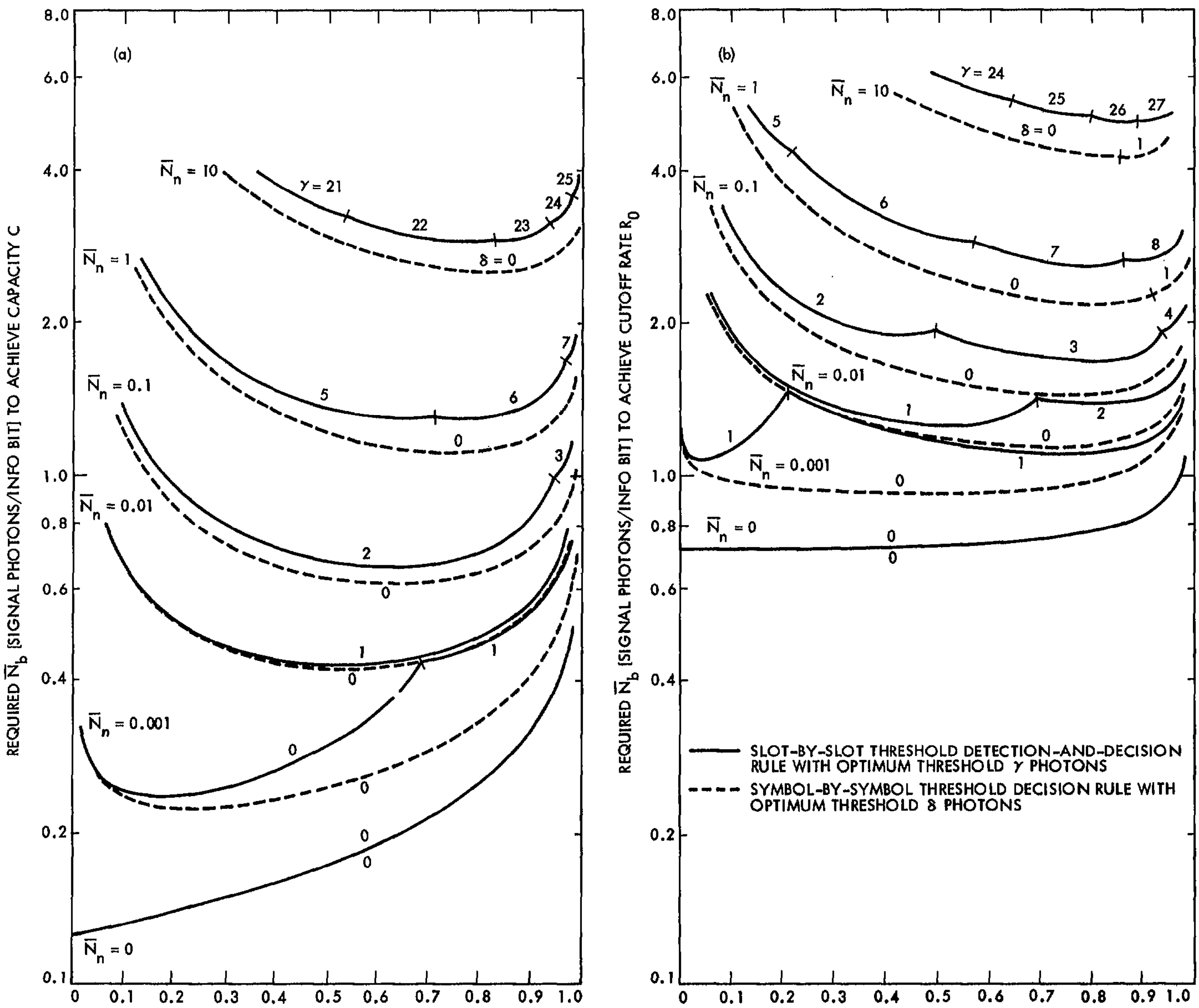

Fig. 4. Required N_b [signal photons/info. bit] to achieve (a) channel capacity C and (b) cutoff rate R_0 vs code rate r for two different (M+1)-ary decision rules on ideally direct-detected M = 256-ary optical PPM channel with several background noise levels N_n [noise photons/PPM slot]

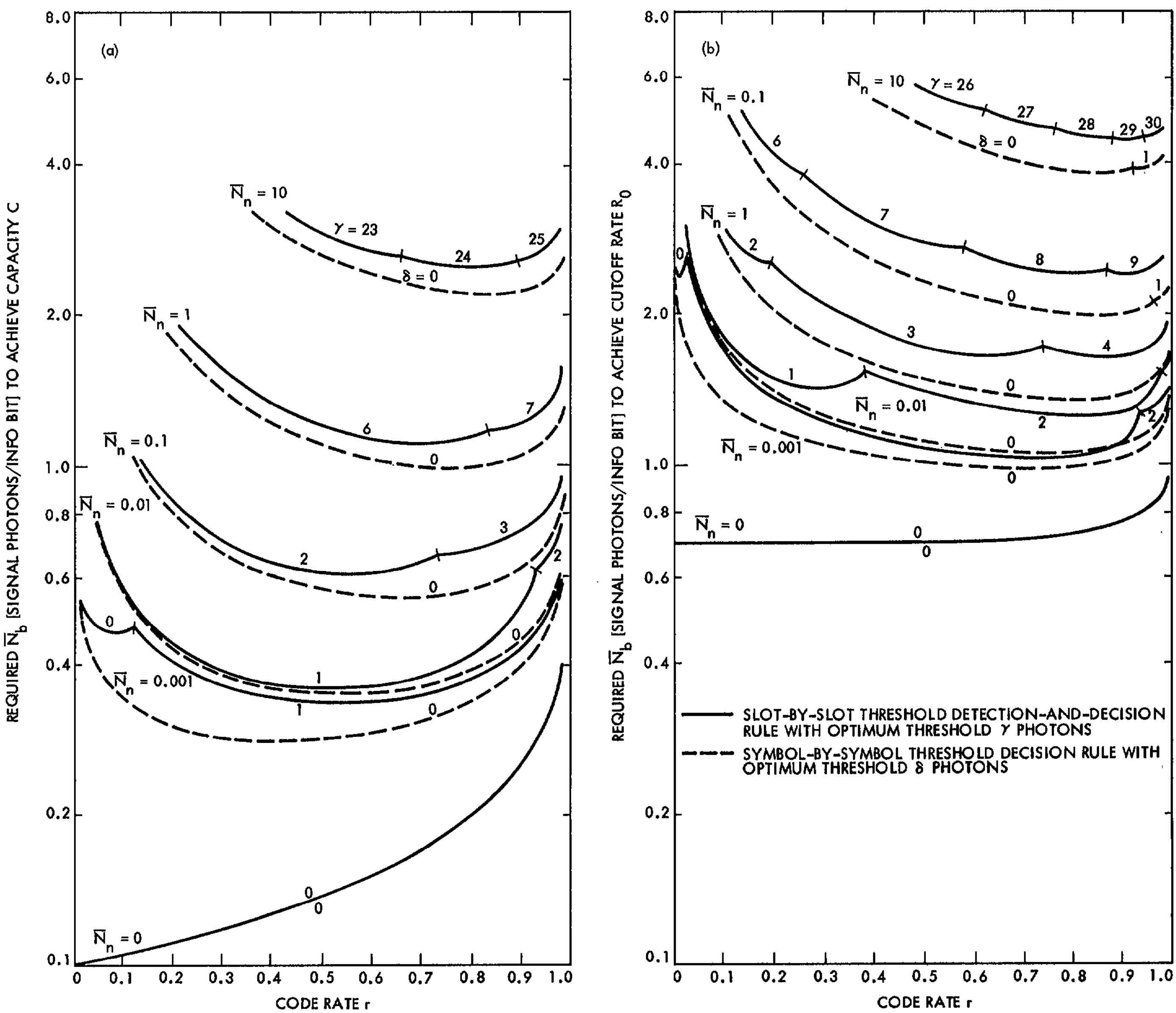

Fig. 5. Required N_b [signal photons/info. bit] to achieve (a) channel capacity C and (b) cutoff rate R_0 vs code rate r for two different (M+1)-ary decision rules on ideally direct-detected M = 1024-ary optical PPM channel with several background noise levels N_n [noise photons/PPM slot]