The Application of the Implicit Alternating-Direction Numerical Technique to Thermal Analysis Involving Conduction and Convection

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A computerized model has been developed for analyzing the temperature distribution of a two-dimensional body which is located at or near the soil surface and is partially exposed to solar radiation. The body may have one or more interior cavities containing air or another fluid. The methodology which evolved is also applicable to a general class of thermal analyses involving a body surrounded by a semi-infinite medium exposed to surface radiation energy. The theoretical analysis, numerical procedure, and a sample case are discussed.

I. Introduction

The problem of determining the temperature and heat flux of bodies which are exposed to ambient environmental conditions and have portions lying below the soil surface occurs in a wide variety of applications. Since determining a closed-form analytical solution is difficult and may only be done to a first-order accuracy, the development of a numerical finite-difference model was necessary to determine the temperature distribution of bodies which fall into this class of problems.

The initial study objective was to develop a thermal model of a concrete cable conduit, which is partially built below the surface of the soil, and has an interior cavity. The conduit is subject to solar radiation (insolation) only on its above-ground surface. As the numerical solution was being developed, its usefulness was extended to enable the analysis of a body in a semi-infinite medium having a more general configuration containing one or more cavities, with or without the external surfaces exposed to insolation.

Examples of such general configurations include: a solar pond, an object floating or submerged in a body of water, a microwave antenna pedestal foundation, a covered excavation for cable conduits, an underground tunnel, or a building without exterior glazing.

The heat transfer formulation is presented in Section II, and the numerical procedure in Section III. The finite difference solution employs the implicit alternating-direction method which is an effective compromise between computational accuracy and speed. This method avoids the stringent step-size requirement of the explicit finite difference method and the computational complexity of the two-dimensional implicit method.

II. Thermal Analysis

A. Formulation of Finite Difference Equations

The governing energy balance equation in this two-dimensional thermal analysis is the nonsteady heat conduction equation, which is:

$$c_p \, \rho \, \frac{\partial T}{\partial t} = \frac{\partial}{\partial X}\left(k \, \frac{\partial T}{\partial X}\right) + \frac{\partial}{\partial Y}\left(k \, \frac{\partial T}{\partial Y}\right) + Q \qquad (1)$$

where

$c_p(X,Y)$ = the local specific heat

$\rho(X,Y)$ = the local density

$T(X,Y,t)$ = temperature

t = time

$k(X,Y)$ = thermal conductivity

$Q(X,Y,t)$ = heat gain (or loss) per unit volume from sources other than conduction, such as internal generation or surface radiation.

In this model, Q consists of surface radiation heat transfer and solar irradiation on the surface.

For a two-dimensional rectangular network of thermal nodes, as shown in Fig. 1, Eq. (1) expressed in the implicit, backward-difference form becomes

$$c_p(i,j)\rho(i,j)\left[\frac{T'(i,j) - T(i,j)}{\Delta t}\right] = k(i,j)\left[\frac{T'(i-1,j) - T'(i,j)}{\Delta X^2}\right.$$

$$+ \frac{T'(i+1,j) - T'(i,j)}{\Delta X^2}$$

$$+ \frac{T'(i,j-1) - T'(i,j)}{\Delta Y^2}$$

$$\left. + \frac{T'(i,j+1) - T'(i,j)}{\Delta Y^2}\right]$$

$$+ Q(i,j) \qquad (2)$$

at node point (i,j), where the thermal conductivity $k(i,j)$ is considered uniform and constant with temperature within the differential element surrounding (i,j), and Δt is the time step. Note that (i,j) denote position in the X- and Y-directions, respectively, and T' represents the unknown temperatures at time $(t + \Delta t)$.

By analogy to an electrical circuit, the thermal energy balance is thought of in terms of heat flows or "currents" into an element centered about the node point (i,j) with "thermal resistances" connecting all neighboring nodes. Then Eq. (2) becomes:

$$C(i,j)\left[\frac{T'(i,j) - T(i,j)}{\Delta t}\right] = \frac{T'(i-1,j) - T'(i,j)}{R(i-1,j)}$$

$$+ \frac{T'(i+1,j) - T'(i,j)}{R(i+1,j)}$$

$$+ \frac{T'(i,j-1) - T'(i,j)}{R(i,j-1)}$$

$$+ \frac{T'(i,j+1) - T'(i,j)}{R(i,j+1)}$$

$$+ q(i,j) \qquad (3)$$

where

$C(i,j)$ = the thermal capacity of the cell

$q(i,j)$ = the rate of heat gain or loss to the cell from other sources

$R(i,j)$ = typical thermal resistance between node $(i-1,j)$ and node (i,j)

$V(i,j)$ = volume of the cell with unit depth

$\Delta X(i,j)$ = X-dimension of the cell

$\Delta Y(i,j)$ = Y-dimension of the cell

Note that for the two-dimensional case, the depth (Z-dimension) of the cell is taken to be unity, giving

$$V(i,j) = \Delta X(i,j)\, \Delta Y(i,j)$$

$$C(i,j) = V(i,j)\, \rho(i,j)\, C_p(i,j)$$

$$q(i,j) = V(i,j)\, Q(i,j)$$

$$R(i-1,j) = \frac{\Delta X^2\,(i,j)}{k(i-1,j)\, V(i,j)}$$

In dealing with interfaces between different media the following cases are considered for the computation of thermal resistance:

(a) For two adjacent cells having different conductivities, the resistance in the X-direction between node $(i-1,j)$ and node (i,j) is:

$$R(i-1,j) = \frac{1}{\Delta Y(i,j)} \left[\frac{\Delta X(i-1,j)}{2k(i-1,j)} + \frac{\Delta X(i,j)}{2k(i,j)}\right]$$

(4)

This relationship applies, for example, to the resistance between nodes (4, 1) and (5, 1) in Fig. 1.

(b) For the case where convection occurs at the boundary between two cells, the resistance between nodes $(i-1,j)$ and (i,j) becomes:

$$R(i-1,j) = \frac{1}{\Delta Y(i,j)} \left[\frac{1}{h(i-1,j)} + \frac{\Delta X(i,j)}{2k(i,j)}\right]$$

(5)

where $h(i-1,j)$ is the convective heat transfer coefficient between the $(i-1,j)$ node and the boundary between the cells. Note that Eqs. (4) and (5) are only valid for an interface that is located halfway between the adjacent nodes along the perpendicular direction.

The general form of the thermal resistance equation combining Eqs. (4) and (5) for convection and different material conductivities at thermal interfaces is: ·

$$R(i-1,j) = \frac{1}{\Delta Y(i,j)} \left[\frac{\Delta X(i-1,j)}{2k(i-1,j)} + \frac{1}{h(i-1,j)} + \frac{\Delta X(i,j)}{2k(i,j)}\right.$$

$$\left. + \frac{1}{h(i,j)}\right]$$

(6)

Appropriate values of thermal conductivity and heat transfer coefficients should be set to match the conditions at each node. "Infinity" values can be used for k or h to selectively reduce appropriate terms to zero in Eq. (6). For instance, if there is convection between the $(i-1,j)$ node and the intercell boundary, then $k(i-1,j)$ should be set to "infinity."

B. Radiation Exchange

The sky-to-surface radiation heat transfer rate per unit area is expressed as:

$$Q_{sky} = \sigma F \epsilon \left[(T_{sky})^4 - T_2^4\right]$$

(7)

where

σ = Stefan-Boltzmann constant = 5.6697×10^{-8} W/m^2 K^4

ϵ = $\epsilon(X, Y)$ = effective emissivity of the surface that is exposed to the environment; appropriate values are discussed in Section IV

T_{sky} = the clear sky temperature

T_2 = surface temperature

F = view factor

The sky temperature is assumed as $0.914\, T_1$, where T_1 = ambient air temperature at the surface.

Equation (7) was applied to flat, horizontal surfaces (where $F = 1$). Other configurations require the determination of radiation shape factor F. The model developed in this report calculates sky-to-surface radiant exchange only for horizontal surfaces.

The rate of solar radiation per unit area absorbed by the surface is represented by the relation:

$$Q_a = \alpha I_t$$

(8)

where

Q_a = solar radiation intensity absorbed by a horizontal surface

$\alpha(X, X)$ = average effective solar radiation absorptance of the exposed surface

I_t = incident solar radiation

Thus the net heat gain or loss from other nearby sources, $q(i,j)$ of Eq. (3), combines Eqs. (7) and (8) and is expressed as:

$$q(i,j) = \Delta X(i,j) \left[Q_{sky}(i,j) + Q_a(i,j)\right]$$

(9)

for two-dimensional surface cells with unit depth exposed to the external environment. For all interior cells, $q(i,j) = 0$.

III. Numerical Procedure

The finite difference formulation given by Eq. (3) provides a set of N simultaneous linear, algebraic equations for N inte-

rior node points in which the unknowns are values of the updated temperature T'.

In some applications, a special one-dimensional version of Eq. (3) needs to be considered. The equation set in this case can be written in matrix form, and the coefficient matrix becomes tridiagonal. A recursive analytical method is available to solve this tridiagonal system. The following discussion refers to the case which consists of a cable conduit structure buried in soil with the top surface exposed to the environment. More design details are given in the sample case in Section IV.

A. One-Dimensional Analysis

In order to provide a boundary condition for the two-dimensional case, the one-dimensional solution is applied to a vertical section of soil which is sufficiently distant from the cable conduit structure so that conduction in the horizontal direction is negligible. The resulting temperature profiles may then be taken as time-dependent boundary conditions for the appropriate boundary of the two-dimensional case.

The one-dimensional form of Eq. (3) is:

$$C(j)\frac{T'(j) - T(j)}{\Delta t} = \frac{T'(j-1) - T'(j)}{R(j-1)} + \frac{T'(j+1) - T'(j)}{R(j+1)} + q(j)$$

$$(q(j) = 0 \quad \text{for } j > 1) \tag{10}$$

which may be expressed as:

$$-\frac{1}{R(j-1)}T'(j-1) + \left[\frac{C(j)}{\Delta t} + \frac{1}{R(j-1)} + \frac{1}{R(j+1)}\right]T'(j)$$

$$-\frac{1}{R(j+1)}T'(j+1) = \frac{C(j)T(j)}{\Delta t} + q(j) \tag{11}$$

The coefficients of $T'(j-1)$, $T'(j)$, and $T'(j+1)$ designates $a(j)$, $b(j)$, and $c(j)$ respectively form a tridiagonal matrix with diagonal vectors a, b, and c, where

$$a(j) = -\frac{1}{R(j-1)}$$

$$b(j) = \frac{C(j)}{\Delta t} + \frac{1}{R(j-1)} + \frac{1}{R(j+1)}$$

$$c(j) = \frac{1}{R(j+1)}$$

The right-hand side of Eq. (11) is a known quantity and is designated as the vector d where:

$$d(j) = \frac{C(j)T(j)}{\Delta t} + q(j)$$

The recursion solution (Ref. 2) is of the form:

$$T'(j) = \gamma(j) - \frac{c(j)}{\beta(j)}T'(j+1) \quad j = N-1, N-2, \cdots, 1$$

$$\tag{12}$$

$$T'(N) = \gamma(N)$$

where

$$\beta(j) = b(j) - \frac{a(j)c(j-1)}{\beta(j-1)} \quad j = 2, 3, \cdots, N \tag{13}$$

$$\gamma(j) = \frac{d(j) - a(j)\gamma(j-1)}{\beta(j)} \quad j = 2, 3, \cdots, N \tag{14}$$

$$\beta(1) = b(1) \quad \text{(node 1 is the surface node)} \tag{15}$$

$$\gamma(1) = \frac{d(1)}{\beta(1)} \tag{16}$$

By this approach, the problems of accumulation of round-off error or iterative convergence which are typical with other methods of matrix equation solution are eliminated.

In applying Eqs. (12-16), the initial and boundary conditions may be introduced by specifying an initial temperature at each node, at time ($t = 0$) and boundary temperatures known as a function of time at nodes $j = 0$ and $j = N + 1$, which represent the ambient air and a "constant" deep earth location, respectively. For this analysis, the ambient air temperature, $T(i,0)$, as a function of time of day was approximated as a sinusoidal function having a given average, amplitude, and phase. These given quantities were determined by curve-fitting to local recorded temperature data for the site being modelled. The one-dimensional analysis was taken to a depth such that the temperature at the bottom node, $j = N + 1$, was essentially constant on both a diurnal and annual basis. Surface effects typically penetrate soil to a depth of about 20 meters throughout the time span of the annual cycle. A separate computer program has been written to perform this one-dimensional analysis.

Since modelling soil temperatures is a cyclic problem, the one-dimensional model has to be run for each day of the year in succession, for about five annual cycles to remove the effects of initial temperature conditions.

B. Two-Dimensional Numerical Procedure

When the two-dimensional form of Eq. (3) is considered, the coefficient matrix is found to no longer be tridiagonal,

and the recursive solution cannot be used to solve the system of equations. However, the implicit alternating-direction method again allows use of the tridiagonal recursion solution and avoids the difficulty of solving the matrix equation by other methods. The implicit alternating-direction method uses two different equations in turn over successive time-steps, each of duration $\Delta t/2$. The first equation is implicit only in the X-direction between times t and $(t + \Delta t/2)$ and the second equation in the Y-direction between times $(t + \Delta t/2)$ and $(t + \Delta t)$ as follows:

$$- \frac{1}{R(i-1,j)} T^*(i-1,j) + \left[\frac{C(i,j)}{\Delta t/2} + \frac{1}{R(i-1,j)} + \frac{1}{R(i+1,j)} \right] T^*(i,j) - \frac{1}{R(i+1,j)} T^*(i+1,j) = \frac{1}{R(i,j-1)} T(i,j-1)$$

$$+ \left[\frac{C(i,j)}{\Delta t/2} - \frac{1}{R(i,j-1)} - \frac{1}{R(i,j+1)} \right] T(i,j) + \frac{1}{R(i,j-1)} \quad T(i,j-1) + q(i) \tag{17}$$

and

$$- \frac{1}{R(i,j-1)} T'(i,j-1) + \left[\frac{C(i,j)}{\Delta t/2} + \frac{1}{R(i,j-1)} + \frac{1}{R(i,j+1)} \right] T'(i,j) - \frac{1}{R(i,j+1)} T'(i,j+1) = \frac{1}{R(i-1,j)} T^*(i-1,j)$$

$$+ \left[\frac{C(i,j)}{\Delta t/2} - \frac{1}{R(i-1,j)} - \frac{1}{R(i-1,j)} \right] T^*(i,j) + \frac{1}{R(i+1,j)} \quad T^*(i+1,j) + q(i) \tag{18}$$

where T is temperature at time t, T^* is temperature at time $(t + \Delta t/2)$, T' is temperature at time $(t + \Delta t)$. The right-hand sides of Eqs. (17) and (18) are known quantities and the temperatures on the left-hand sides are the unknowns. By using tridiagonal matrix techniques, Eq. (17) is solved for the intermediate values T^*, which are then used in Eq. (18) to similarly solve for T' at the end of the time interval Δt.

The boundary conditions are prescribed along each of the four boundaries as follows:

(1) The top boundary condition is the temperature of ambient air above the soil surface as a function of time. The air temperature is determined as previously described for the one-dimensional case.

(2) The right-hand boundary is the temperature profile of homogenous soil as a function of time. The results of the one-dimensional analysis are used for this boundary condition.

(3) The left-hand boundary is the vertical line of symmetry of the structure. A symmetric boundary condition implies thermal gradients of zero in the normal direction to the boundary.

(4) The bottom boundary condition is a prescribed temperature which is constant along the bottom row of nodes, but can vary with time. The value is taken from the corresponding node on the right-hand boundary at each time step. This boundary must be located at a sufficient depth so that it can be assumed unaffected by the thermal influence of the structure. A trial-and-error approach may be necessary here to avoid unacceptably large nodal networks.

Although the boundary conditions have been included in the computer model in the form described, they can be modified to fit another problem with little difficulty.

IV. Sample Case

The sample case configuration consisted of a hypothetical concrete trench designed to house communication cables which require a thermally stable environment. The trench, shown in cross-section in Fig. 2, is covered by steel plates which are coated with a highly reflective paint and lined with styrofoam on the inside. Ambient conditions at the Goldstone Deep Space Communications Complex were used for this case.

A. One-Dimensional Analysis

The incident solar radiation I_t is computed by a subroutine, SOLAR, which uses the ASHRAE formulation described in Ref. 1 to compute direct and diffuse sky components. Input

parameters to this subroutine include site latitude, day of the year, and angle of the surface with respect to the horizon (zero in this model). Diffuse sky radiation and direct solar radiation are the only components which are included in computing I_t. Solar radiation reflected from nearby surroundings can be determined by the SOLAR subroutine, but is not used in this model.

The input conditions for the one-dimensional analysis were: latitude = 34°N, effective surface absorptivity α = 0.4, surface emissivity ϵ = 0.45, time step = 1 hour, initial temperature = 20.6°C, soil density ρ = 2.05 g/cm^3, soil specific heat = 1.84 J/g-k, soil conductivity k = 9.5 × 10^{-3} W/cm-K, surface convection coefficient h = 2.27 × 10^{-3} W/cm^2-K; for ambient air temperature: annual average = 18.9°C, amplitude of annual temperature wave = 11.4°C, daily amplitude = 12.2°C, and the constant soil temperature at a depth of 17 m was taken to be 20.6°C.

No cloud cover factors were used, and no direct means was included to take into account the precipitation-evaporation cycle which causes heat loss at the soil surface. Thus an "effective" absorptivity was defined which accounts for these factors. A typical set of calculated temperature profiles is shown in Fig. 3, indicating conditions at four different times on a particular day. The left-most point on the curves is the ambient air temperature. Note that the effect of diurnal fluctuation in ambient temperature is damped to nearly zero at a depth of about 0.5 m below the surface. The remainder of the profile is characterized by annual temperature cycle effects. The validity of the model result is confirmed by the fact that the average temperature profile for the entire year was effectively uniform and equal to the constant soil temperature of 20.6°C, and the behavior of the solution in terms of time lag and damping of temperature amplitude as a function of depth is analogous to published analytical solutions.

B. Two-Dimensional Analysis

The additional input conditions for the two-dimensional analysis consisted of handbook values for thermal properties

of the trench materials, absorptivity of the cover plate = 0.1, and emissivity of the cover plate = 0.12. The analysis was done for the 170th day of the year. The model was cycled for three 24-hour periods to overcome the effect of initial conditions (all the columns of nodes were initially set equal to the right-hand-side boundary condition). Figure 4 shows the nodal network configuration which was input to the computer model.

Results are shown at times 0100 and 1200 PST in Table 1. Ten nodes were used in the vertical direction and eight in the horizontal. Note that the temperatures in row 9 are nearly uniform, showing that the assumption of uniform temperature along the bottom boundary (row 10) is valid. Also, temperatures in the horizontal direction in columns 5-8 are nearly uniform, showing that the assumption of one-dimensional heat transfer for the right-hand boundary is valid.

V. Discussion

The modelling procedure outlined in this report is applicable to a general class of problems and offers several advantages over using large general-purpose thermal analysis computer programs. Among these advantages are the compact size of the software, the economy of operation of the software, and the simplicity of applying the modelling procedure. However, there are some restrictions inherent in the model as it presently exists: (1) the thermal node mesh must be rectangular, (2) the left-hand boundary must be a line of symmetry, (3) radiant heat exchange inside the cavity is neglected, and (4) no cloud cover factors are included in the solar model.

All of these restrictions may be overcome without great difficulty by additional development of the modelling software. This thermal model will become an increasingly general analytical tool when these additional capabilities are implemented.

References

1. Hughes, R. D., "A Simplified Solar Cell Array Modelling Program," *TDA Progress Report 42-68*, pp. 167-185, Jet Propulsion Laboratory, Pasadena, Calif., Apr. 15, 1982.

2. Carnahan, B., Luther, H. A., and Wilkes, J. O., *Applied Numerical Methods*, John Wiley & Sons, Inc., N.Y., 1969.

Table 1. Soil temperatures for Day 170 at two representative times
(Two-dimensional temperature profiles, °C)

Time = 0100

Vertical node	Horizontal node							
	1	2	3	4	5	6	7	8
1	25.63	25.63	25.63	25.63	25.63	25.63	25.63	25.63
2	27.03	24.42	27.03	26.71	26.22	26.30	26.30	26.88
3	30.60	26.33	30.60	31.57	31.71	31.78	31.77	31.61
4	30.07	27.04	30.07	29.76	29.91	29.98	30.00	29.83
5	27.25	26.40	27.25	27.20	27.28	27.31	27.30	27.13
6	25.09	24.10	25.09	25.08	25.09	25.09	25.07	24.88
7	23.51	22.54	23.51	23.57	23.59	23.59	23.57	23.40
8	22.79	22.28	22.79	22.81	22.82	22.82	22.80	22.65
9	21.87	21.75	21.87	21.88	21.88	21.88	21.86	21.70
10	20.74	20.74	20.74	20.74	20.74	20.74	20.74	20.74

Time = 1200

Vertical node	Horizontal node							
	1	2	3	4	5	6	7	8
1	28.80	28.80	28.80	28.80	28.80	28.80	28.80	28.80
2	30.55	32.79	30.55	37.20	37.48	37.42	37.43	36.79
3	27.28	30.44	27.28	28.90	29.41	29.42	29.44	29.59
4	27.51	28.10	27.51	28.54	28.91	28.98	28.98	28.92
5	27.12	25.67	27.12	27.24	27.31	27.35	27.33	27.17
6	25.12	23.58	25.12	25.12	25.13	25.13	25.10	24.93
7	23.52	22.45	23.52	23.60	23.63	23.63	23.60	23.44
8	22.81	22.22	22.81	22.84	22.86	22.86	22.83	22.69
9	21.89	21.73	21.89	21.90	21.91	21.92	21.88	21.74
10	20.78	20.78	20.78	20.78	20.78	20.78	20.78	20.78
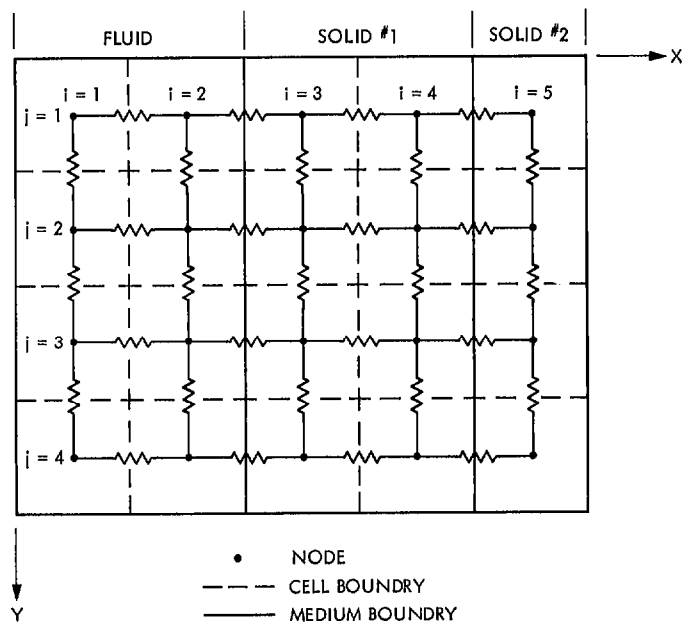
Fig. 1. Nodal representation of two-dimensional heat conduction

NODE •
CELL BOUNDRY ─ ─ ─
MEDIUM BOUNDRY ───

FLUID | SOLID #1 | SOLID #2

i = 1   i = 2   i = 3   i = 4   i = 5

j = 1
j = 2
j = 3
j = 4
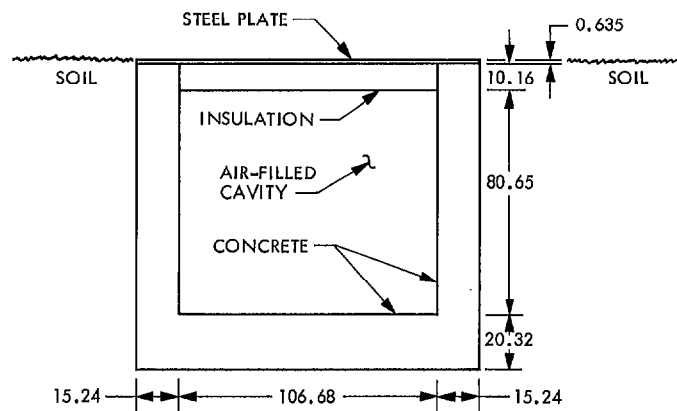
NOTE: ALL DIMENSIONS ARE IN CENTIMETERS

Fig. 2. Cable trench configuration

STEEL PLATE
SOIL
INSULATION
AIR-FILLED CAVITY
CONCRETE
SOIL

0.635
10.16
80.65
20.32
15.24
106.68
15.24
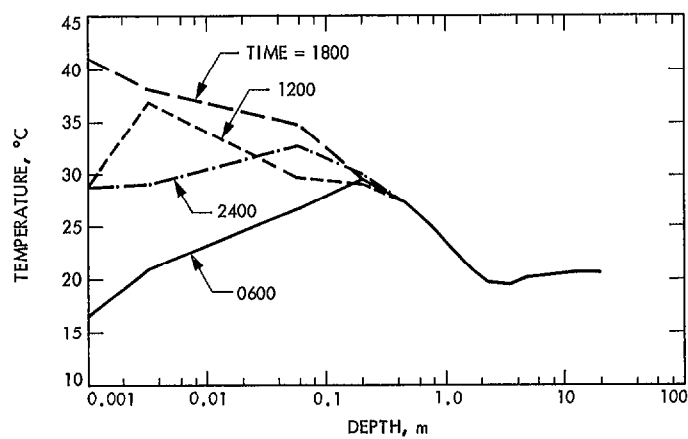
Fig. 3. One-dimensional soil temperature profiles, Day 170

TIME = 1800
1200
2400
0600

TEMPERATURE, °C
DEPTH, m
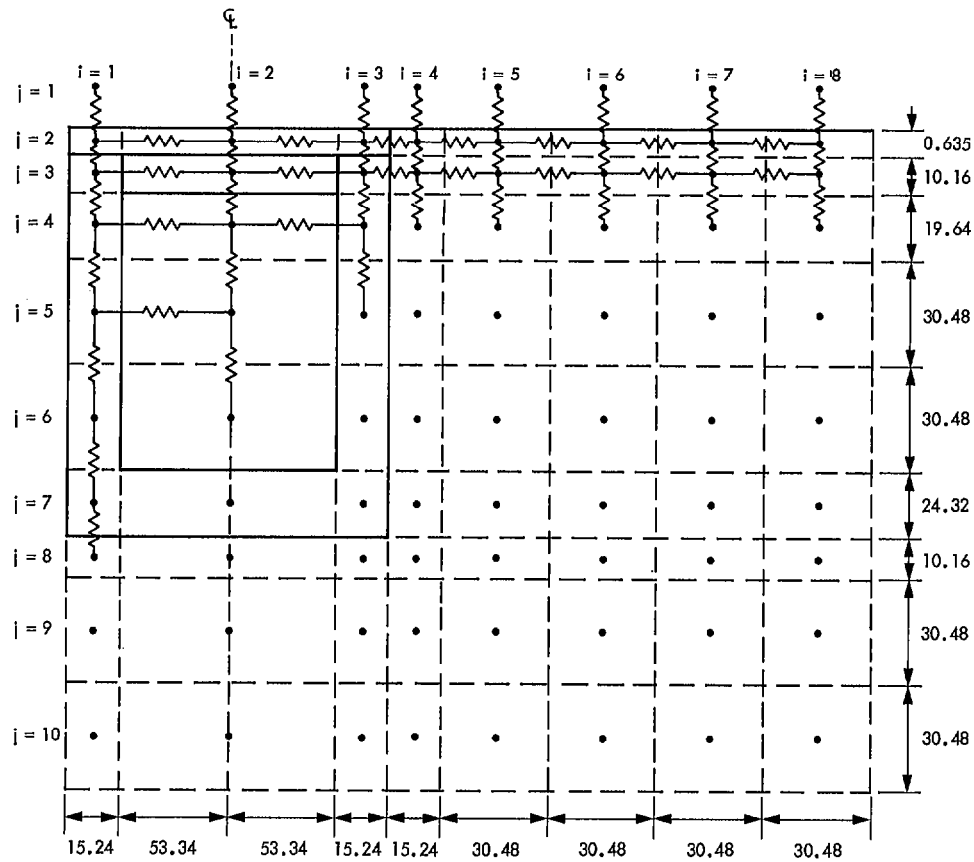
NOTE: ALL DIMENSIONS IN CENTIMETERS
ONLY SOME TYPICAL RESISTANCES ARE INDICATED

Fig. 4. Nodal representation of cable trench and surrounding soil