Symbol Stream Combining Versus Baseband Combining for Telemetry Arraying

D. Divsalar
Communications Systems Research Section

The objectives of this article are to investigate and analyze the problem of combining symbol streams from many Deep Space Network stations to enhance bit signal-to-noise ratio and to compare the performance of this combining technique with baseband combining. Symbol stream combining (SSC) has some advantages and some disadvantages over baseband combining (BBC). The SSC suffers almost no loss in combining the digital data and no loss due to the transmission of the digital data by microwave links between the stations. The BBC suffers 0.2 dB loss due to alignment and combining the IF signals and 0.2 dB loss due to transmission of signals by microwave links. On the other hand, the losses in the subcarrier demodulation assembly (SDA) and in the symbol synchronization assembly (SSA) for SSC are more than the losses in the SDA and SSA for BBC. It is shown that SSC outperforms BBC by about 0.35 dB (in terms of the required bit energy-to-noise spectral density for a bit error rate of 10^{-3}) for an array of three DSN antennas, namely 64 m, 34 m(T/R) and 34 m(R).

I. Introduction

To capture signals from the Voyager spacecraft and potentially other space probes, signals from more than one antenna are combined. To reduce the cost of bringing the received signals from the Deep Space Network (DSN) and non-DSN facilities together, J. W. Layland of JPL has suggested combining quantized symbol streams in place of telemetry baseband combining.

The analysis of this article shows that symbol stream combining is superior to baseband combining. In baseband combining we require that the baseband signals from each antenna, after carrier demodulation, be brought to a signal processing center. Here the signals are delay adjusted, weighted, summed and then passed through the subcarrier demodulation assembly (SDA) and symbol synchronization assembly (SSA). The output of this processing is a quantized symbol stream which enters the maximum likelihood convolutional decoder (MCD). Since prior to decoding by the MCD all operations on the data signal are linear, it seems we might not be gaining anything by combining these signals before the SSA processing; i.e., symbols could be combined just before input to the MCD. This has the advantage of combining digital signals, which will be easier than combining baseband signals since the problems of time alignment, weighting and summing the symbol streams no longer exist. On the other hand, each station will have lower SNRs at its SDA and SSA than the SNRs from baseband combining. This article analyzes these methods of combining. It shows that these lower SNRs at the

SDA and SSA prior to combining do not present as much loss as was previously expected. On the contrary, when we compare these two methods under the same conditions, namely the same loop bandwidths for carrier tracking loop, SDA and SSA, and the same mod index, etc., we find that the overall total loss in symbol signal-to-noise ratio (SNR) for symbol stream combining is actually less than the loss in symbol SNR for baseband combining.

II. Symbol Stream Combining

A system for combining symbol streams from N stations is depicted in Fig. 1. The telemetry signal is an RF carrier that is phase-modulated by a squarewave subcarrier ($\mathrm{Sin}\, \omega_{sc}t$) at a peak modulation index θ. The subcarrier is bi-phase modulated with a binary data stream $D(t)$. This telemetry signal is received by N ground stations. The received telemetry signal at the ith station is

$$r_i(t) = \sqrt{2P_i} \sin(\omega_c t + \Phi_{ci} + D(t+\tau_i)\theta$$

$$\mathrm{Sin}(\omega_{sc}t + \Phi_{sci})) + \approx{n}_i(t) \tag{1}$$

where P_i is the total received power at this station, ω_c is the carrier radian frequency, Φ_{ci} is the carrier phase, ω_{sc} is the subcarrier radian frequency, Φ_{sci} is the subcarrier phase and $\approx{n}_i(t)$ is the additive white Gaussian noise with two-sided spectral density $N_{0i}/2$. The subscript i refers to the ith station throughout this article. At the output of the receiver and carrier tracking loop (CTL) the signal can be represented as

$$s_i(t) = \sqrt{2P_i} \sin\theta\, D(t+\tau_i)\,\mathrm{Sin}(\omega_{sc}t + \Phi_{sci})$$

$$\cos(\omega_{IF}t + \phi_{ci}) + \tilde{n}_i(t) \tag{2}$$

where ω_{IF} is IF radian frequency, $\phi_{ci} = \Phi_{ci} - \hat{\Phi}_{ci}$ is carrier phase error and $\hat{\Phi}_{ci}$ is the phase locked loop (PLL) estimate of the carrier phase. $\tilde{n}_i(t)$ is white Gaussian noise with two-sided spectral density $N_{0i}/2$. The IF carrier reference signal is

$$r_{ci}(t) = \sqrt{2}\cos(\omega_{IF}t) \tag{3}$$

The subcarrier squarewave reference signal generated by the SDA is

$$r_{sci}(t) = \mathrm{Sin}(\omega_{sc}t + \hat{\Phi}_{sci}) \tag{4}$$

where $\hat{\Phi}_{sci}$ is an estimate of Φ_{sci}. After demodulating the signal $S_i(t)$ by the reference signals in (3) and (4), we obtain

$$W_i(t) = \sqrt{P_i}\sin\theta\, D(t+\tau_i)\left[1 - \frac{2}{\pi}|\phi_{sci}|\right]\cos\phi_{ci} + n_i(t) \tag{5}$$

which enters the SSA. The subcarrier phase error is $\phi_{sci} = \Phi_{sci} - \hat{\Phi}_{sci}$ and $n_i(t)$ is the baseband white Gaussian noise with two-sided spectral density $N_{0i}/2$.

The data waveform can be represented as

$$D(t) = \sum_{n=-\infty}^{\infty} a_n p(t - (n-1)T_s) \tag{6}$$

where $p(t)$ is unit power rectangular pulse shape with duration T_s (symbol time) and a_n is a binary channel symbol taking on values ± 1. Passing this signal through the SDA, assuming ϕ_{ci} and ϕ_{sci} are very slowly varying with respect to symbol time T_s, we get

$$Q_{k+m_i,i} = \frac{1}{T_s}\int_{(k-1)T_s+\epsilon_i}^{kT_s+\epsilon_i} W_i(t)\,p(t-(k-1)T_s - \epsilon_i)\,dt$$

$$= \frac{\sqrt{P_i}}{T_s}\sin\theta\left[1 - \frac{2}{\pi}|\phi_{sci}|\right]\cos\phi_{ci}$$

$$\int_{(k-1)T_s+\epsilon_i}^{kT_s+\epsilon_i} D(t+\tau_i)\,p(t-(k-1)T_s-\epsilon_i)\,dt$$

$$+ n_{k+m_i} \tag{7}$$

Let $\tau_i = m_i T_s + \epsilon_i'$ where $0 \leqslant \epsilon_i' < T_s$ for some integer m_i and $\bar{\epsilon}_i \triangleq \epsilon_i - \epsilon_i'$; then

$$\int_{(k-1)T_s+\epsilon_i}^{kT_s+\epsilon_i} D(t+\tau_i)\,p(t-(k-1)T_s-\epsilon_i)\,dt$$

$$= \begin{cases} (T_s - \bar{\epsilon}_i)\,a_{k+m_i} + \bar{\epsilon}_i a_{k+m_i+1} & 0 \leqslant \bar{\epsilon}_i < T_s/2 \\ (T_s + \bar{\epsilon}_i)\,a_{k+m_i} - \bar{\epsilon}_i a_{k+m_i-1} & -T_s/2 \leqslant \bar{\epsilon}_i < 0 \end{cases} \tag{8}$$

The worst case occurs if the symbol sequence consists of alternate symbol values ± 1, because whenever there is no symbol transition, a time synchronization error will not affect the signal amplitude. In this case, we have

$$Q_{k+m_i,i} = \sqrt{P_i}\sin\theta\left[1-\frac{2}{\pi}|\phi_{sci}|\right]\cos\phi_{ci}\,[1-2|\lambda_i|]\,a_{k+m_i} + n_{k+m_i} \tag{9}$$

where

$$\lambda_i = \bar{\epsilon}_i/T_s \text{ and } -\frac{1}{2} < \lambda_i < \frac{1}{2}$$

Assuming the time delay for each station is perfectly estimated, then each symbol stream $Q_{k+m_i,i}$ can be delayed by m_i seconds.

Samples of the signal at the output of the combiner for arraying of N antennas are

$$z_k = \sum_{i=1}^{N} \beta_i Q_{k,i} \tag{10}$$

where β_i's are weighting factors. The optimum values of β_i's will be derived shortly. Now let us find the mean and variance of z_k. Given

$$\underline{\phi}_c \triangleq \{\phi_{c1}, \phi_{c2}, \ldots, \phi_{cN}\}, \underline{\phi}_{sc} = \{\phi_{sc1}, \phi_{sc2}, \ldots, \phi_{scN}\}$$

$$\underline{\lambda} \triangleq \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$$

and a_k, we have the conditional mean and the conditional variance of z_k, respectively, as

$$\bar{z}_k = \sum_{i=1}^{N} \beta_i\sqrt{P_i}\sin\theta\, a_k\left[1-\frac{2}{\pi}|\phi_{sci}|\right]\cos\phi_{ci}\,[1-2|\lambda_i|] \tag{11}$$

and

$$\sigma^2_{z_k} = \frac{1}{2T_s}\sum_{i=1}^{N}\beta_i^2 N_{0i} \tag{12}$$

Then the conditional symbol SNR, conditioned on $\underline{\phi}_c$, $\underline{\phi}_{sc}$ and $\underline{\lambda}$ is

$$\text{conditional symbol SNR} = \frac{\bar{z}_k^2}{\sigma^2_{z_k}} \tag{13}$$

and the conditional bit SNR is

Conditional bit SNR = 2 × conditional symbol SNR

$$= \frac{\left[\sum_{i=1}^{N}\beta_i\sqrt{E_{bi}}\left(1-\frac{2}{\pi}|\phi_{sci}|\right)\cos\phi_{ci}\left(1-2|\lambda_i|\right)\right]^2}{\sum_{i=1}^{N}\beta_i^2 N_{0i}} \tag{14}$$

where $E_{bi} = 2P_iT_s\sin^2\theta$.

The bit error rate for N arrayed antennas, given $\underline{\phi}_c$, $\underline{\phi}_{sc}$ and $\underline{\lambda}$, is

$$P_b(\underline{\phi}_c, \underline{\phi}_{sc}, \underline{\lambda}) = f(\text{conditional bit SNR}) \tag{15}$$

where

$$f(x) = \begin{cases} e^{\alpha_0-\alpha_1 x} & ; \quad x \geqslant \dfrac{\alpha_0+\ln 2}{\alpha_1} \\ 0.5 & ; \quad x < \dfrac{\alpha_0+\ln 2}{\alpha_1} \end{cases}$$

$$\alpha_0 = 4.4514$$

$$\alpha_1 = 5.7230 \tag{16}$$

Letting $\beta_1 = 1$ and optimizing β_i's; $i = 2, 3, \ldots, N$, in order to minimize the bit error rate, we get the optimum values for β_i's:

$$\beta_i^* = \sqrt{\frac{E_{bi}}{E_{b1}}}\ \frac{N_{01}}{N_{0i}} \tag{17}$$

Let

$$\rho_i \triangleq \frac{P_i}{P_1}\cdot\frac{N_{01}}{N_{0i}} \tag{18}$$

and use (17) and (15); we obtain

$$P_b(\underline{\phi}_c, \underline{\phi}_{sc}, \underline{\lambda}) = f\left(\sum_{i=1}^{N}\frac{E_{bi}}{N_{0i}}y^2\right) \tag{19}$$

where

$$y \triangleq \frac{\sum_{i=1}^{N} \rho_i \left(1 - \frac{2}{\pi}|\phi_{sci}|\right) \cos\phi_{ci}(1 - 2|\lambda_i|)}{\sum_{i=1}^{N} \rho_i} \tag{20}$$

Note that when carrier phase, subcarrier phase and symbol time are all perfectly synchronized, the value of y is one.

Assume ϕ_{ci}'s, ϕ_{sci}'s and λ_i's are independent of each other, having density functions $p(\phi_{ci})$, $p(\phi_{sci})$ and $p(\lambda_i)$ respectively. Then the bit error rate can be expressed as

$$P_b = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2} P_b(\underline{\phi}_c, \underline{\phi}_{sc}, \underline{\lambda}) \left[\prod_{i=1}^{N} p(\phi_{ci})\, p(\phi_{sci})\, p(\lambda_i)\right] d\underline{\phi}_c\, d\underline{\phi}_{sc}\, d\underline{\lambda} \tag{21}$$

III. Baseband Combining

A system for arraying baseband signals out of N stations is shown in Fig. 2. Consider the baseband signals at the output of carrier tracking loops given by

$$s_i(t) = \sqrt{2P_i}\sin\theta\, D(t + \tau_i)\, \mathrm{S}\mathrm{in}\,(\omega_{sc}t + \Phi_{sci}) \cos(\phi_{ci}) + \tilde{n}_i(t) \tag{22}$$

After estimating the time delays, we can align the signals in time (in practice there is about 0.2 dB loss for this alignment) and then combine them. For simplicity of analysis, assume that alignment is perfect; then at the output of the signal combiner we have

$$S(t) = \sum_{i=1}^{N} \beta_i S_i(t - \tau_i) \tag{23}$$

At SDA, after demodulating $S(t)$ by reference subcarrier squarewave reference signal,

$$r_{sc} = \mathrm{S}\mathrm{in}\,(\omega_{sc}t + \hat{\Phi}_{sc}) \tag{24}$$

where $\hat{\Phi}_{sc}$ is the estimate of

$$\Phi_{sc} = \Phi_{sci} - \omega_{sc}\tau_i \tag{25}$$

we get

$$W(t) = \sum_{i=1}^{N} \beta_i \sqrt{P_i}\sin\theta\, D(t)\left[1 - \frac{2}{\pi}|\phi_{sc}|\right]\cos\phi_{ci} + n_i(t) \tag{26}$$

entering the SSA. The $\phi_{sc} = \Phi_{sc} - \hat{\Phi}_{sc}$ is the subcarrier phase error and $n_i(t)$ is baseband white Gaussian noise with two-sided spectral density $N_{0i}/2$.

Now if we proceed similarly to the previous section, at the output of SSA we get (for alternate symbol sequence)

$$Q_k = \sum_{i=1}^{N} \beta_i \sqrt{P_i}\sin\theta\left[1 - \frac{2}{\pi}|\phi_{sc}|\right] \cos\phi_{ci}\,[1 - 2|\lambda|]\, a_k + n_k \tag{27}$$

where λ is symbol time error and $-1/2 < \lambda < 1/2$.

Continuing, also as in the previous section, we get

$$P_b(\underline{\phi}_c, \phi_{sc}, \lambda) = f\left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}} y^2\right) \tag{28}$$

where

$$y \triangleq \frac{\sum_{i=1}^{N} \rho_i \left(1 - \frac{2}{\pi}|\phi_{sc}|\right) \cos\phi_{ci}(1 - 2|\lambda|)}{\sum_{i=1}^{N} \rho_i} \tag{29}$$

Then the bit error rate for the BBC case is

$$P_b = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-1/2}^{1/2} P_b(\underline{\phi}_c, \phi_{sc}, \lambda) \left(\prod_{i=1}^{N} p(\phi_{ci})\right) p(\phi_{sc})\, p(\lambda)\, d\underline{\phi}_c\, d\phi_{sc}\, d\lambda \tag{30}$$

IV. Carrier Tracking Loop Performance

The performance of carrier tracking loop (CTL) can be summarized as follows. The density function for carrier phase error can be expressed as (Ref. 1)

$$p(\phi_c) = C \frac{\exp\{\rho_{CTL} \cos\phi_c + \rho_{CTL}\, \phi_c \phi_{cs}\}}{2\pi I_0(\rho_{CTL})}; \quad -\pi < \phi_c < \pi \tag{31}$$

where C is a normalization factor such that

$$\int_{-\pi}^{\pi} p(\phi_c) d\phi_c = 1.$$

The ϕ_c is carrier phase error, ϕ_{cs} is carrier static phase error and ρ_{CTL} is carrier tracking loop SNR given by (Ref. 1)

$$\rho_{CTL} = \frac{P \cos^2\theta}{N_0 B_L \Gamma_c} \tag{32}$$

In (32) P is power, and B_L is CTL loop bandwidth given by (Ref. 1)

$$B_L = B_{L0} \left(\frac{1 + r_0 \alpha/\alpha_0}{1 + r_0} \right), \tag{33}$$

where B_{L0} is loop bandwidth at threshold, r_0 is the damping parameter at threshold, and α is the loop suppression factor given by (Ref. 2)

$$\alpha = \sqrt{\frac{0.7854\,\rho_{in} + 0.4768\,\rho_{in}^2}{1 + 1.024\,\rho_{in} + 0.4768\,\rho_{in}^2}}, \tag{34}$$

and ρ_{in} is the input signal-to-noise ratio to the bandpass limiter and an IF filter having bandwidth B_{IF}. Defining carrier margin (CM) by (Ref. 1)

$$CM = \frac{P \cos^2\theta}{N_0 (2 B_{L0})}, \tag{35}$$

then ρ_{in} can be expressed as

$$\rho_{in} = CM \frac{2 B_{L0}}{B_{IF}} \tag{36}$$

Finally Γ_c is the limiter performance factor given by (Ref. 2)

$$\Gamma_c = \frac{1 + \rho_{in}}{0.862 + \rho_{in}} \tag{37}$$

For second-order loop with transfer function

$$F(s) = \frac{1 + \tau_2 S}{1 + \tau_1 S} \tag{38}$$

where τ_1 and τ_2 are loop filter time constants, assume that the instantaneous Doppler offset is expressed by

$$d(t) = \frac{\omega_c}{c}(\Omega_0 + \Lambda_0 t), \tag{39}$$

where c is speed of light, Ω_0 is spacecraft speed and Λ_0 is spacecraft acceleration. Then the static phase error ϕ_{cs} can be expressed as (Ref. 3)

$$\phi_{cs} \approx \frac{\omega_c}{c} \left[\frac{\Omega_0 + \Lambda_0 t + \Lambda_0 \tau_1}{G} \right], \tag{40}$$

where G is the loop gain

$$G = \frac{\sqrt{8}}{\pi} \alpha K \tag{41}$$

For a perfect second-order loop, or when $\tau_2 << r\tau_1$, expression (40) for ϕ_{cs} reduces to

$$\phi_{cs} \approx \frac{\omega_c}{c} \frac{\Lambda_0}{r} \left(\frac{1 + r}{4 B_L} \right)^2 \tag{42}$$

Note that Λ_0 at Uranus is about 0.204 m/sec^2 and at Neptune is about 0.434 m/sec^2. If at threshold the desired static phase error is ϕ_{cs0} then

$$\phi_{cs} = \frac{\alpha_0}{\alpha} \phi_{cs0} \tag{43}$$

where α_0 is suppression factor at threshold.

V. Performance of the SDA

The performance of the subcarrier demodulation assembly (Fig. 3) can be summarized as follows. The density function for subcarrier phase error can be expressed as (Ref. 1)

$$p(\phi_{sc}) = C \frac{\exp\{\rho_{SDA} \cos\phi_{sc} + \rho_{SDA}\,\phi_{sc}\phi_{ss}\}}{2\pi I_0(\rho_{SDA})}; \quad -\pi \leqslant \phi_{sc} \leqslant \pi \tag{44}$$

where

C is a normalization constant as before,

ϕ_{sc} is the subcarrier phase error, ϕ_{ss} is the static phase error,

ρ_{SDA} is subcarrier loop SNR given by (Ref. 1)

$$\rho_{SDA} = \frac{E_s}{N_0} \cdot \frac{R_s}{B_{LS}} \cdot \frac{1}{\Gamma_{SL}} \left(\frac{2\alpha'}{\pi}\right)^2 \tag{45}$$

The E_s is symbol energy, R_s is symbol rate and $\alpha' = \langle D(t)\hat{D}(t)\rangle$ is a suppression factor due to data aiding. This suppression factor for various symbol transition densities is given by (Ref. 4)

Symbol transition density

0%
$$\alpha' = \mathrm{erf}\left[\sqrt{\frac{2}{3}\left(\frac{E_s}{N_0}\right)_e}\,\right] \tag{46}$$

50%
$$\alpha' = 0.769 \left[\frac{0.887 + 0.2\left(\frac{E_s}{N_0}\right)_e^{1\cdot 2}}{1 + 0.2\left(\frac{E_s}{N_0}\right)_e^{1\cdot 2}}\right] \mathrm{erf}\left(\sqrt{\frac{2}{3}\left(\frac{E_s}{N_0}\right)_e}\right) \tag{47}$$

100%
$$\alpha' = 0.538 \left[\frac{0.687 + 0.28\left(\frac{E_s}{N_0}\right)_e^{1\cdot 2}}{1 + 0.28\left(\frac{E_s}{N_0}\right)_e^{1\cdot 2}}\right] \mathrm{erf}\left(\sqrt{\frac{2}{3}\left(\frac{E_s}{N_0}\right)_e}\right) \tag{48}$$

for filter time constant to symbol time ratio 1/3.

The soft limiter "softness" parameter which depends on the physical parameters of the limiter and the input noise power is given by (Ref. 5)

$$D = \frac{\pi}{4}\nu^2\, SNR \tag{49}$$

The soft limiter factor ν^2 is

$$\nu^2 = \frac{1}{4} \quad \text{for Block III}$$
$$\nu^2 = \frac{1}{16} \quad \text{for Block IV,} \tag{50}$$

and the SNR is given by

$$SNR = \frac{E_s}{N_0}\,\frac{R_s}{B_{IFs}} \tag{51}$$

where B_{IFs} is the bandwidth of the bandpass limiter

$$B_{IFs} = 500 \text{ Hz} \quad \text{for Block III}$$
$$B_{IFs} = 1000 \text{ Hz} \quad \text{for Block IV}. \tag{52}$$

Then Γ_{SL}, the soft limiter performance factor given in Ref. 6 (Eq. 65) can be approximated (Ref. 5) by

$$\Gamma_{SL} = \frac{1+D}{0.862+D} \tag{53a}$$

or by (Ref. 7)

$$\Gamma_{SL} = \frac{1 + 0.345\,\alpha'\,SNR + 50\,(\alpha'\,SNR)^5}{0.862 + 0.690\,\alpha'\,SNR + 50\,(\alpha'\,SNR)^5} \tag{53b}$$

The loop gain of SDA is

$$G' = \alpha'\,\alpha_{SL}\,G'_{MAX} \tag{54}$$

where G'_{MAX} is the maximum gain ($G'_{MAX} = 400$) and α_{SL}, the soft limiter average slope when the carrier tracking loop phase error is neglected, is given by (Ref. 8)

$$\alpha_{SL} = \sqrt{\frac{D}{1+D}}\exp\left[-\frac{R_{SL}}{2(1+D)}\right]\left\{I_0\left[\frac{R_{SL}}{2(1+D)}\right] + I_1\left[\frac{R_{SL}}{2(1+D)}\right]\right\} \tag{55}$$

where (Ref. 8)

$$R_{SL} = \left(\frac{2\alpha'}{\pi}\right)^2 SNR\,(\phi_{ss}^2 + \sigma_{sc}^2)\,. \tag{56}$$

Since R_{SL} is small, α_{SL} can be approximated as

$$\alpha_{SL} \approx \sqrt{\frac{D}{1+D}} \tag{57a}$$

or in Refs. 7 and 9 α_{SL} has been approximated as

$$\alpha_{SL} = \mathrm{erf}\left[\frac{2\alpha'}{\pi}\sqrt{2D}\right]. \tag{57b}$$

B_{LS} is the SDA loop bandwidth given by

$$B_{LS} = B_{LS0}\,\frac{1 + r_0'\,G'/G_0'}{1 + r_0'} \tag{58}$$

where B_{LS0}, r_0', and G_0' are SDA loop parameters at threshold. For the second-order loop with transfer function

$$F'(s) = \frac{1 + \tau_2' S}{1 + \tau_1' S}, \tag{59}$$

as for CTL, the SDA static phase error can be expressed as

$$\phi_{ss} \approx \frac{\omega_{sc}}{c}\left[\frac{\Omega_0 + \Lambda_0 t + \Lambda_0 \tau_1'}{G'}\right]. \tag{60}$$

For a perfect second-order loop, or when $\tau_2 << r'\tau_1$, (60) reduces to

$$\phi_{ss} = \frac{\omega_{sc}}{c}\,\frac{\Lambda_0}{r'}\left(\frac{1 + r'}{4 B_{LS}}\right) \tag{61}$$

If we set the desired static phase error at threshold as ϕ_{ss0} then

$$\phi_{ss} = G_0'\,\phi_{ss0}/G' \tag{62}$$

Finally, for SSC, the loss in data SNR due to carrier tracking loop is (Ref. 8)

$$L_{CTL,i} = \overline{\cos\phi_{ci}}^{\,2} \approx \left(\frac{I_1(\rho_{CTL,i})}{I_0(\rho_{CTL,i})}\cos\phi_{csi}\right)^2, \tag{63}$$

and the loss in data SNR due to subcarrier tracking loop is (Refs. 7, 8)

$$L_{SDA,i} = \left[1 - \left(\frac{2}{\pi}\right)^{1.5} \exp\left\{-\frac{\phi_{ssi}^2}{2\sigma_{sci}^2}\right\} \sigma_{sci} - \frac{2}{\pi}\phi_{ssi}\,\mathrm{erf}\left(\frac{\phi_{ssi}}{\sqrt{2}\,\sigma_{sci}}\right)\right]^2 \tag{64}$$

Then

$$\left(\frac{E_{si}}{N_0}\right)_e = \frac{E_{si}}{N_0} \times L_{SDA,i} \times L_{CTL,i}\,. \tag{65}$$

For BBC the loss in data SNR due to all carrier tracking loops is

$$L_{CTL} = \left[\frac{\sum_{i=1}^{N} \rho_i \frac{I_1(\rho_{CTL,i})}{I_0(\rho_{CTL,i})}\cos\phi_{csi}}{\sum_{i=1}^{N}\rho_i}\right]^2 \tag{66}$$

and loss in data SNR due to the SDA loop is

$$L_{SDA} = \left[1 - \left(\frac{2}{\pi}\right)^{1.5} \exp\left\{-\frac{\phi_{ss}^2}{2\sigma_{sc}^2}\right\} \sigma_{sc} - \frac{2}{\pi}\phi_{ss}\,\mathrm{erf}\left(\frac{\phi_{ss}}{\sqrt{2}\,\sigma_{sc}}\right)\right]^2 \tag{67}$$

Thus

$$\left(\frac{E_s}{N_0}\right)_e = \left(\sum_{i=1}^{N}\frac{E_{bi}}{N_{0i}}\right) \times L_{SDA} \times L_{CTL} \tag{68}$$

VI. Performance of the SSA

The performance of the symbol synchronization assembly (Fig. 4) can be summarized as follows. The probability density function for symbol time error λ can be expressed as

$$p(\lambda) = C\,\frac{\exp\left\{\dfrac{4\pi^2\lambda\lambda_s + \cos 2\pi\lambda}{(2\pi\sigma_\lambda)^2}\right\}}{I_0\left(\dfrac{1}{(2\pi\sigma_\lambda)^2}\right)};\ -\frac{1}{2} < \lambda < \frac{1}{2} \tag{69}$$

where λ_s is the static time shift of the loop and σ_λ^2 can be expressed as (Ref. 2)

$$\sigma_\lambda^2 = \frac{\xi_0 B_{LSS}}{2\left(\frac{E_s}{N_0}\right)_e R_s} \mathscr{L} \tag{70}$$

where ξ_0 is the window size ($\xi_0 = 1/4$), and the SSA loop bandwidth is

$$B_{LSS} = B_{LSSo} \frac{1 + r_0'' \sqrt{\left(\frac{E_s}{N_0}\right)_e \Big/ \left(\frac{E_s}{N_0}\right)_0}}{1 + r_0''} \tag{71}$$

$(E_s/N_0)_e$ is the effective input SNR to the SSA and is given by (65) for SSC and (68) for BBC. $(E_s/N_0)_0$ is the data SNR at the design point and r_0'' is the SSA loop damping parameter at the design point ($r_0'' = 2$). $\mathscr{L}$ is the squaring loss, given by Ref. 2,

$$\mathscr{L} = \frac{h(0)}{k_g^2} \tag{72}$$

where

$$h(0) = 1 + \frac{\xi_0 \left(\frac{E_s}{N_0}\right)_e}{2} - \frac{\xi_0}{2}\left[\frac{1}{\sqrt{\pi}} e^{-\left(\frac{E_s}{N_0}\right)_e} + \sqrt{\left(\frac{E_s}{N_0}\right)_e}\; erf\left(\sqrt{\left(\frac{E_s}{N_0}\right)_e}\right)\right]^2 \tag{73}$$

and

$$K_g = erf\left(\sqrt{\left(\frac{E_s}{N_0}\right)_e}\right) - \frac{\xi_0}{2}\sqrt{\frac{1}{\pi}\left(\frac{E_s}{N_0}\right)_e}\; e^{-\left(\frac{E_s}{N_0}\right)_e} \tag{74}$$

For a perfect second-order loop the static time error

$$\lambda_s = \frac{R_s}{c}\frac{\Lambda_0}{r_{ss}}\left(\frac{1 + r_{ss}}{4 B_{LSS}}\right) \tag{75}$$

where r_{ss} is the loop damping parameter.

The loss in data SNR due to the SSA alone is

$$L_{SSA} = \overline{(1 - 2|\lambda|)}^2$$

$$= \left[1 - 2\sqrt{\frac{2}{\pi}}\, e^{-\frac{\lambda_s^2}{2\sigma_\lambda^2}}\, \sigma_\lambda - 2\lambda_s \operatorname{erf}\left(\frac{\lambda_s}{\sqrt{2}\sigma_\lambda}\right)\right]^2 \tag{76}$$

VII. Computation of Bit Error Rate

In order to compute Eq. (21) or (30), first we examine an approximate result for the large signal-to-noise ratio case, where

$$y \approx 1 \tag{77}$$

Therefore, we can make use of Taylor expansion for Eq. (19) as

$$f\left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}} y^2\right) = f\left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}}\right) + 2(y-1) f^1\left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}}\right) \tag{78}$$

But

$$f^1(x) = \begin{cases} -\alpha_1 f(x) \;; & x > \dfrac{\alpha_0 + \ln 2}{\alpha_1} \triangleq T' \\ 0 \;; & x < \dfrac{\alpha_0 + \ln 2}{\alpha_1} \end{cases} \tag{79}$$

Therefore,

$$P_b = \overline{f\left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}} y^2\right)}$$

$$= \begin{cases} f\left(\sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}}\right)\left[1 + 2\alpha_1 (1 - \bar{y})\right] \;; & \sum_{i=1}^{N} \frac{E_{bi}}{N_0} > T' \\ 0.5 \;; & \sum_{i=1}^{N} \frac{E_{bi}}{N_0} < T' \end{cases} \tag{80}$$

For SSC,

$$\overline{y} = \frac{\sum_{i=1}^{N} \rho_i \left(1 - \frac{2}{\pi}\overline{|\phi_{sci}|}\right)\overline{\cos\phi_{ci}}\left(1 - 2\overline{|\lambda_i|}\right)}{\sum_{i=1}^{N} \rho_i} \tag{81}$$

For BBC,

$$\overline{y} = \frac{\sum_{i=1}^{N} \rho_i \left(1 - \frac{2}{\pi}\overline{|\phi_{sc}|}\right)\overline{\cos\phi_{ci}}\left(1 - 2\overline{|\lambda|}\right)}{\sum_{i=1}^{N} \rho_i} \tag{82}$$

Note

$$\overline{\cos\phi_{ci}} \approx \frac{I_1(\rho_{CTL,i})}{I_0(\rho_{CTL,i})}\cos\phi_{csi} \tag{83}$$

for large SNR

$$\overline{|\phi_{sc}|} \approx \sqrt{\frac{2}{\pi}}\, e^{\frac{-\phi_{ss}^2}{2\sigma_{sc}^2}}\sigma_{sc} + \phi_{ss}\,\mathrm{erf}\left(\frac{\phi_{ss}}{\sqrt{2}\sigma_{sc}}\right) \tag{84}$$

$$\overline{|\lambda|} \approx \sqrt{\frac{2}{\pi}}\, e^{\frac{-\lambda_s^2}{2\sigma_\lambda^2}}\sigma_\lambda + \lambda_s\,\mathrm{erf}\left(\frac{\lambda_s}{\sqrt{2}\sigma_\lambda}\right) \tag{85}$$

VIII. Computation of Bit Error Rate Using Moment Technique

Note that the direct evaluation of Eq. (21) for N antennas needs $(3N)$-tuple integration; for even the simple case $N = 3$ we need 9-tuple integration! Similarly, for evaluation of Eq. (30) we need $(N + 2)$-tuple integration. Using direct method of computation is very hard and consumes a lot of computer time (days and maybe weeks). Therefore, to solve the problem we use the moment technique of Ref. 10. Here we want to find

$$P_b = E\{f(x)\} = E\{g(y)\} \tag{86}$$

where

$$x = \sum_{i=1}^{N} \frac{E_{bi}}{N_{0i}} y^2 \tag{87}$$

and

$$g(y) = f(x). \tag{88}$$

The y is given by Eq. (20) for SSC and by Eq. (29) for BBC, and the expection E is over all random variables-contained in y. Suppose we have $M + 1$ moments of y

$$\mu_k = E\{y^k\}; \quad k = 0, 1, 2, \cdots, M \tag{89}$$

Suppose we could expand $g(y)$ as

$$g(y) \approx \sum_{i=0}^{N} \alpha_i y^i \tag{90}$$

Then

$$E\{g(y)\} = \sum_{i=0}^{N} \alpha_i E\{y^i\} = \sum_{i=0}^{N} \alpha_i \mu_i \tag{91}$$

Unfortunately, expansion of $g(\cdot)$ given by (90) results in an alternating sum of moments with large coefficients α_i; this results in numerical inaccuracies in summing. Thus, we cannot simply expand $g(y)$ and use moments of y. Therefore, we would like to find the smallest number of points $y_1, y_2, \cdots, y_\nu$ and weights $w_1, w_2, \cdots, w_\nu$ so that the approximate discrete probability distribution

$$\hat{p}_r\{y = y_\ell\} = \omega_\ell; \quad \ell = 1, 2, \cdots, \nu \tag{92}$$

satisfies the given moment constraints

$$\mu_k = \hat{E}(y_k) = \sum_{\ell=1}^{\nu} \omega_\ell y_\ell^k; \quad k = 0, 1, 2, \cdots, M \tag{93}$$

More details about the following summary of the moment technique can be found in Ref. 10.

Next define the polynomial

$$C(D) = \prod_{\ell=1}^{\nu}(1 - Dy_\ell) = C_0 + C_1 D + C_2 D^2 + \cdots + C_\nu D^\nu$$

$$C_0 = 1 \tag{94}$$

We can show

$$\mu_n = -\sum_{j=1}^{\nu} C_j \mu_{n-j} \tag{95}$$

This form of the relationship between moments can be interpreted as a linear feedback shift register generating the moments (Fig. 5). Note that the Berlekamp-Massey linear feedback shift register synthesis algorithm can find a smallest length feedback shift register that generates $\mu_0, \mu_1, \cdots, \mu_M$. This enables us to find $C(D)$. Having $C(D)$, we can find $y_1, y_2, \cdots, y_\nu$. Define the polynomial

$$P(D) = \sum_{\ell=1}^{\nu} \omega_\ell \prod_{\substack{j=1\\ j\neq \ell}}^{\nu} (1 - Dy_j) = P_0 + P_1 D + \cdots + P_{\nu-1} D^{\nu-1} \tag{96}$$

Define the moment generating function polynomial as

$$\mu(D) \triangleq \sum_{k=0}^{\infty} \mu_k D^k \tag{97}$$

Then we can show

$$p(D) = \mu(D) C(D) \tag{98}$$

Finally it can be shown that

$$\omega_k = -\frac{y_k P(y_k^{-1})}{C'(y_k^{-1})}; \quad k = 1, 2, \cdots, \nu \tag{99}$$

where

$$C'(D) \triangleq \frac{d}{dD} C(D) \tag{100}$$

A summary of finding $E\{g(y)\}$ using the moment technique is shown in Fig. 6.

Define y for SSC as

$$y \triangleq y_N = \left[\sum_{i=1}^{N} \rho_i \left(1 - \frac{2}{\pi} |\phi_{sci}|\right) \cos\phi_{ci} (1 - 2|\lambda_i|)\right] \Bigg/ \left[\sum_{i=1}^{N} \rho_i\right] \tag{101}$$

Then

$$y_N = y_{N-1} + \rho_N \left(1 - \frac{2}{\pi} |\phi_{SN}|\right) \cos\phi_{CN} (1 - 2|\lambda_N|) \Bigg/ \left(\sum_{i=1}^{N} \rho_i\right) \tag{102}$$

Finally

$$E\{y_N^k\} = \sum_{i=0}^{k} \binom{k}{i} E\{y_{N-1}^i\} \rho_N^{k-i} E\left\{\left(1 - \frac{2}{\pi} |\phi_{SN}|\right)^{k-i}\right\}$$

$$\times E\{\cos^{k-i}\phi_{CN}\} E\{(1 - 2|\lambda_N|)^{k-i}\} \tag{103}$$

Therefore, having certain moments of $[1 - 2/\pi |\phi_{si}|]$, $\cos\phi_{ci}$ and $(1 - 2|\lambda_i|)$, by iteration using Eq. (103) we can find all required moments of y. A faster method for computation of moments of y is to use the so-called semi-invariants method given in Ref. 10. Having all moments of y we can use the Berlekamp-Massey algorithm as discussed before to find required points y_i and the weights w_i. Thus

$$P_b = E\{g(y)\} = \sum_{i=1}^{\nu} \omega_i \, g(y_i) \tag{104}$$

Example: Consider the arraying of three antennas, namely 64 m, 34 m (T/R) and 34 m (R). For a modulation index of 72°, a data symbol rate of 20 kbps (bit rate = 10 kbps), loop bandwidths at threshold of

$$2B_{L0}\ (\text{CTL}) = 30\ \text{Hz}$$

$$B_{LS}(\text{SDA}) = 0.1\ \text{Hz}$$

$$B_{LSS}\ (\text{SSA}) = 0.05\ \text{Hz},$$

and window size = 1/4 in SSA, we have tabulated all data SNR losses and loop SNRs for SSC and BBC. From Table 1, if we include the losses of 0.2 dB from the microwave link and of 0.2 dB due to alignment in the BBC case, then the total loss in the data SNR in SSC will be 0.35 dB less than the data SNR loss in BBC. Finally, the bit error probability curve using the moment technique is given in Figs. (7) and (8) for the above case for SSC and BBC.

IX. Conclusion

In this article, a system performance analysis of symbol stream and baseband combining is given. The performances of the carrier tracking loop, the subcarrier demodulation assembly, and the symbol synchronization assembly have been determined. Numerical results are given for an example of arraying three antennas, namely a 64 m, 34 m (T/R) and 34 m (R) antennas of DSN. Final results show that symbol stream combining outperforms baseband combining by about 0.35 dB.

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Table 1. Comparison of performance of SSC and BBC

$$\text{at} \sum_{i=1}^{3} \frac{E_{bi}}{N_{oi}} = 3.2 \text{ dB}$$

(all losses and loop SNR's in dB)

Type of arraying	Antennas	Carrier tracking loop		SDA		SSA		Microwave loss	Combining loss	Total loss in data SNR
		L_{CTL}	ρ_c	L_{SDA}	ρ_s	L_{SSA}	ρ_{ss}			
SSC	64 m	−0.22	13	−0.044	40	−0.01	44	0	0	−0.43
	34 m (T/R)	−0.56	9.2	−0.16	28	−0.05	32	0	0	
	34 m (R)	−0.46	9.9	−0.12	31	−0.04	35	0	0	
BBC	64 m	−0.22	13	−0.03	43	−0.02	47	−0.2	−0.2	−0.78
	34 m (T/R)	−0.56	9.2							
	34 m (R)	−0.46	9.9							

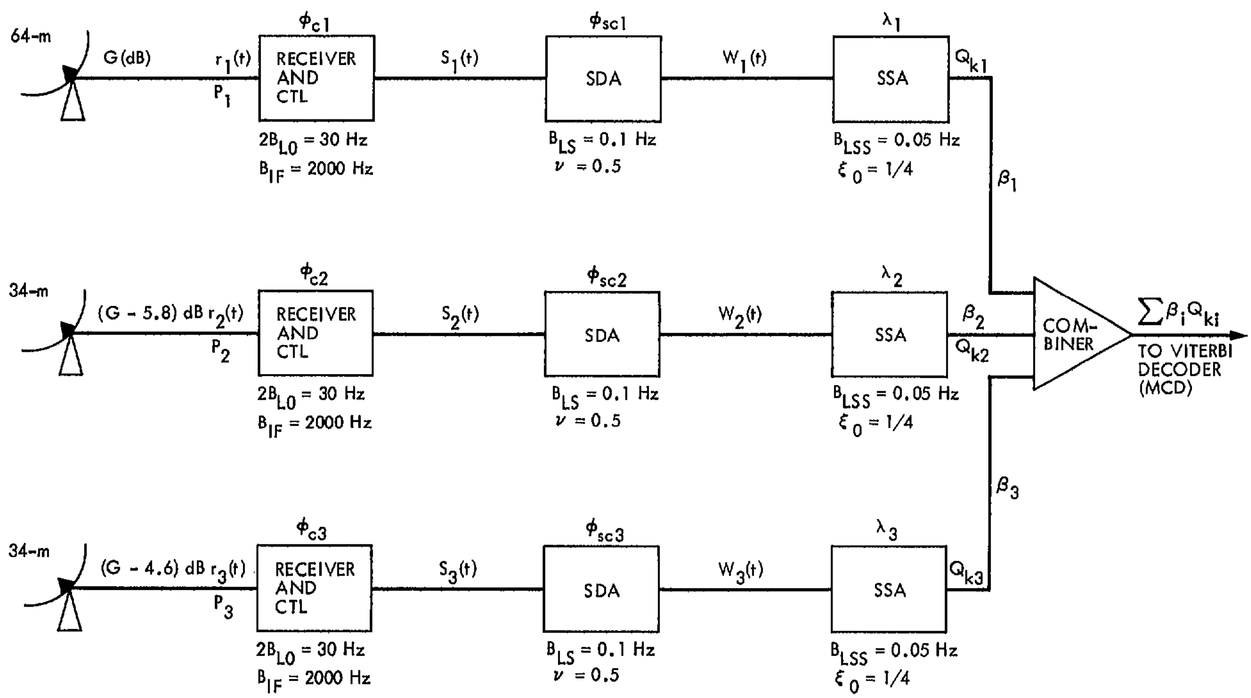

Fig. 1. Block diagram of symbol stream combining system

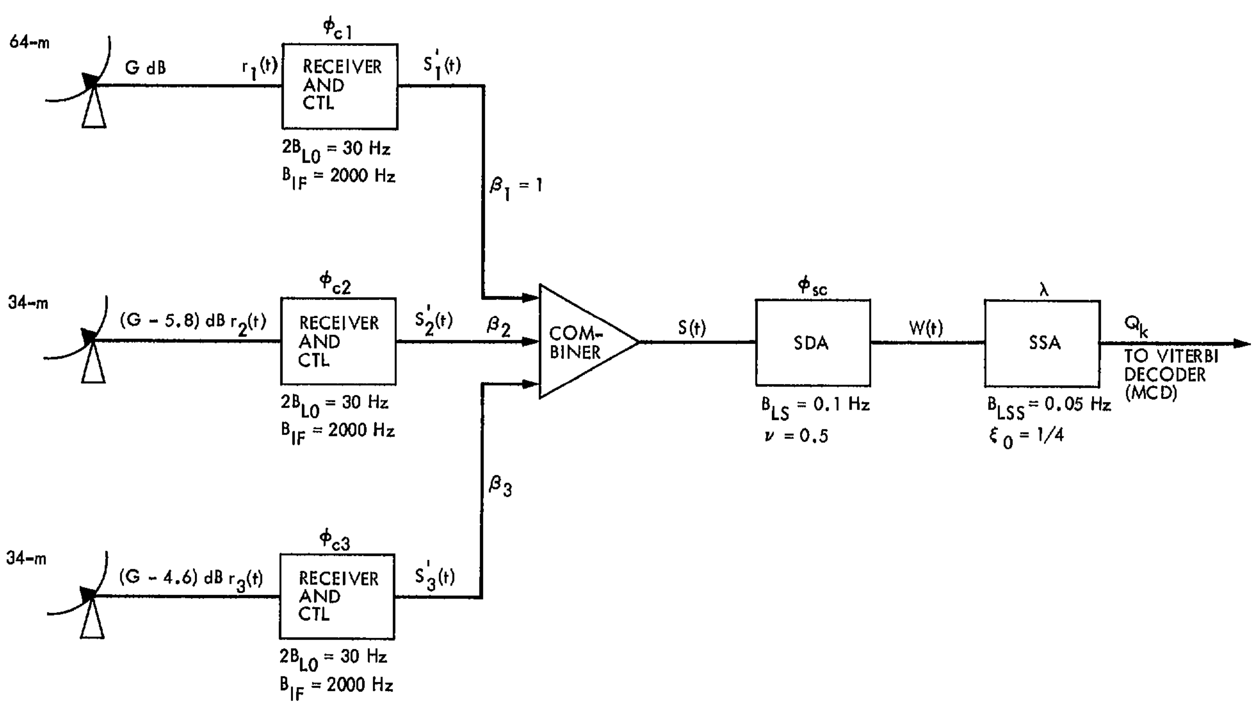

Fig. 2. Block diagram of baseband combining system

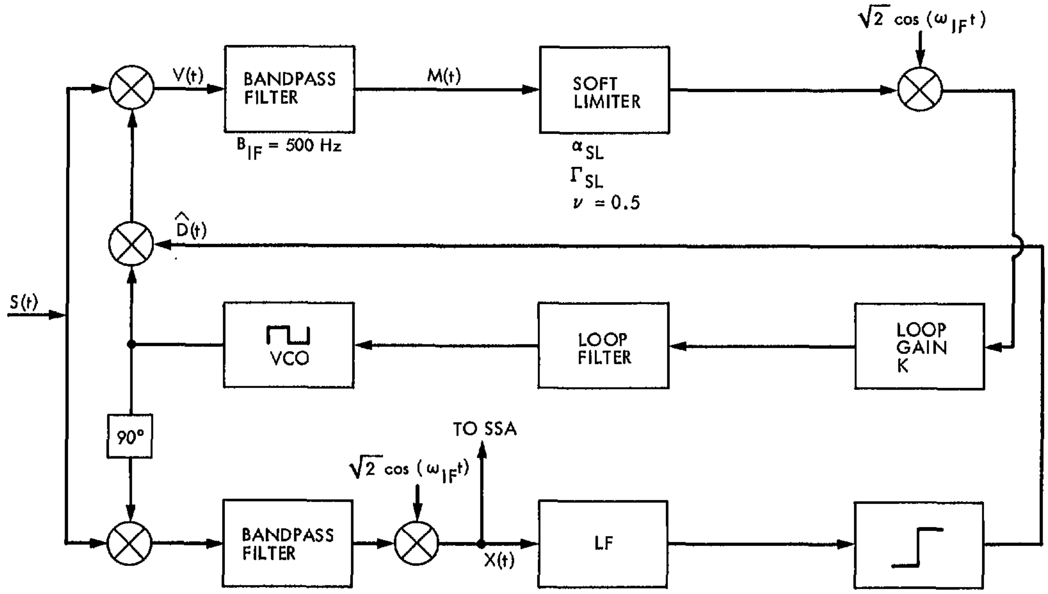

Fig. 3. Block diagram of subcarrier demodulation assembly (SDA)

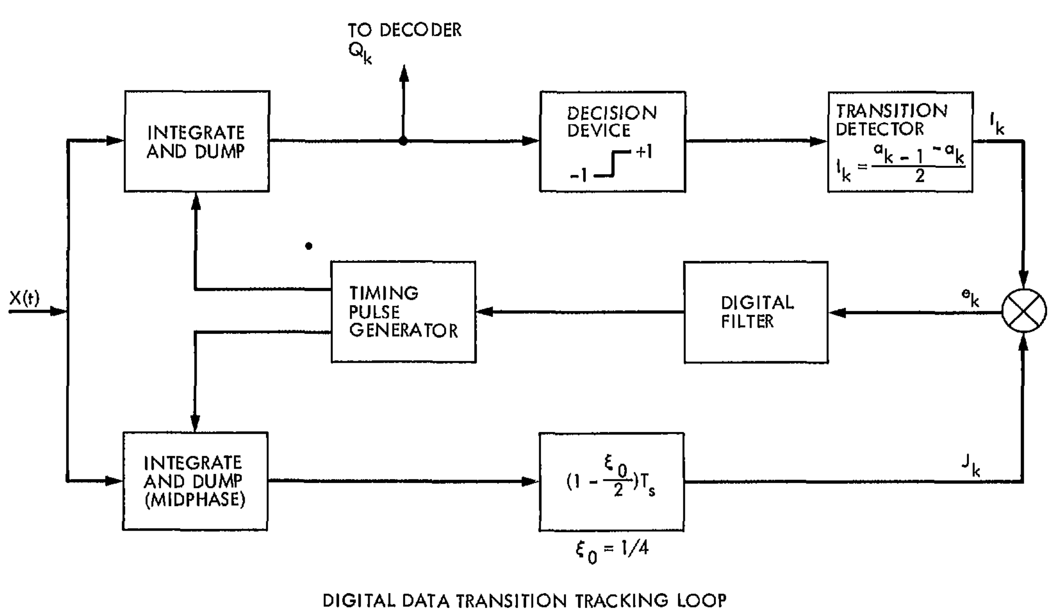

Fig. 4. Block diagram of symbol synchronization assembly (SSA)

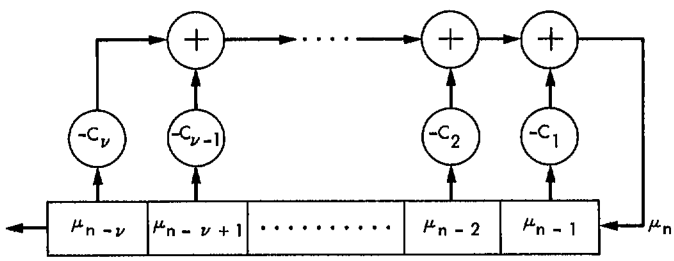

Fig. 5. Moment generating linear feedback shift register

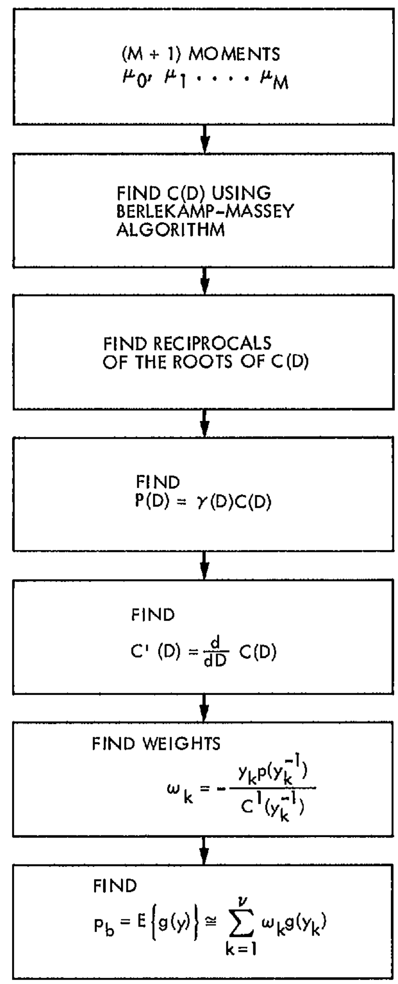

Fig. 6. Flow chart for computation of $E\{g(y)\}$ using moment technique

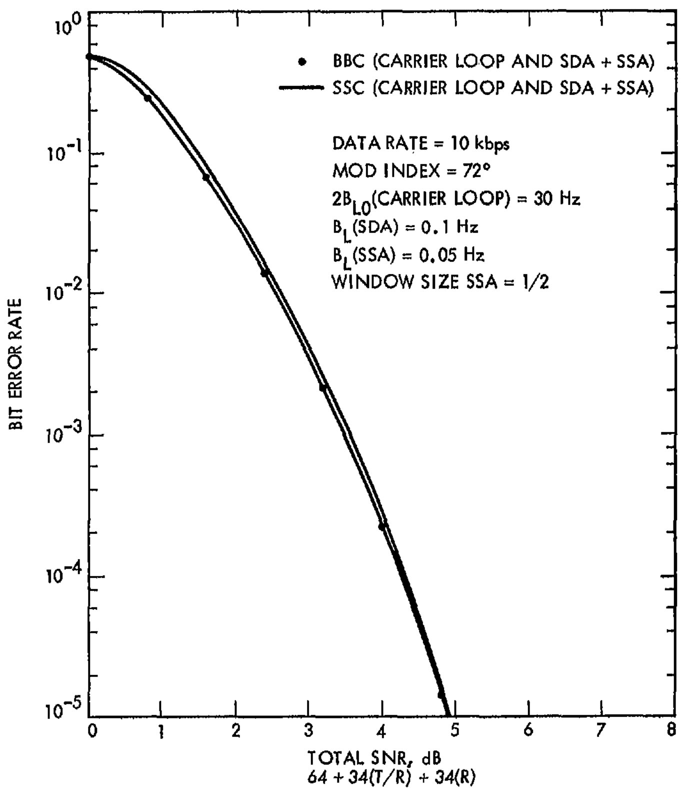

Fig. 7. Bit error rate vs total SNR for symbol stream and baseband combining for window size 1/2 (microwave and combiner losses are not included)

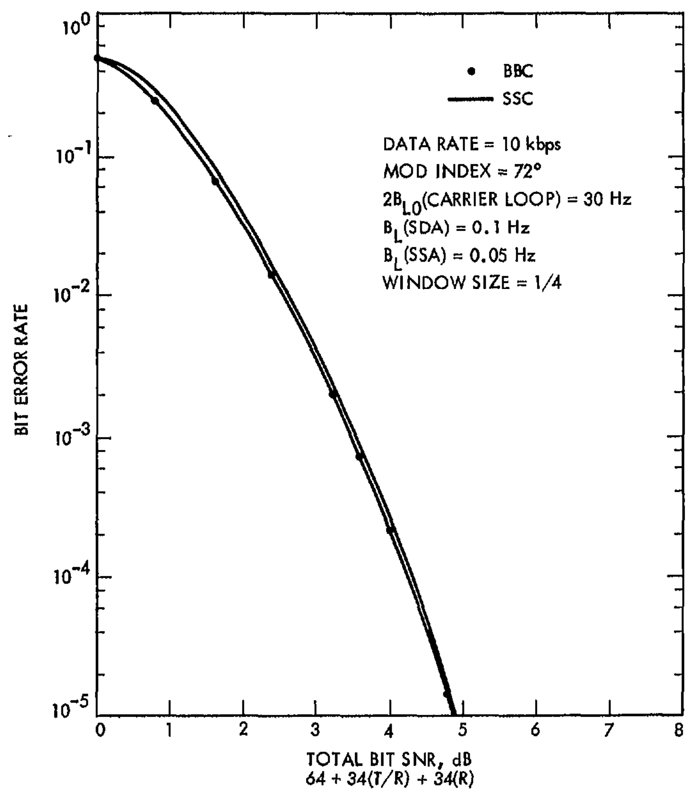

Fig. 8. Bit error rate vs total SNR for symbol stream and baseband combining for window size 1/4 (microwave and combiner losses are not included)