A Sequential Decoding Performance Analysis for International Comet Explorer

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This article analyzes the effect of a noisy reference carrier on the performance of International Comet Explorer (ICE) Sequential decoder. Ideal reference models for frame deletion probability are given for the Telemetry Processing Assembly (TPA) and the Linkabit (LS4815) sequential decoders. Based on these ideal reference models the deletion probability in the presence of noisy reference carrier is computed for both the high and the low data rate cases. A medium data rate performance model is then derived using an interpolation method. The derived medium rate performance model depends on the phase locked loop bandwidth-integration time product. Results are obtained for frame length of 1024 bits, data bit rates of 1024 bps and 512 bps, modulation index of 60°, and threshold loop bandwidths of 10 Hz and 3 Hz. The medium rate model agrees with experimental data to within measurement uncertainty. For the 10 Hz loop bandwidth and modulation index of 60° at frame deletion rate of 10^{-4}, the required total received power to noise ratio is 36 dB.

The analysis given in this article is general and applicable to any sequential decoder, provided that the ideal reference model of the decoder is available.

I. Introduction

Sequential decoding is a useful technique for communicating at low undetected error rates from deep space probes. But a failure mode known as computational overflow or erasure limits the operation of the decoder at very high data rates. The erasure (deletion) of a data frame occurs when the decoder has not finished decoding that frame at the time that it must be output. Increasing the speed of the decoding computation and the size of the decoder buffer provides some improvement in performance, but only linearly. On the other hand, if the buffer size is increased too much, erasures occur in bursts rather than singly. Such bursts decrease the expected improvement. Also if the speed of decoding is increased too much with respect to data rate, then the decoder idle time waiting for incoming data will be increased. Therefore much of the increased capability is wasted.

Based on the received symbol sequence and past decoded bits, the sequential decoder makes local best estimates of the current data sequence. If the received symbols are relatively noiseless, decoding proceeds rapidly with no searching. On the other hand, if the received symbols are noisy, some of the local estimates of current data will be wrong. The decoder must eventually recognize that an error has been made and search systematically backward through the local estimates to correct those in error. The amount of searching depends on the amount of noise in the received symbol sequence.

In this article, the effect of receiver phase-locked loop (PLL) phase jitter on the frame deletion probability P_d of the sequential decoder is modeled for the International Comet Explorer (ICE) link. The approach is first to obtain an ideal-reference (baseline) model for the deletion probability of the sequential decoder. This has been obtained using models and simulation results in Ref. 1. These simulation and measurement results do not completely match our problem. Therefore approximation was used for the ideal-reference models. For the Linkabit (LS4815) sequential decoder, there are only two measurement points available. So the ideal-reference model used for LS4815 is also an approximation. Using these baseline models, first the expected deletion probability is calculated by averaging over the deletion probability conditioned on the PLL phase error, with the assumption that the PLL phase error is varying sufficiently slowly over a frame. The expected deletion probability, under this assumption, serves as an upper bound on the actual deletion probability and will be called the high data rate model for frame deletion probability. Then a lower bound on the deletion probability is obtained, which corresponds to the situation when the PLL phase error is varying rapidly over one-bit time. For this case the effective SNR has been computed and is inserted in the ideal reference model. The result is called the low data rate model. With these extreme models, an interpolation method similar to that in Refs. 2 and 3 has been used to interpolate a more realistic performance approximation between these bounds which is called the medium data rate model.

II. System Model

The ICE telemetry system is shown in Fig. 1. The convolutional code is rate 1/2, the constraint length $K = 24$, and the data frame length 1024 bits. The output symbol sequence of the encoder is Manchester coded before it phase modulates the carrier. At the receiver, the channel noise with two sided power spectral density $N_0/2$ is added to the received signal. Let P_T be the total received power. Then P_T/N_0 represents the total received power to noise spectral density ratio. Throughout the deletion probability calculation, we assume nominal 0.5 dB degradation loss due to signal conditioning, symbol synchronization and Manchester decoding assembly.

III. Ideal Reference Frame Deletion Probability Models

Based on simulation results in Ref. 1, the frame deletion probability has been approximated by a function for Helios frame size of 1152 bits (Refs. 1 and 4) as

$$P_d = \exp\left\{ \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} R^{i-1} (\ell n N)^{j-2} \right\} \qquad (1)$$

where matrix $\mathbf{A}$ with coefficients A_{ij} is given as

$$\mathbf{A} = \begin{bmatrix} 2.397 & 8.824 & -0.9887 \\ -0.5331 & -6.788 & 1.569 \\ 0.02303 & 0.8848 & -0.8543 \end{bmatrix}$$

In Eq. (1), $R = E_b/N_0$ is the bit signal-to-noise ratio and N is the average number of computations per bit. If the computational speed of the decoder is C computations/s, the decoder buffer size is B bits, the frame size is F and the information rate is r_b bps, then

$$N = \frac{C \cdot B}{F \cdot r_b} = \frac{C \cdot B \cdot T_b}{F} \qquad (2)$$

where T_b is the bit time.

In order to find noisy reference deletion probability, we should first find the ideal reference (perfect carrier reference) frame deletion probability $P_d(R)$. Using as a model Eq. (1) and noting that ICE frame length size is close to Helios frame length of 1152, with some approximation the following models for frame deletion for TPA and LS4815 decoders are proposed:

$$P_d(R) = \exp\{\alpha_0 + \alpha_1 R + \alpha_2 R^2\} \qquad (3)$$

where α_0, α_1 and α_2 are given in Table 1 for various cases. To find the noisy reference medium data rate model for deletion probability, we first find two extreme cases, namely the high data rate model and the low data rate model.

IV. The High Data Rate Model

For the high data rate model we assume that the carrier phase error is constant during one frame computation. Therefore the expected frame deletion probability can be defined by

$$P_{dh} = \int_{-\pi}^{\pi} P_d(R \cos^2 \phi \mid \phi) p(\phi) d\phi \qquad (4)$$

where $P_d(R \cos^2 \phi \mid \phi)$ is the conditional deletion probability, for a given phase error ϕ, which can be found by replacing R with $R \cos^2 \phi$ in Eq. (3), the ideal reference deletion probability model. In Eq. (4), the $p(\phi)$ is the PLL phase error probability density, which is given (Ref. 5) approximately by

$$p(\phi) = \frac{\exp\{\rho \cos \phi\}}{2\pi I_0(\rho)} \qquad |\phi| \leqslant \pi \qquad (5)$$

where ρ is the PLL signal-to-noise ratio, given by (Ref. 5)

$$\rho = \frac{P_T \cos^2\theta}{N_0 B_L \Gamma_C} \qquad (6)$$

and $I_0(\cdot)$ is the modified Bessel function of order zero. In Eq. (6) θ is the modulation index; B_L is the PLL bandwidth given by (Ref. 5)

$$B_L = B_{LO} \left(\frac{1 + r_0 \, \alpha/\alpha_0}{1 + r_0} \right) \qquad (7)$$

where B_{LO} is the loop bandwidth at threshold, r_0 is the damping parameter at threshold, and α is the loop suppression factor given by (Ref. 5)

$$\alpha = \sqrt{\frac{0.7854 \, \rho_{in} + 0.4768 \, \rho_{in}^2}{1 + 1.024 \, \rho_{in} + 0.4768 \, \rho_{in}^2}} \qquad (8)$$

and ρ_{in} is the input signal-to-noise ratio to the bandpass limiter and an IF filter having bandwidth B_{IF}. Defining the Carrier Margin (CM) by

$$CM = \frac{P_T \cos^2 \phi}{N_0 \, 2 \, (B_{LO})} \qquad (9)$$

then ρ_{in} can be expressed as

$$\rho_{in} = CM \times \rho_0 \qquad (10)$$

with

$$\rho_0 \overset{\Delta}{=} \frac{2B_{LO}}{B_{IF}} \qquad (11)$$

In Eq. (7) α_0 is loop suppression factor evaluated by Eq. (8) with $\rho_{in} = \rho_0$. Finally Γ_C in Eq. (6) is the limiter performance factor given by (Ref. 5)

$$\Gamma_C = \frac{1 + \rho_{in}}{0.862 + \rho_{in}} \qquad (12)$$

V. The Low Data Rate Model

For the low data rate model, we assume that the carrier phase error is varying rapidly during the bit time. This is true if loop bandwidth-bit time product is much larger than 1, i.e.,

$$B_L \, T_b \gg 1 \qquad (13)$$

To compute degraded deletion probability for this case we should replace the ideal reference bit signal-to-noise ratio R in Eq. (3) by the effective bit signal-to-noise ratio $R \, \bar{x}^2$, where $\bar{x}$ is given by

$$\bar{x} = E \, \{\cos \phi\} = \frac{I_1(\rho)}{I_0(\rho)} \qquad (14)$$

In Eq. (14), $I_1 \, (\cdot)$ is the modified Bessel function of first order. Then we can get the low data rate model for frame deletion probability as

$$P_{d\ell} = P_d \, (R \, \bar{x}^2) \qquad (15)$$

where $P_d \, (\cdot)$ is given by Eq. (3).

VI. The Medium Data Rate Model

In sequential decoding, when the data rates are roughly between $2B_{LO}$ and $2B_{LO}F$ bps, they should be considered as medium data rates. Therefore the performance of the sequential decoder under the noisy reference for the medium data rates should lie somewhere between the high data rate and the low data rate performance models. In order to have an accurate medium data rate model, we should predict the effective memory duration or the effective integration time for the decoder to make a decision on each decoded bit. Let T_m denote the average integration time (time required on average to test different braches through the tree diagram). It is true that

$$T_b \leqslant T_m \leqslant F \, T_b \qquad (16)$$

But if a long search occurs, the search pattern is more likely to look like a full tree search. In Ref. 1 the effective integration time T_m has been found to be well approximated by

$$T_m = 2T_b \left(1 - \frac{1}{N} \left(\log_2 \left(1 + \frac{N}{2} \right) \right) \right) \qquad (17)$$

The following computation has been used in Ref. 1 to derive Eq. (17). If ϕ_n denotes the number of tests required in a full binary tree with branch depth of n, then ϕ_n can be found from recursion

$$\phi_n = 1 + 2 \, \phi_{n-1} \qquad (18)$$

The solution to Eq. (18) is

$$\phi_n = 2^n - 1 \tag{19}$$

Similarly if we consider both forward and backward moves contributing to a search of N steps then the branch depth n is the solution to the equation

$$\phi_n = \frac{N}{2} \tag{20}$$

or

$$n = \log_2 \left(1 + \frac{N}{2}\right) \tag{21}$$

On the other hand the accumulated length of tested branches θ_n can be found from recursion

$$\theta_n = n + 2\theta_{n-1} \tag{22}$$

The solution to Eq. (22) is

$$\theta_n = 2^{n+1} - n - 2 \tag{23}$$

Then

$$T_m = \frac{\theta_n}{\phi_n} T_b \tag{24}$$

where n satisfies Eq. (20) or (21). Using Eqs. (19), (21) and (23) in Eq. (24) results in Eq. (17). For large N from Eq. (17)

$$T_m \simeq 2 T_b \tag{25}$$

As follows, we represent the medium data rate deletion probability model as an interpolation between the high and the low data rate models. We will find the interpolation parameter in terms of loop bandwidth-integration time product $B_L \, T_m$. This approach is similar to a technique used for uncoded telemetry and block codes (Ref. 2). Let's define the conditional effective bit signal-to-noise ratio as

$$R_{\text{eff}} = R \, x^2 \tag{26}$$

where

$$x \overset{\Delta}{=} \frac{1}{T_m} \int_0^{T_m} \cos \phi(t) \, dt \tag{27}$$

Note that for $B_L \, T_m \gg 1$ the phase error $\phi(t)$ changes rapidly over a bit time; thus

$$x \xrightarrow[B_L \, T_m \to \infty]{} \bar{x} = E \{\cos \phi\} \tag{28}$$

where we have assumed $\phi(t)$ is a stationary ergodic process. When $B_L \, T_m \ll 1$, $\phi(t)$ remains essentially constant over the integration time T_m and thus

$$x \xrightarrow[B_L \, T_m \to 0]{} \cos \phi \tag{29}$$

Clearly these two extreme cases result in the low data and the high data rate models for deletion probabilities, respectively. Now for anything between these two extremes we can write

$$P_{dm} = \int P_d \left(Rx^2 \mid x\right) p(x) \, dx \tag{30}$$

where x is defined by Eq. (27) and $p(x)$ is the probability density function of the random variable x. The approximate probability density function of x is given in Ref. 1, which enables us to evaluate P_{dm} from Eq. (30). But rather than doing so, we prefer to find the high and low data rate models by using more accurate phase error distributions and then finding the interpolation parameter approximately. Using the ideal reference frame deletion probability models in Eq. (3), we have

$$P_d \left(Rx^2 \mid x\right) = \exp \{\alpha_0 + \alpha_1 \, Rx^2 + \alpha_2 \, R^2 \, x^4\} \tag{31}$$

Using the Taylor's series expansion of Eq. (31) around $\bar{x} = E \{\cos \phi\}$, we obtain

$$P_d \left(Rx^2 \mid x\right) = P_d \left(R\bar{x}^2\right) \left(1 + \beta_1 \, (x - \bar{x}) + \beta_2 \, (x - \bar{x})^2 + \dots\right) \tag{32}$$

where

$$\beta_1 = 2 \alpha_1 \, R\bar{x} + 4 \alpha_2 \, R^2 \, \bar{x}^3 \tag{33}$$

and

$$\beta_2 = \alpha_1 \, R + \alpha_1 \, R\bar{x} + 6 \alpha_2 \, R^2 \, \bar{x}^2 + 2 \alpha_2 \, R^2 \, \bar{x}^3 \tag{34}$$

Noting that $P_{d\ell} = P_d(R\overline{x}^2)$ and using Eq. (32) in Eq. (30) we get

$$P_{dm} = P_{d\ell} (1 + \beta_2 \, \sigma_x^2 + \ldots) \qquad (35)$$

Also if we use Eq. (32) in Eq. (30) with assumption that $B_L \, T_m \ll 1$, then we get

$$P_{dh} = P_{d\ell} (1 + \beta_2 \, \sigma_{\cos\phi}^2 + \ldots) \qquad (36)$$

From Eqs. (35) and (36), we conclude that if σ_x^2 and $\sigma_{\cos\phi}^2$ are small and if other central moment terms are ignored, we obtain the required interpolation formula as

$$P_{dm} = \left(1 - \frac{\sigma_x^2}{\sigma_{\cos\phi}^2}\right) P_{d\ell} + \frac{\sigma_x^2}{\sigma_{\cos\phi}^2} \, P_{dh} \qquad (37)$$

At this point we model $\phi(t)$ as a Gaussian process having the same variance and bandwidth as the actual $\phi(t)$ process and with an autocorrelation function derived from a first-order PLL (Ref. 4):

$$R_\phi(\tau) = \sigma_\phi^2 \exp\{-4B_L \, |\tau|\} \qquad (38)$$

Hence,

$$\sigma_x^2 = E(x - \overline{x})^2$$

$$= \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} R_{\cos\phi - \overline{x}}(t_1 - t_2) \, dt_1 \, dt_2 \qquad (39)$$

where

$$R_{\cos\phi - \overline{x}}(t_1 - t_2) = E \, \{(\cos \phi(t_1) - \overline{x})(\cos \phi(t_2) - \overline{x})\} \qquad (40)$$

Using the approximation

$$\cos \phi(t) \approx 1 - \phi^2(t)/2 \qquad (41)$$

we get

$$R_{\cos\phi - \overline{x}}(t_1 - t_2) \approx \frac{[E \, \{\phi^2(t_1) \, \phi^2(t_2)\} - \sigma_\phi^4]}{4} \qquad (42)$$

Since we modeled $\phi(t)$ as a Gaussian process with the auto-correlation given in Eq. (38), we can easily find the joint density function of $\phi(t_1)$ and $\phi(t_2)$. Having this joint density function, we can derive the following:

$$E \, \{\phi^2 \, (t_1) \, \phi^2 \, (t_2)\} = \sigma_\phi^4 + 2R_\phi^2 \, (t_1 - t_2) \qquad (43)$$

and

$$R_{\cos\phi - \overline{x}}(t_1 - t_2) \approx \frac{1}{2} \, R_\phi^2 \, (t_1 - t_2) \qquad (44)$$

Note that

$$\sigma_{\cos\phi}^2 = R_{\cos\phi - \overline{x}}(0) \approx \frac{1}{2} R_\phi^2 \, (0) \qquad (45)$$

Using Eq. (44) in Eq. (39) with Eq. (45) we get

$$\frac{\sigma_x^2}{\sigma_{\cos\phi}^2} \approx \frac{1}{T_m^2} \int_0^{T_m} \int_0^{T_m} \exp \, \{-8B_L \, | \, t_1 - t_2 \, |\} \, dt_1 \, dt_2$$

$$= \frac{1}{4 \, B_L \, T_M} - \frac{1}{32 \, B_L^2 \, T_m^2} \, (1 - \exp \, (-8 \, B_L T_m)) \qquad (46)$$

Substituting Eq. (46) in Eq. (37), we have the medium data rate frame deletion probability P_{dm}.

VII. Conclusion and Numerical Results for ICE

In this article, we have obtained models for ideal reference frame deletion probabilities, based on results in Ref. 1, for TPA and LS4815 sequential decoders. Then we have derived the high and low data rate models for noisy reference frame deletion probabilities. Finally, using an interpolation method, we drived the medium data rate model for noisy reference frame deletion probability. The results are shown in Figs. 3 through 8. In Fig. 2, the PLL bandwidth vs P_T/N_0 is depicted. In Figs. 3 through 6 P_d vs P_T/N_0 for various models is shown for two-sided threshold loop bandwidth of 10 Hz and 3 Hz, and for bit rates 1024 and 512 bps, for TPA decoder. In Figs. 7 and 8, the results are given for the LS4815 sequential decoder with 1024 bps. These theoretical models are in good agreement with measurement results (J. W. Layland, private communication).

Acknowledgment

References

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Table 1. Model coefficients for frame-deletion rate

	C, Computational speed, comp/sec	B, Buffer size, bits	r_b, Bit rate, bps	α_0	α_1	α_2
TPA	92,500	3 × 1024	1024	3.71	1.9	−3.89
	92,500	3 × 1024	512	2.98	3	−4.49
LS4815	10^6	4 × 1024	1024	0.936	6.12	−6.177
	10^6	4 × 1024	512	0.23	7.2	−6.77
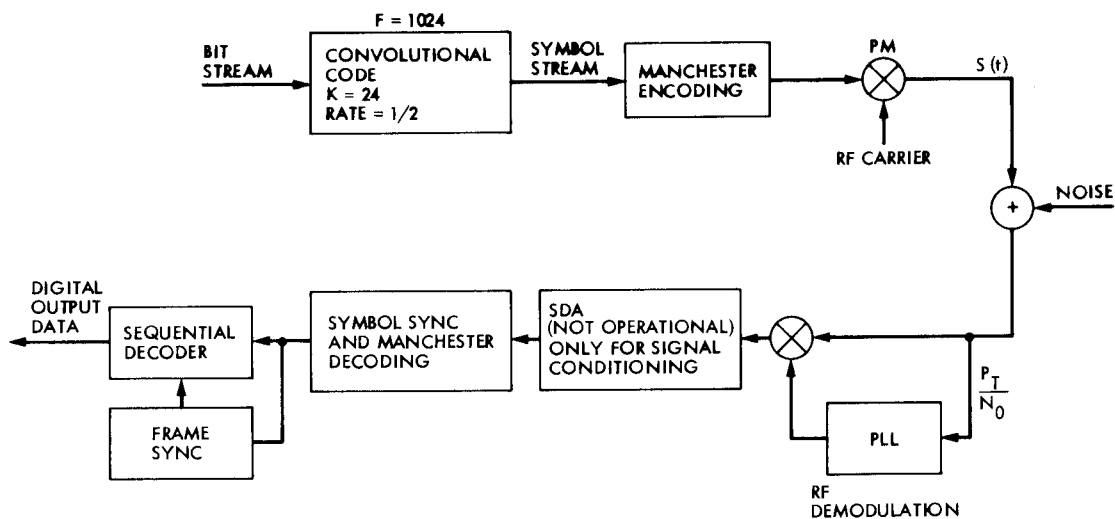
Fig. 1. Telemetry link configuration
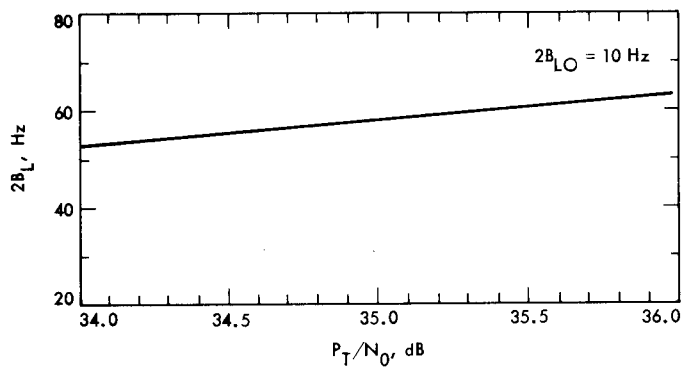
Fig. 2. PLL bandwidth vs P_T/N_0 for 2 B_{L0} = 10 Hz
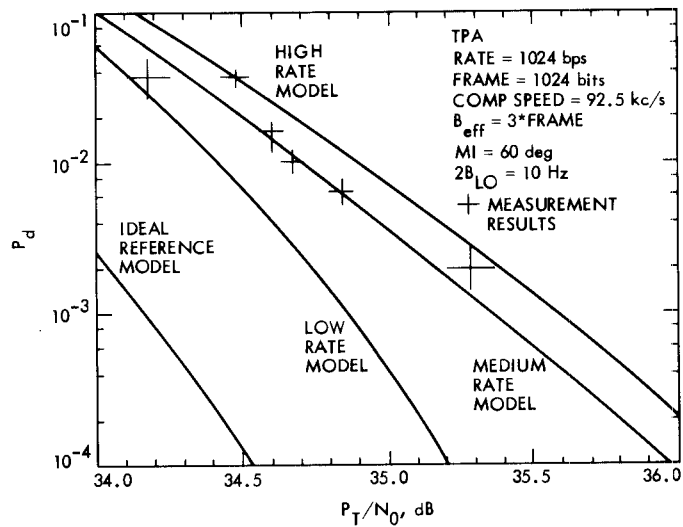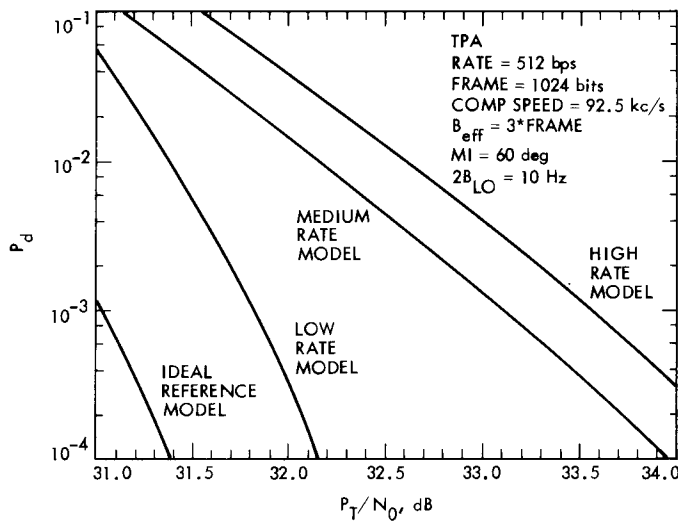

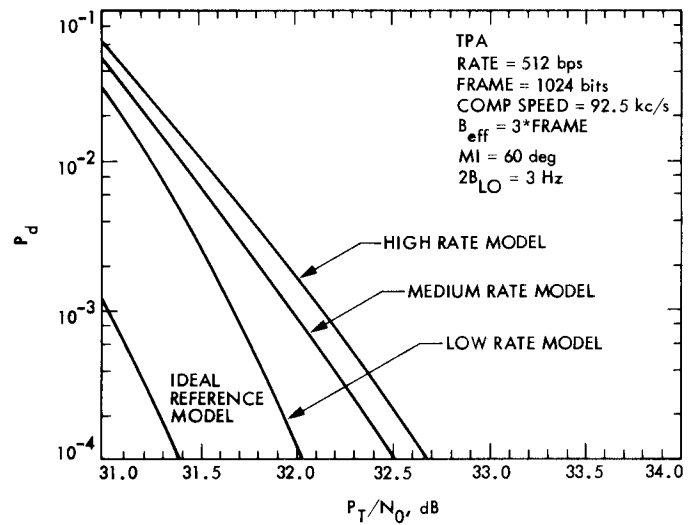Fig. 3. Frame deletion probability models vs P_T/N_0 for TPA, bit rate = 1024 bps and 2 B_{L0} = 10 Hz
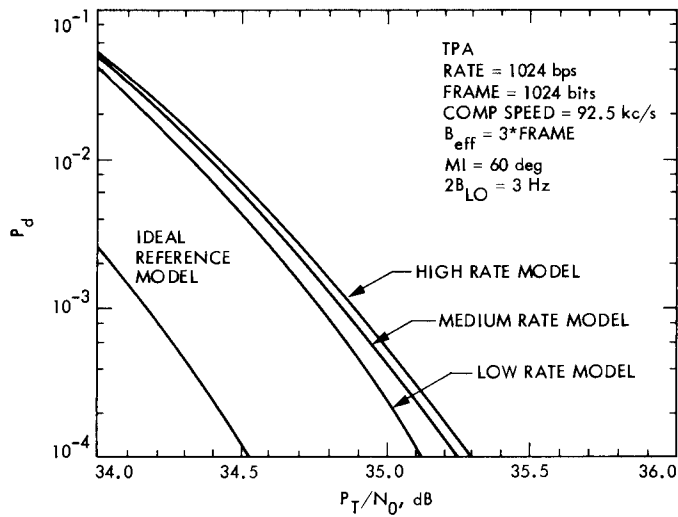
Fig. 4. Frame deletion probability models vs P_T/N_0 for TPA, bit rate = 1024 bps and 2 B_{L0} = 3 Hz

Fig. 5. Frame deletion probability models vs P_T/N_0 for TPA, bit rate = 512 bps and 2 B_{L0} = 10 Hz

Fig. 6. Frame deletion probability models vs P_T/N_0 for TPA, bit rate = 512 bps and 2 B_{L0} = 3 Hz
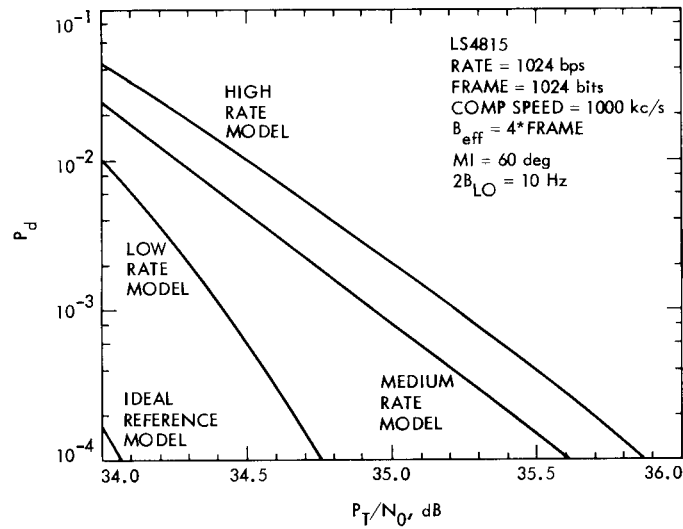
Fig. 7. Frame deletion probability models vs P_T/N_0 for LS 4815, bit rate = 1024 bps and 2 B_{LO} = 10 Hz
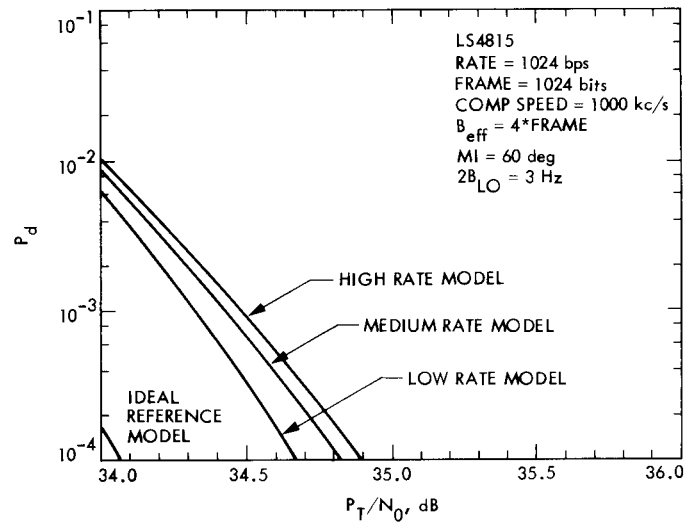

Fig. 8. Frame deletion probability models vs P_T/N_0 for LS 4815, bit rate = 1024 bps and 2 B_{LO} = 3 Hz