A Note on the Wideband Gaussian Broadcast Channel

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It is well known that for the Gaussian broadcast channel, timeshared coding is not as efficient as more sophisticated "broadcast" coding strategies. However, in this article we will show that the relative advantage of broadcast coding over timeshared coding is small if the signal-to-noise ratios of both receivers are small. One surprising consequence of this is that for the wideband Gaussian broadcast channel, which we shall define, broadcast coding offers no advantage over timeshared coding at all, and so timeshared coding is optimal.

I. Introduction

T. M. Cover (Ref. 1) introduced the "broadcast channel" with one transmitter and two (or more) different receivers. Following Ref. 4, we ask the following question about a broadcast channel: certain common information is to be communicated simultaneously to both receivers. How much additional information can be communicated to the better receiver at the same time?

For channels like Gaussian channels, where one receiver is just a degraded version of the other, one obvious approach is timeshared coding: devote a fixed fraction of the total transmission time to sending the common information, coded for the weaker channel. This information will be comprehensible to both receivers. During the remaining time, transmit additional information coded for the stronger receiver. This information will not be comprehensible to the weaker receiver.

But in Ref. 1, Cover introduced a technique called broadcast coding, and showed that, in general, broadcast coding achieves greater, often much greater, data rates than time sharing. Later El Gamal and Cover (Ref. 2) found that broadcast coding cannot be further improved upon.

In this article we will discuss the Gaussian broadcast channel. For this channel, we will show that the relative advantage of broadcast coding over timeshared coding is small if the signal-to-noise ratios of both receivers are small. One surprising consequence of this is that for the wideband Gaussian broadcast channel, which we shall define, broadcast coding offers no advantage over timeshared coding at all, and so timeshared coding is optimal.

II. The Gaussian Channel: A Review

A Guassian channel is a discrete-time memoryless channel model whose input X and output Y are related by $Y = X + Z$, where Z is a mean zero Gaussian random variable independent from X. If the input is constrained by $E(X^2) \leqslant S$, and if the variance of Z is denoted by σ^2, it is well known that the channel capacity depends only on the ratio $x = S/\sigma^2$, which is called the signal-to-noise ratio, and is given by the formula

$$C(x) = \frac{1}{2}\log(1+x) \tag{1}$$

In Eq. (1), $C(x)$ represents the maximum possible amount of information (measured in bits, nats, etc., depending on the base of the logarithm) which can be reliably transmitted per channel use; in the usual physical sense, $C(x)$ is dimensionless.

Equation (1) can be used to derive the following formula for the capacity of a continuous-time, band- and power-limited Guassian channel model:

$$C = B\log\left(1+\frac{P}{N_0 B}\right) \tag{2}$$

where B is the channel bandwidth in Hertz, P is the average transmitter power in Watts, and N_0 is the noise spectral density in Watts per Hertz. The transition from Eq. (1) to Eq. (2) is explained in Ref. 3 (Chapter 4), for example. In Eq. (2), C represents the maximum possible information which can be reliably transmitted per unit of time; the physical dimensions of C are $\sec^{-1}$.

If in Eq. (2) we assume natural logarithms and pass to the limit as the bandwidth B approaches infinity, we obtain

$$C = \frac{P}{N_0} \tag{3}$$

which is the well-known formula for the capacity of the infinite bandwidth white Gaussian channel. The units in Eq. (3) are nats per second.

III. The Gaussian Broadcast Channel

In Ref. 1, Cover introduced a discrete-time memoryless channel model with one transmitter and two receivers, which he called a Gaussian broadcast channel. This channel has one input X, and two outputs Y_1 and Y_2, related by

$$Y_1 = X + Z_1$$

$$Y_2 = X + Z_2$$

where now Z_1 and Z_2 are mean zero Gaussian random variables, and X, Z_1, and Z_2, are independent. Let us denote by σ_1^2 and σ_2^2 the variances of Z_1 and Z_2, respectively, and assume that $\sigma_1^2 \leqslant \sigma_2^2$, so that Y_1 is received more reliably than Y_2. If the channel input X is constrained as in Section I by $E(X^2) \leqslant S$, then separately channels 1 and 2 have capacity $C(x_1)$ and $C(x_2)$, respectively, where $x_1 = S/\sigma_1^2$ and $x_2 = S/\sigma_2^2$.

In Ref. 4 a Gaussian broadcast channel was used to model deep-space communications in the presence of weather uncertainties; the high signal-to-noise ratio corresponds to good weather, and the low signal-to-noise ratio, to bad weather. The problem posed there was the following. Suppose the weather on earth is unknown to a distant spacecraft, and that data must be sent to earth so that even in bad weather, certain minimal but critical information will get through; but if the weather is good, additional information will be received.

Motivated by this point of view, we state the fundamental question about broadcast channels in the following somewhat unusual way. Suppose we wish to send certain information, called the common information, simultaneously to both receivers. If we do this, how much extra information, called bonus information, can we send to the better receiver at the same time?

One approach to this problem is the timesharing approach, in which the transmitter devotes a fixed fraction $1-\rho$ (where $0 < \rho < 1$) of the total transmission time to sending the common information. During this time the information is coded for the weaker receiver. This information will also be comprehensible to the stronger receiver. By Eq. (1), during this common time, information can be transmitted at a maximum rate of $C(x_2)$. During the remaining fraction ρ of the transmission time, the transmitter sends bonus information to the stronger channel, at the rate $C(x_1)$. This will not be comprehensible to the weaker receiver.

It follows that for the timesharing strategy, the data rates will be

$$\begin{aligned} \text{Common Rate} &= (1-\rho)\,C(x_2) \\ \text{Bonus Rate} &= \rho C(x_1) \end{aligned} \tag{4}$$

and the parameter ρ can be selected arbitrarily by the transmitter.

Cover showed, however, that it is possible to do better than timesharing. Using a technique called broadcast coding, he showed that for any choice of the parameter $\alpha, 0 < \alpha < 1$, the following rates are achieveable:

$$\begin{aligned} \text{Common Rate} &= C(x_2) - C(\alpha x_2) \\ \text{Bonus Rate} &= C(\alpha x_1) \end{aligned} \tag{5}$$

(Actually Cover gave the common rate in the form

$$C\{[(1-\alpha)x_2]/(1+\alpha x_2)\},$$

but it is an easy exercise to show that this is the same as we have given in Eq. [5].) Later El Gamal and Cover (Ref. 2) showed that in fact no improvement over Eq. (5) is possible, so that the region of the first quadrant bounded by the curve given parametrically by Eq. (5) is now called the capacity region of the Gaussian Broadcast Channel (see Fig. 1).

Motivated by the discussion in Section I, let us pass from the discrete-time Gaussian broadcast channel to the continuous-time band- and power-limited Gaussian broadcast channel. The resulting expressions are for timesharing:

$$\begin{aligned} \text{Common Rate} &= (1-\rho) B \log \left(1 + \frac{P}{N_2 B}\right) \\ \text{Bonus Rate} &= \rho B \log \left(1 + \frac{P}{N_1 B}\right) \end{aligned} \tag{6}$$

and for broadcast coding:

$$\begin{aligned} \text{Common Rate} &= B \log \left(1 + \frac{P}{N_2 B}\right) - B \log \left(1 + \alpha \frac{P}{N_2 B}\right) \\ \text{Bonus Rate} &= B \log \left(1 + \alpha \frac{P}{N_1 B}\right) \end{aligned} \tag{7}$$

where P is the transmitter power, B is the transmission bandwidth, and N_1, N_2, are the noise spectral densities for the two receivers. In Eqs. (6) and (7) the units are nats per second.

To investigate wideband Gaussian broadcast channels, we pass as before to the limit as $B \to \infty$. The results follow easily from Eqs. (6) and (7) for wideband timesharing:

$$\begin{aligned} \text{Common Rate} &= (1-\rho) \frac{P}{N_2} \\ \text{Bonus Rate} &= \rho \frac{P}{N_1} \end{aligned} \tag{8}$$

and for wideband broadcast coding:

$$\begin{aligned} \text{Common Rate} &= (1-\alpha) \frac{P}{N_2} \\ \text{Bonus Rate} &= \alpha \frac{P}{N_1} \end{aligned} \tag{9}$$

We thus reach the surprising conclusion that for wideband Gaussian broadcast channels, broadcast coding offers no advantage over timesharing. (Actually, this was mentioned but not further investigated in Ref. 4.) We investigate this interesting phenomenon more closely in the next section.

IV. A More Detailed Analysis

In this section we will see that the reason wideband broadcast coding offers no advantage over wideband timesharing is that, for a given common rate, the bonus rates in Eqs. (4) and (5) are nearly equal, when the "good" SNR x_1 is small. More precisely, we have the following:

Theorem: If α and ρ are chosen so that the common rates in Eqs. (4) and (5) are equal, then

$$\begin{aligned} \frac{\text{Broadcast bonus rate (BBR)}}{\text{Timesharing bonus rate (TBR)}} &= \frac{C(\alpha x_1)}{\rho C(x_1)} \\ &\leqslant \frac{x_1}{C(x_1)} \cdot \frac{C(x_2)}{x_2} \\ &= \frac{x_1}{x_2} \cdot \frac{\log(1+x_2)}{\log(1+x_1)} \end{aligned}$$

Corollary 1: Since $\log(1+x_2) \leqslant x_2$, we also have

$$\frac{\text{BBR}}{\text{TBR}} \leqslant \frac{x_1}{\log(1+x_1)}$$

independent of x_2. Thus also

$$\lim_{x_1 \to 0} \frac{\text{BBR}}{\text{TBR}} = 1$$

again independent of x_2.

Corollary 2: For the continuous time channel, the corresponding result is

$$\frac{\mathrm{BBR}}{\mathrm{TBR}} \leqslant \frac{N_2}{N_1} \cdot \frac{\log(1+P/N_2 B)}{\log(1+P/N_1 B)}$$

$$< \frac{P/N_1 B}{\log(1+P/N_1 B)}$$

$$\to 1 \text{ as } B \to \infty$$

Proof of Theorem: For the two common rates to be equal, we have, from Eqs. (4) and (5), that

$$C(\alpha x_2) = \rho C(x_2) \tag{10}$$

On the other hand, the ratio of the bonus rates is

$$\frac{C(\alpha x_1)}{\rho C(x_1)} \tag{11}$$

Combining Eqs. (10) and (11), we see that, for a fixed common rate, the ratio of the bonus rates is

$$\frac{C(\alpha x_1)}{C(\alpha x_2)} \cdot \frac{C(x_2)}{C(x_1)}$$

The desired result now follows from the fact that the function $C(\alpha x_1)/C(\alpha x_2)$ is a decreasing function of α, and approaches x_1/x_2 as $\alpha \to 0$. ■

We conclude with a brief discussion of the shape of the broadcast capacity regions as a function of x_1 and x_2. It is useful to normalize the parametric curves described by Eq. (5) by dividing the common rate by its maximum value $C(x_2)$, and the bonus rate by its maximum value $C(x_1)$:

$$\begin{aligned} &\text{Normalized Common Rate (NCR)} = 1 - \frac{C(\alpha x_2)}{C(x_2)} \\ &\text{Normalized Bonus Rate (NBR)} = \frac{C(\alpha x_1)}{C(x_1)} \end{aligned} \tag{12}$$

For a given value of x_1, the parametric curves described by Eq. (12) vary monotonically outward from $x_2 = x_1$, in which case they reduce to

$$\mathrm{NCR} = 1 - \frac{C(\alpha x_2)}{C(x_2)}$$

$$\mathrm{NBR} = \frac{C(\alpha x_1)}{C(x_1)}$$

which is just the timesharing straight line, to $x_2 = 0^+$, in which case they reduce to

$$\mathrm{NCR} = 1 - \alpha$$

$$\mathrm{NBR} = \frac{C(\alpha x_1)}{C(x_1)}$$

Thus for a given good SNR x_1, broadcast coding offers the largest relative advantage over timesharing when $x_2 \to 0$, and the smallest relative advantage (none at all) when $x_2 = x_1$. Of course, as we have seen, when x_1 is small, even the largest relative advantage is quite small. In Fig. 2, we have graphed the outer ($x_2 = 0$) and inner ($x_2 = x_1$) envelopes for several values of x_1.

References

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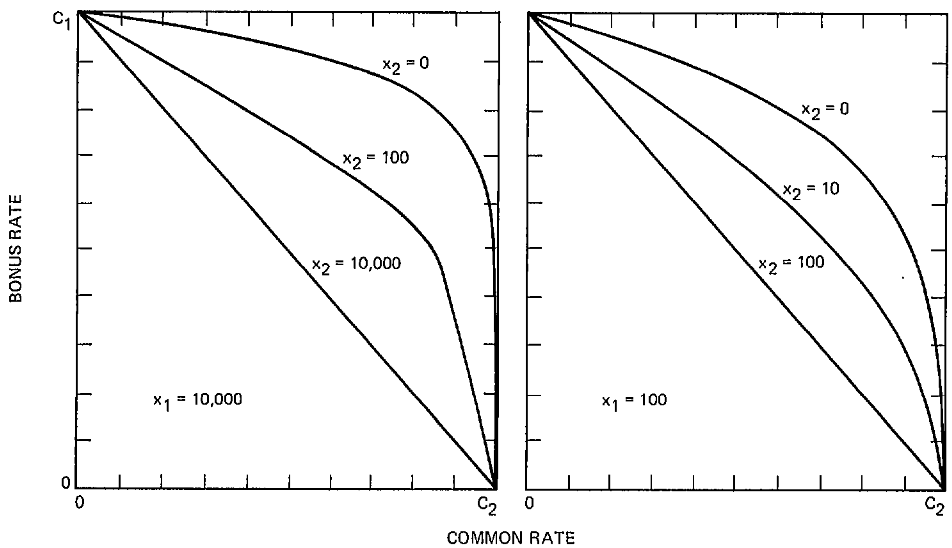

Fig. 1. The capacity region of some Gaussian broadcast channels

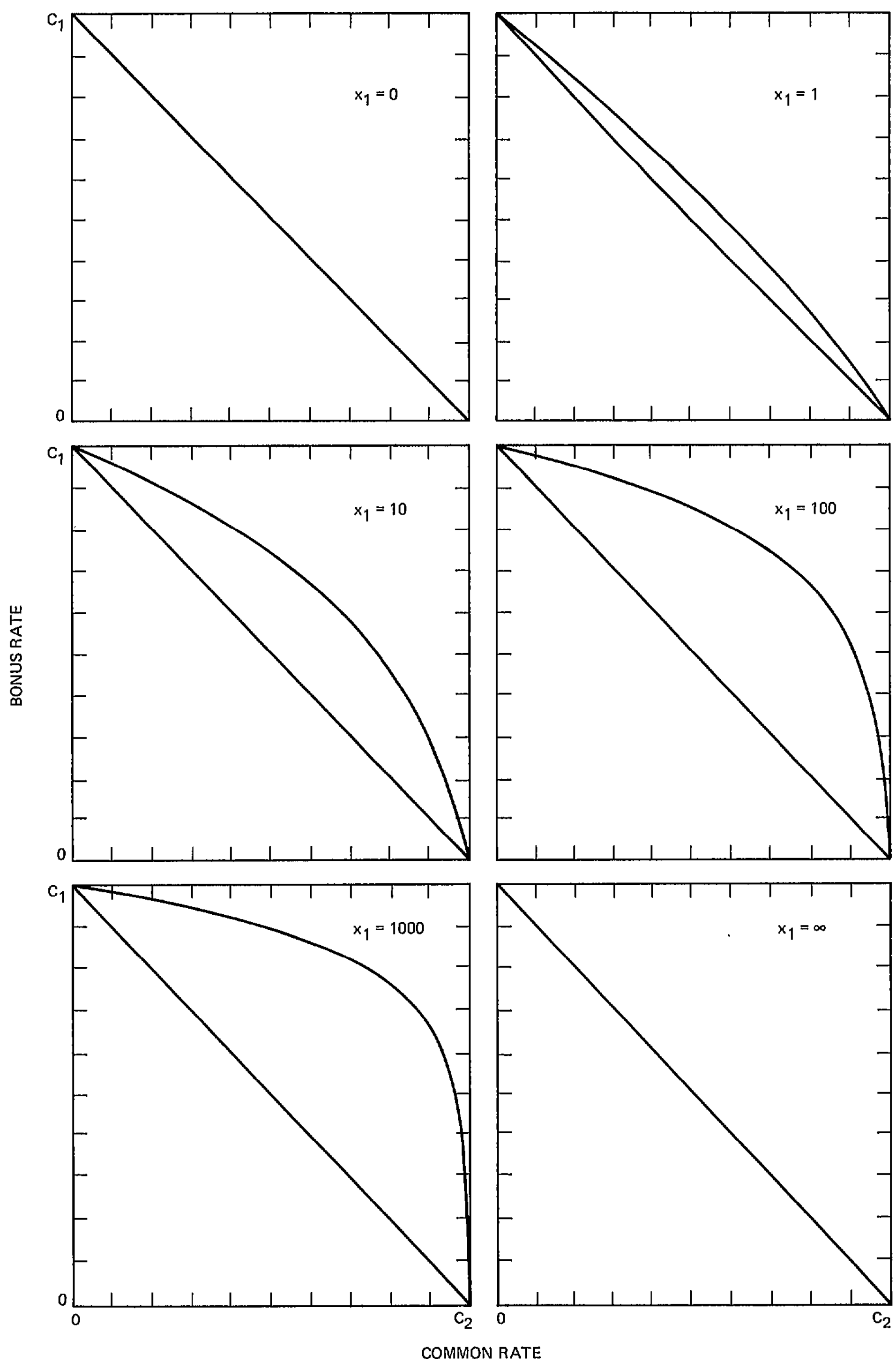

Fig. 2. The extreme capacity regions ($x_2 = 0$ and $x_2 = x_1$) for several values of x_1