SNR Estimation for the Baseband Assembly

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The expected value and the variance of the Baseband Assembly symbol signal-to-noise ratio (SNR) estimation algorithm are derived. The SNR algorithm treated here is designated as the Split Symbol Moments Estimator (SSME). It consists of averaging the first two moments of the integrated half symbols. The SSME is a biased, consistent estimator. The SNR degradation factor due to the jitter in the subcarrier demodulation and symbol synchronization loops is taken into account. Curves of the expected value of the SNR estimator versus the actual SNR are shown.

I. Introduction

The Baseband Assembly[1] uses a Split Symbol Moments Estimator (SSME) algorithm to estimate the symbol signal-to-noise ratio (SNR) of the input signal. Here we describe the SSME algorithm and give the expected value and the variance of the SNR estimator. Two numerical examples corresponding to the Voyager and the Pioneer missions are included to illustrate its performance. As in previous Baseband Assembly analyses (Refs. 1 and 2), Nyquist sampling rate is assumed.

II. Statistics of the SNR Estimator

Figure 1 is a flow chart representation of the SSME algorithm. Referring to this figure, the input to the SNR estimator is a string of signal samples modeled as

$$y_{ij} = s_{ij} + n_{ij} \tag{1}$$

where

s_{ij} = unitless random variable whose amplitude is proportional to the information signal voltage

n_{ij} = unitless random variable whose amplitude is proportional to the rms noise voltage

The first two moments of y_{ij} are

$$E\{y_{ij}\} = \sqrt{S} \tag{2}$$

$$E\{(y_{ij})^2\} = S + \sigma_n^2 \tag{3}$$

It is assumed that $E\{n_{ij}\} = 0$

$i = 1, 2, \ldots, N_s$ Nyquist samples per symbol

$j = 1, 2, \ldots, n$ symbols

The variance of the noise process is assumed to be

$$\sigma_n^2 = N_0 B_n \tag{4}$$

[1] *Deep Space Network/Flight Project Interface Design Handbook*, JPL internal document 810-5, Rev. D, 1981.

where N_0 is the one-sided noise spectral density, and B_n is the one-sided baseband noise-equivalent bandwidth.

As shown in Fig. 1, in the upper "arm" the samples from the first half of a symbol are summed to produce $Y_{\alpha j}$. In the lower "arm" the samples of the second half of a symbol are summed to produce $Y_{\beta j}$. In this analysis, it will be assumed that the number of samples in both summers are equal at the instants when $Y_{\alpha j}$ and $Y_{\beta j}$ are sampled. For this reason, $Y_{\alpha j}$ and $Y_{\beta j}$ have identical statistics. Making $Y_{\alpha j} = Y_{\beta j} = Y_j$, the mean value and the variance of Y_j will be, assuming that the samples are independent,

$$\overline{Y}_j \triangleq E\{Y_{\alpha j}\} = E\{Y_{\beta j}\} = \frac{N_s}{2} \sqrt{S} \sqrt{d_j} \tag{5}$$

$$\sigma_j^2 \triangleq E\{(Y_{\alpha j} - \overline{Y}_{\alpha j})^2\} = E\{(Y_{\beta j} - \overline{Y}_{\beta j})^2\} = \frac{N_s}{2} \sigma_n^2 \tag{6}$$

The factor d_j, designated as the SNR degradation factor, is due to the phase jitter and timing jitter in the subcarrier demodulation and symbol synchronization loops, respectively. In general,

$$0 < d_j < 1 \tag{7}$$

It can be shown that

$$d_j = \left(1 - \frac{|\phi_j|}{\pi/2}\right)^2 \left(1 - 2p_T \frac{|\tau_j|}{T_s}\right)^2 \tag{8}$$

where

ϕ_j = phase error in the subcarrier demodulation loop during the jth symbol

τ_j = timing error in the symbol synchronization loop during the jth symbol

p_T = probability of symbol transition

T_s = symbol time

In this preliminary analysis, it will be assumed that there is no doppler stress in the tracking loops and that ϕ_j and τ_j are functions of the phase and timing jitter only. With this assumption, ϕ_j and τ_j will be constant during one update interval, and the subscript j can be dropped, i.e., we will assume that during the estimation interval

$$d_j = d_{j+1} = d \tag{9}$$

and, consequently, the statistics of Y_j will be equal to those of Y_{j+1}.

In the SSME algorithm, the random variables Y_α and Y_β are combined to create two new random variables X_p and X_{ss} in the following way:

$$X_p = Y_\alpha Y_\beta \tag{10}$$

and

$$X_{ss} = (Y_\alpha + Y_\beta)^2 \tag{11}$$

Then, as shown in Fig. 1, n samples of X_p and X_{ss} are averaged in the second pair of summers to produce m_p' and m_{ss}'. Finally, m_p' and m_{ss}' are scaled and combined to produce the random variable R^*, which is the SNR estimator of the SSME algorithm, namely,

$$R^* = \frac{m_p'}{2\left(\frac{1}{4} m_{ss}' - m_p'\right)} \tag{12}$$

The statistics of R^* can be determined from the statistics of the random variables along the two paths in Fig. 1. These statistics are obtained in what follows.

Using Eqs. (6) and (7) and the fact that Y_α and Y_β are independent, the first two moments of their product defined in Eq. (10) will be

$$\overline{X}_p = N_s^2 \, Sd/4 \tag{13}$$

$$\overline{X_p^2} = (N_s^2 \, Sd/4 + N_s \sigma_n^2/2)^2 \tag{14}$$

The first two moments of X_{ss} defined by Eq. (11) are obtained using Eq. (A-2) of Appendix A with $\mu = N_s \sqrt{Sd}$ and $\sigma^2 = N_s \sigma_n^2$, namely,

$$\begin{aligned} \overline{X}_{ss} &= E(X_s^2) \\ &= N_s^2 Sd + N_s \sigma_n^2 \end{aligned} \tag{15}$$

$$\begin{aligned} \overline{X_{ss}^2} &= E(X_{ss}^4) \\ &= 3N_s^2 \sigma_n^4 + 6N_s^3 \, Sd \, \sigma_n^2 + N_s^4 S^2 d^2 \end{aligned} \tag{16}$$

Referring to Fig. 1, and using Eq. (A-5), the first and second moments at the outputs of the second pair of summers will be

$$\overline{m_p'} = \overline{X}_p \tag{17}$$

$$\overline{(m_p')^2} = \overline{X_p^2} \tag{18}$$

$$\overline{m_{ss}'} = \overline{X_{ss}} \tag{19}$$

$$\overline{(m_{ss}')^2} = \overline{X_{ss}^2} \tag{20}$$

The variances of m_p' and m_{ss}' are obtained using Eq. (A-6) with the moments obtained in Eqs. (13) through (16), namely,

$$\mathrm{var}\,(m_p') = \frac{1}{n}\,(\overline{X_p^2} - (\overline{X_p})^2)$$

$$= \frac{N_s^2\,\sigma_n^2}{4n}\,(N_s\,Sd + \sigma_n^2) \tag{21}$$

$$\mathrm{var}\,(m_{ss}') = \frac{1}{n}\,[\overline{X_{ss}^2} - (\overline{X_{ss}})^2]$$

$$= \frac{2N_s^2\sigma_n^2}{n}\,(2N_s\,Sd + \sigma_n^2) \tag{22}$$

The covariance of X_{ss} and X_p is, by definition,

$$\mathrm{cov}\,(X_{ss}, X_p) \triangleq E\{(X_{ss} - \overline{X_{ss}})\,(X_p - \overline{X_p})\} \tag{23}$$

$$= E\{[(Y_\alpha + Y_\beta)^2 - (\overline{Y_\alpha + Y_\beta})]\;[Y_\alpha\,Y_\beta - \overline{Y}_\alpha\,\overline{Y}_\beta]$$

$$= E\{[Y_\alpha^2 - \overline{Y_\alpha^2}) + (Y_\beta^2 - \overline{Y_\beta^2})$$

$$+ 2\,(Y_\alpha\,Y_\beta - \overline{Y_\alpha}\,\overline{Y_\beta})]\;[Y_\alpha\,Y_\beta - \overline{Y_\alpha}\,\overline{Y_\beta}]\}$$

$$= \overline{Y}_\beta\,(\overline{Y_\alpha^3} - \overline{Y}_\alpha\,\overline{Y_\alpha^2}) + \overline{Y}_\alpha\,(\overline{Y_\beta^3} - \overline{Y}_\beta\,\overline{Y_\beta^2})$$

$$+ 2\,(\overline{Y_\alpha^2}\,\overline{Y_\beta^2} - (\overline{Y}_\alpha)^2\,(\overline{Y}_\beta)^2) \tag{24}$$

Using Eqs. (6) and (7) in Eq. (A-2), the third moment of Y_α and Y_β is

$$\overline{Y_\alpha^3} = \overline{Y_\beta^3} = \frac{3}{4}\,N_s^2\,\sqrt{Sd}\,\sigma_n^2 + \frac{1}{8}\,N_s^3\,(Sd)^{3/2} \tag{25}$$

Inserting Eqs. (6), (7), and (25) in Eq. (24) and dividing by n, we obtain the covariance of m_p' and m_{ss}', namely,

$$\mathrm{cov}\,(m_p', m_{ss}') = \frac{N_s^2\,\sigma_n^2}{2\,n}\,(2N_s\,S_d + \sigma_n^2) \tag{26}$$

Having obtained the moments of m_p' and m_{ss}', we now are ready to determine the statistics of the estimator R^*. Using Eq. (A-9), the expected value of R^* defined by Eq. (12) is

$$\overline{R^*} = R^*\Bigg|_{\substack{\overline{m_p'}\\ \overline{m_{ss}'}}} + \frac{1}{2}\left[\frac{\partial^2 R^*}{\partial m_p'^2}\,\mathrm{var}\,(m_p') + \frac{\partial^2 R^*}{\partial m_{ss}'^2}\,\mathrm{var}\,(m_{ss}')\right]\Bigg|_{\substack{\overline{m_p'}\\ \overline{m_{ss}'}}}$$

$$+ \frac{\partial^2 R^*}{\partial m_p'\,\partial m_{ss}'}\,\mathrm{cov}(m_p', m_{ss}')\Bigg|_{\substack{\overline{m_p'}\\ \overline{m_{ss}'}}} + \ldots \tag{27}$$

Inserting Eqs. (A-16), (A-18), (A-19), (21), (22), and (26) in Eq. (27) and ignoring higher order terms, we obtain

$$\overline{R^*} = \hat{R} + \frac{1}{n}(2\hat{R} + 1) \tag{28}$$

where

$$\hat{R} \triangleq \frac{\overline{m_p'}}{2\left(\frac{1}{4}\,\overline{m_{ss}'} - \overline{m_p'}\right)} = \frac{N_s\,Sd}{2\sigma_n^2} = R\,d \tag{29}$$

is the degraded symbol SNR at the input to the SNR estimator. From Eq. (28) we observe that R^* is a biased but consistent estimator (i.e., the bias goes to zero when n goes to infinity).

The variance of R^* is obtained using Eq. (A-10), namely,

$$\mathrm{var}\,(R^*) = \left(\frac{\partial R^*}{\partial m_p'}\right)^2\Bigg|_{\substack{m_p'\\ m_{ss}'}}\mathrm{var}\,(m_p') + \left(\frac{\partial R^*}{\partial m_{ss}'}\right)^2\Bigg|_{\substack{m_p'\\ m_{ss}'}}\mathrm{var}(m_{ss}')$$

$$+ 2\,\frac{\partial R^* \partial R^*}{\partial m_p'\,\partial m_{ss}'}\,\mathrm{cov}\,(m_p', m_{ss}') \tag{30}$$

Inserting Eqs. (A-15), (A-17), (A-19), (15), (16), and (26) in Eq. (30), we obtain

$$\mathrm{var}\,(R^*) = \frac{1}{n}\,(1 + 4\hat{R} + 2\hat{R}^2) \tag{31}$$

By defining the SNR of our estimator as the ratio

$$\mathrm{SNR}\,(R^*) = \frac{(\overline{R^*})^2}{\mathrm{var}\,(R^*)} \tag{32}$$

we see that

$$\lim_{R\to 0} \mathrm{SNR}\,(R^*) = \frac{1}{n} \tag{33}$$

$$\lim_{R\to\infty} \mathrm{SNR}\,(R^*) \cong \frac{n}{2} + 2 \tag{34}$$

III. Evaluation of *d*

Assuming that there are no doppler or quantization errors, the SNR degradation factor defined in Eq. (8) is a function of the phase jitter in the subcarrier demodulation loop and the timing jitter in the symbol synchronization loop. Both jitter processes, ϕ and τ, are modeled as Gaussian random variables having zero mean and variance σ_ϕ^2 and σ_τ^2, respectively.

According to Ref. 2, the variance of the phase error in the subcarrier demodulation loop at update instants is

$$\sigma_\phi^2 = \left(\frac{T_L B_{L1}}{4K}\right) \frac{\pi^2}{\left(\frac{E_s}{N_0}\right)^2} \left(1 + \frac{E_s}{N_0}\right) \tag{35}$$

Repeating the steps of Ref. 2, it can be shown that the variance of the timing error in the symbol synchronization loop at update instants is

$$\sigma_\tau^2 = \left(\frac{T_L B_{L2} a_2}{8 K a_1}\right) \frac{T_s^2}{\left(\frac{E_s}{N_0}\right)^2} \left[1 + 2\left(\frac{E_s}{N_0}\right)\cdot(a_1 + a_2)\right] \tag{36}$$

where

B_{Lj} = one sided noise-equivalent bandwidth, j = 1 for subcarrier loop, j = 2 symbol synch loop

T_L = loop update time, assumed to be identical for both loops

K = number of symbols between updates

T_s = symbol time = $1/r$

E_s/N_0 = ratio of energy per symbol to noise spectral density

$$\triangleq R = \frac{N_s S}{2\sigma_n^2} \tag{37}$$

a_1 = M/N_s = ratio of the width of the middle portion of a symbol to the total symbol length (typically 1/2)

a_2 = L/N_s = ratio of the width of the transition portion of a symbol to the total symbol length (typically 1/4)

The expected value of d in Eq. (8) will be

$$d = \frac{1}{\sqrt{2\pi}\,\sigma_\phi} \int_{-\infty}^{\infty} \left(1 - \frac{|\phi|}{\pi/2}\right)^2 \exp\left(-\frac{1}{2}\frac{\phi^2}{\sigma_\phi^2}\right) d\phi$$

$$\times \frac{1}{\sqrt{2\pi}\,\sigma_\tau} \int_{-\infty}^{\infty} \left(1 - 2p_T \frac{|\tau|}{T_s}\right)^2 \exp\left(-\frac{1}{2}\frac{\tau^2}{\sigma_\tau^2}\right) d\tau \tag{38}$$

$$= \left[1 - 4\sqrt{\frac{2}{\pi}}\left(\frac{\sigma_\phi}{\pi}\right) + 4\left(\frac{\sigma_\phi}{\pi}\right)^2\right]\left[1 - 4\sqrt{\frac{2}{\pi}}\, p_T\left(\frac{\sigma_\tau}{T_s}\right) + 4p_T^2\left(\frac{\sigma_\tau}{T_s}\right)^2\right] \tag{39}$$

In Appendix B two numerical examples are given for parameter values typical of the Voyager and Pioneer missions.

In general, the bias in $\overline{R^*}$ can be reduced by increasing n (number of symbols in the estimator). Of course, we can improve our knowledge of R if we compensate for the effects of the bias and the degradation factor in Eq. (28), i.e., we may assume that the actual input SNR is

$$\widetilde{R} = \frac{\langle R^* \rangle}{\widetilde{d}\left(1 + \frac{2}{n}\right) + \frac{1}{n}} \tag{40}$$

where $\langle R^* \rangle$ is the average value of many R_i^* and $\widetilde{d}$ is our estimate of d.

IV. Conclusions

In this article the expected value and the variance of the SSME SNR estimator was derived. This estimator was shown to be biased and consistent.

Figures 2 and 3 illustrate the numerical results for the Voyager and Pioneer missions. At high signal SNR, the positive bias of the estimator dominates over the degradation effect due to phase jitter in the tracking loops. At low SNR, it is the other way around. Figure 4 is for the ideal case when there is no jitter in the tracking loops ($d = 1$).

Acknowledgment

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References

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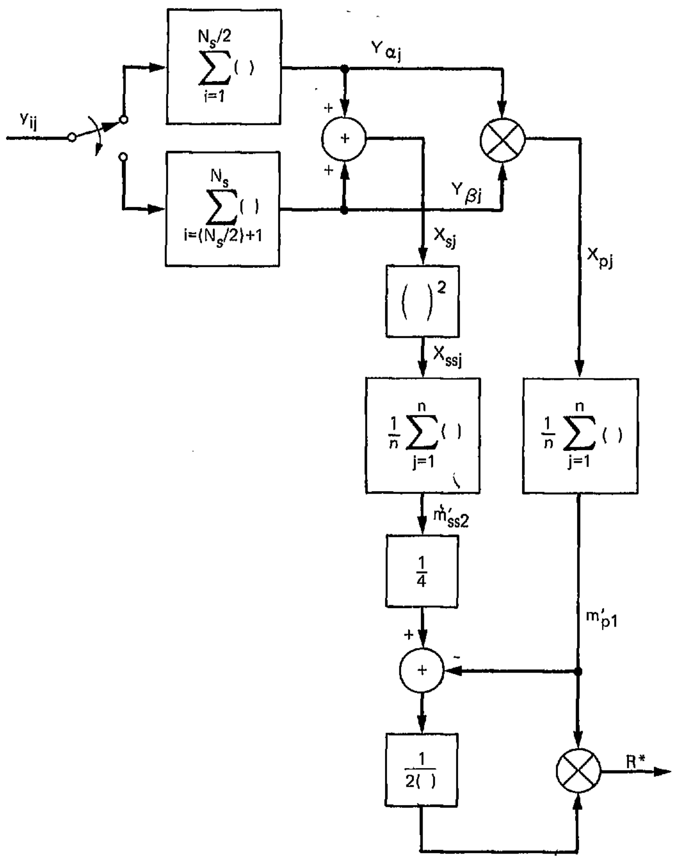

Fig. 1. Split symbol SNR estimator algorithm

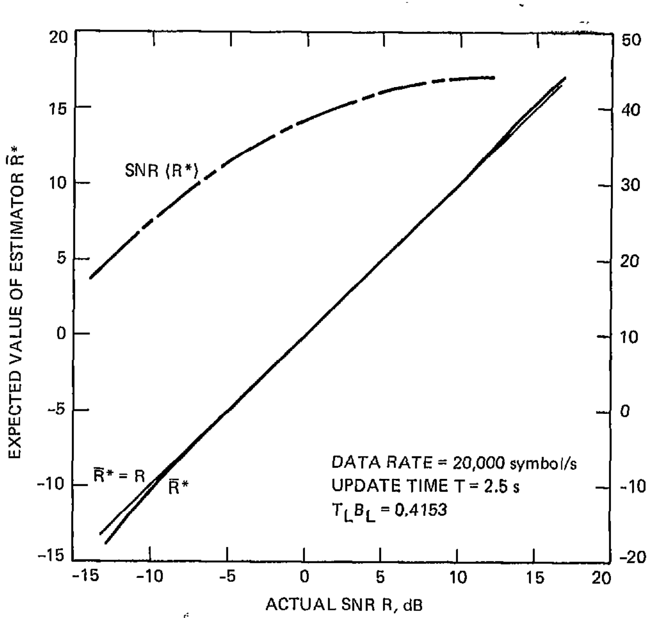

Fig. 2. Mean value and SNR of SNR estimator vs actual SNR: Voyager mission

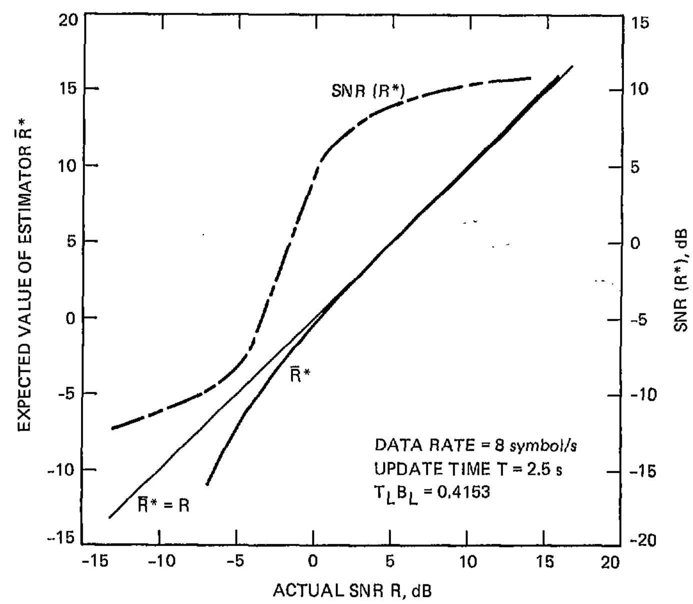

Fig. 3. Mean value and SNR of SNR estimator vs actual SNR: Pioneer mission

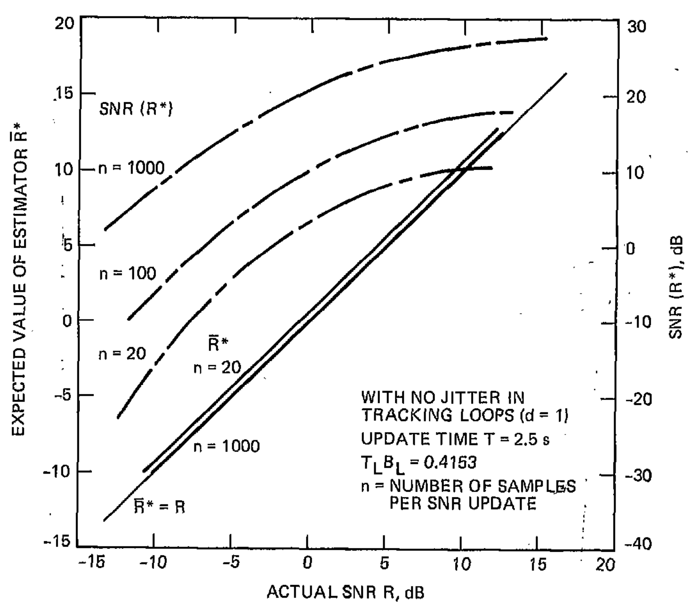

Fig. 4. Mean value and SNR of SNR estimator vs actual SNR: general

Appendix A

Gaussian Moments

1. Relation Between Statistical and Probabilistic Moments

Given a random variable x with Gaussian pdf $G(\mu,\sigma^2)$ and defining the rth moment as

$$\mu_r' = E\{x^r\} \tag{A-1}$$

the first four *probabilistic* moments of x will be

$$\begin{aligned}
\mu_1' &= \mu \\
\mu_2' &= \mu^2 + \sigma^2 \\
\mu_3' &= 3\sigma^2\mu + \mu^3 \\
\mu_4' &= 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4
\end{aligned} \tag{A-2}$$

Defining m_r' as the rth *statistical* moment of a random variable

$$m_r' = \frac{1}{n} \sum_{j=1}^{n} (x_j)^r \tag{A-3}$$

and the variance of m_r' as

$$\operatorname{var}(m_r') = E\left\{\frac{1}{n} \sum_{j=1}^{n} (x_j)^r - \mu_r'\right\}^2 \tag{A-4}$$

Chapter 10 of Ref. 3 shows that

$$E\{m_r'\} = \mu_r' \tag{A-5}$$

and

$$\operatorname{var}(m_r') = \frac{1}{n}\left[\mu_{2r}' - (\mu_r')^2\right] \tag{A-6}$$

This is an exact result.

Given a function g of K random variables x_k,

$$g(\mathrm{x}) = g(x_1, x_2, \ldots, x_K) \tag{A-7}$$

with means

$$E\{x_K\} = \theta_K$$

$$\theta \triangleq \theta_1, \theta_2, \ldots, \theta_K \tag{A-8}$$

it can be shown (Prob 10.17, Ref. 3) that

$$E\{g(\mathrm{x})\} = g(\theta) + \frac{1}{2} \sum_{k=1}^{K} \frac{\partial^2}{\partial x_k^2} g(\mathrm{x}) \bigg|_{\mathrm{x}=\theta} \operatorname{var}(x_k) + \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{K} \sum_{j=1}^{K} \frac{\partial g(\mathrm{x})}{\partial x_i\, \partial x_j} \bigg|_{\mathrm{x}=\theta} \operatorname{cov}(x_i, x_j) + \ldots \tag{A-9}$$

The variance of $g(\mathrm{x})$ will be (Eq. (10.12) of Ref. 3)

$$\operatorname{var}\{g(\mathrm{x})\} = \sum_{k=1}^{K} \left[\frac{\partial}{\partial x_k} g(\mathrm{x})\right]^2 \bigg|_{\mathrm{x}=\theta} \operatorname{var}(x_k) + \sum_{\substack{i=1 \\ i \neq j}}^{K} \sum_{j=1}^{K} \frac{\partial}{\partial x_i} g(\mathrm{x}) \frac{\partial}{\partial x_j} g(\mathrm{x}) \bigg|_{\mathrm{x}=\theta} \operatorname{cov}(x_i, x_j) + \ldots \tag{A-10}$$

2. Evaluation of the Derivatives of the Estimator

In the SSME algorithm R^* is computed from m_p' and m_{ss}', namely,

$$R^* = \frac{m_p'}{2\left(\frac{1}{4} m_{ss}' - m_p'\right)} \tag{A-11}$$

It can be shown that

$$E\{m_p'\} = \overline{m_p'} = \frac{1}{4} N_s^2\, Sd \tag{A-12}$$

and

$$E\{m_{ss}'\} = \overline{m_{ss}'} = N_s^2\, Sd + N_s\, \sigma_n^2 \tag{A-13}$$

Let

$$\widehat{R} \triangleq \frac{E\,\{m_p'\}}{2\,E\left\{\frac{1}{4}\,m_{ss}' - m_p'\right\}} = \frac{N_s\,Sd}{2\sigma_n^2} \qquad \text{(A-14)}$$

The following derivatives of the estimator R^* are evaluated:

$$\left.\frac{\partial R^*}{\partial m_p'}\right|_{\overline{m_p'},\ \overline{m_{ss}'}} = \frac{2}{\sigma_n^2\ N_s}\,(1 + 2\widehat{R}) \qquad \text{(A-15)}$$

$$\left.\frac{\partial^2 R^*}{\partial m_p'^2}\right|_{\overline{m_p'},\ \overline{m_{ss}'}} = \left(\frac{4}{\sigma_n^2 N_s}\right)^2 (1 + 2\widehat{R}) \qquad \text{(A-16)}$$

$$\left.\frac{\partial R^*}{\partial m_{ss}'}\right|_{\overline{m_p'},\ \overline{m_{ss}'}} = -\frac{1}{\sigma_n^2 N_s}\widehat{R} \qquad \text{(A-17)}$$

$$\left.\frac{\partial^2 R^*}{\partial m_{ss}'^2}\right|_{\overline{m_p'},\ \overline{m_{ss}'}} = 2\left(\frac{1}{\sigma_n^2 N_s}\right)^2 \widehat{R} \qquad \text{(A-18)}$$

$$\left.\frac{\partial^2 R^*}{\partial m_p'\,\partial m_{ss}'}\right|_{\overline{m_p'},\ \overline{m_{ss}'}} = -2\left(\frac{1}{\sigma_n^2 N_s}\right)^2 (1 + 4\widehat{R}) \qquad \text{(A-19)}$$

Appendix B

Constants Used to Derive Figs. 2–4

In order to illustrate the performance of the SSME estimator, two cases are considered.

(1) Voyager

Data rate r = 20,000 symbols/second

Update time T_L = 2.5 seconds

$K = n$ = 2.5 × 20,000 symbols/loop update

$T_L B_L$ = 0.4153 (from Table 1, Ref. 1 for both tracking loops

Noise bandwidth B_n = 3.75 MHz

(2) Pioneer

r = 8 symbols/second

T_L = 2.5 seconds

$K = n$ = 2.5 × 8 symbols/loop update

$T_L B_L$ = 0.4153 for both tracking loops

B_n = 135 kHz

The performance of R^* for the Magellan mission will be better than for Voyager.

Using Eqs. (35), (36), (39), (28), (29), (31), and (32), Figs. 2 and 3 are obtained.