# A Recursive Solution for a Fading Memory Filter Derived From Kalman Filter Theory

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*A simple recursive solution for a class of fading memory tracking filters is presented. A fading memory filter provides estimates of filter states based on past measurements, similar to a traditional Kalman filter. Unlike a Kalman filter, an exponentially decaying weight is applied to older measurements, discounting their effect on present state estimates. It is shown that Kalman filters and fading memory filters are closely related solutions to a general least squares estimator problem. Closed form filter transfer functions are derived for a time invariant, steady state, fading memory filter. These can be applied in loop filter implementation of the DSN Advanced Receiver carrier phase locked loop (PLL).*

## I. Introduction

The problem of estimating system state based on measurements is usually addressed by some form of a least squares estimator (LSE), where a Kalman filter is the common choice. The Kalman filter (Ref. 1) offers a recursive solution for state estimates, as well as for a state estimate covariance matrix. However, most Kalman filter implementations are sensitive to errors in modeling, both in generating a linearized model and in selecting model parameters. The effect of mismodeling is more severe when the system varies with time. Then, as the Kalman filter attempts to fit all past data to a single model, large errors in state estimates occur.

One approach to reducing effects of past data is to use an LSE that applies an exponentially decaying weight to older measurements. This "fading memory" approach, introduced in Ref. 3, overcomes most mismodeling and instability problems associated with standard Kalman filters, at a cost of losing some of the information associated with the discounted past measurements. In addition, the fading memory filter has the advantage that an exponential decay of past data is an intuitively clear concept to most electrical engineers, or anyone who works with basic electrical circuits. Sorensen (Ref. 2) presents a general recursive solution to the linear LSE problem, that with an appropriate selection of parameters can be either a fading memory filter or a standard Kalman filter. Unfortunately, the computational load associated with the fading memory filter implementation of Ref. 2 is excessive compared to what can be achieved.

In this article, the recursive LSE solution proposed in Ref. 2 is simplified for fading memory filters, assuming a particular form of process noise covariance matrix, $Q_n$. The resulting filter performs well for simulations of real life situations. The ensuing simplicity of the filter equations, based on a largely intuitive selection of $Q_n$, may incur some cost in terms of optimality. The simplified equations are then applied to a time invariant system, where filter gains have reached their steady state value, and closed form filter input-to-output transfer functions are derived.

## II. System Model and Recursive Solution

The linearized model of a system under consideration is given by Eqs. (1) through (3). Equation (1) describes how the system state vector is propagated from one time point to the next. Equation (2) defines the relationship between system state and present measurement. All differences between state propagation model and behavior of the actual system are represented by the random variable $\nu_{n-1}$, while measurement noise is represented by $u_n$. Usually, $\{\nu_{n-1}\}$and $\{u_n\}$are modeled as zero-mean, white Gaussian random processes, with covariance matrices given by Eq. (3).

$$x_n = \phi_{n,n-1}\, x_{n-1} + \nu_{n-1} \tag{1}$$

$$y_n = H_n\, x_n + u_n \tag{2}$$

$$\left.\begin{aligned} E\,[\nu_{n-1}\,\nu_{m-1}^T] &= Q_{n-1}\,\delta_{m,n} \\ E\,[u_n\, u_m^T] &= R_n\,\delta_{m,n} \\ E\,[\nu_n u_m^T] &= 0 \end{aligned}\right\} \tag{3}$$

where

| | |
|---|---|
| $\delta_{m,n}$ | The Kronecker delta function |
| $x_n$ | System state vector (at time $n$) |
| $\phi_{n,n-1}$ | State transition matrix (from time $n-1$ to time $n$) |
| $\nu_{n-1}$ | Process (or state) noise |
| $y_n$ | Vector of measurements |
| $H_n$ | Measurement transformation matrix |
| $u_n$ | Measurement noise |
| $Q_{n-1}$ | Process noise covariance matrix |
| $R_n$ | Measurement noise covariance matrix |
| $E(\cdot)$ | Statistical expected value |

The LSE problem can be stated as follows. Given a set of measurements $\{y_i,\ i = 1 \ldots n\}$ and a weight matrix $S_n$, find an estimate of the state, $\bar{x}_{n,n}$, that minimizes $J_n$:

$$J_n = U_n S_n^{-1} U_n^T \tag{4}$$

where $U_n$ is the column vector composed of the individual measurement noise vectors $\{u_i,\ i = 1 \ldots n\}$, and $S_n$ is a non-negative definite matrix. In this formulation, the covariance associated with initial state uncertainty is ignored.

The matrix $S_n$ is often defined as a diagonal, or quasi-diagonal matrix, reflecting the stationary nature of the measurement noise processes. When $u_n$ is a scaler, $S_n$ is a diagonal matrix, while when $u_n$ is a $k$-length vector, $S_n$ is a block diagonal matrix consisting of $k$ by $k$ matrices along the block diagonal, with zeroes elsewhere. Let us first explore the scalar measurement case, with constant measurement noise variance. There are two approaches for selecting the elements of $S_n$. The first approach is to assign equal weight to all measurements, i.e., $S_n$ is an identity matrix. This approach leads to a standard Kalman filter. The second approach is to degrade older measurements, accounting for less validity of older measurements. In this case, the diagonal elements of $S_n$, denoted $s_i$, satisfy:

$$s_i > s_{i-1} \quad i = 2 \ldots n$$

This approach results in a fading memory filter. The two approaches can be easily extended to the cases where each measurement is a $k$-length vector and measurement noise covariance changes from one time to the next.

As the number of measurements increases, a complete LSE solution (requiring inversion of an $nk$ by $nk$ matrix) becomes computationally unattractive and a recursive form of the algorithm is used. Sorensen proposed Eqs. (5) through (8) as an optimum recursive solution to the LSE problem:

$$\bar{x}_{n,n} = \phi_{n,n-1}\,\bar{x}_{n-1,n-1} + K_n(y_n - H_n\phi_{n,n-1}\,\bar{x}_{n-1,n-1}) \tag{5}$$

$$P_{n,n-1} = \phi_{n,n-1}\,P_{n-1,n-1}\,\phi_{n,n-1}^T\,e^{c_{n-1}} + Q_{n-1} \tag{6}$$

$$K_n = P_{n,n-1}\,H_n^T\,(H_n P_{n,n-1}\,H_n^T + R_n)^{-1} \tag{7}$$

$$P_{n,n} = P_{n,n-1} - K_n H_n P_{n,n-1} \tag{8}$$

where

$\bar{x}_{n,n}$ State estimate (at time $n$)

| | |
|---|---|
| $K_n$ | Filter gain |
| $P_{n,n}$ | State estimate covariance matrix (at time $n$, including $y_n$) |
| $P_{n,n-1}$ | State estimate covariance matrix (at time $n$, but without $y_n$) |
| $e^{c_n}$ | Decay factor |

Typical filter update is as follows. In the $n$th step, Eq. (6) is evaluated, extrapolating the state estimate covariance matrix in time, and accounting for the process noise covariance matrix. This equation also performs a time decay function, using the multiplier $e^{c_{n-1}}$. Next, filter gain is computed by Eq. (7). Finally, the new measurement is incorporated into the state estimate and the state covariance matrix, using Eqs. (5) and (8), respectively.

When the decay factor is unity, i.e. no degradation of past measurements is used, Eqs. (5) through (8) represent a standard Kalman filter (Ref. 1). On the other hand, when $e^{c_{n-1}} > 1$ for all $n$, there is actual decay of past measurements, resulting in a fading memory filter.

As seen in the above equations, the computational complexity associated with this implementation of a fading memory filter is at least as high as that of a Kalman filter. However, a significantly simpler filter implementation is derived in Appendix A, for the design value of $Q_{n-1}$ selected according to Eq. (12). The resulting recursion formulas are:

$$\bar{x}_{n,n} = \phi_{n,n-1}\,\bar{x}_{n-1,n-1} + K_n\,(y_n - H_n\,\phi_{n,n-1}\,\bar{x}_{n-1,n-1}) \quad (9)$$

$$M_n = H_n^T R_n^{-1} H_n + \alpha_n\,\phi_{n,n-1}^{-T}\,M_{n-1}\,\phi_{n,n-1}^{-1} \quad (10)$$

$$K_n = M_n^{-1} H_n^T R_n^{-1} \quad (11)$$

$$Q_{n-1} = \beta_n\,\phi_{n,n-1}\,P_{n-1,n-1}\,\phi_{n,n-1}^T \qquad \beta_n \geqslant 0 \quad (12)$$

where

| | |
|---|---|
| $M_n$ | Inverse of state estimate covariance matrix |
| $\alpha_n = \dfrac{1}{e^{c_{n-1}} + \beta_n}$ | Filter decay factor, $0 < \alpha_n < 1$ |

Filter update for the $n$th measurement consists of updates of $M_n$, $K_n$, and $\bar{x}_{n,n}$, in that order. Computations associated with this form of filter update are simpler than corresponding computations for a Kalman filter. A measure of the simplification is that steady state value of $P_{n,n}$ can be obtained from a linear set of equations, rather than the (quadratic) Riccati equation required for a Kalman filter. The filter decay factor, used in Eq. (10), can be viewed as an exponential:

$$\alpha_n = e^{-\tau_n/T} \quad (13)$$

where $T$ is the filter sample time and $\tau_n$ is the filter time constant. With this definition, the filter response to input impulse is somewhat similar to the response of an RC electrical circuit to an impulse, with an exponentially decaying transient. From the above equations it is clear that $\alpha_n$ has a dual role. First, it represents the exponential increase in the state estimate covariance matrix, $e^{c_{n-1}}$. In addition, it also includes the effect of the assumed $Q_{n-1}$, via a $\beta_n$ component.

## III. Results for Time Invariant, Steady State Filter Gain

Often, the system model assumes that the state transition and measurement matrices and the measurement noise covariance matrix are time invariant, and measurement samples are uniformly spaced in time. It is also assumed that the filter decay factor is constant. In this case, the filter update equations are:

$$\bar{x}_{n,n} = \phi\bar{x}_{n-1,n-1} + K_n\,(y_n - H\,\phi\bar{x}_{n-1,n-1}) \quad (14)$$

$$M_n = H^T R^{-1} H + \alpha\phi^{-T} M_{n-1}\,\phi^{-1} \quad (15)$$

$$K_n = M_n^{-1} H^T R^{-1} \quad (16)$$

In these equations, the unsubscripted $\phi$, $R$, $H$, and $\alpha$ are the time invariant versions of the corresponding subscripted variables. After a sufficiently long time, the matrices $M_n$ approach a steady state value, $M$, that depends only on $\phi$, $H$, $R$, and $\alpha$. When $M_n$ is approximated by this steady state value, the filter gain defined in Eq. (15) can be precomputed. This results in a significant reduction in the computational load associated with filter updates. Of course there is some loss of flexibility in using constant, steady state, filter gains.

When using steady state filter gain, $K$, transfer functions from filter input to filter output can be evaluated. The transfer function, in matrix form, is:

$$C(z) = (zI - \phi + KH\phi)^{-1}\,Kz \quad (17)$$

Note that for an $m$-input, $n$-state filter, the matrix $C(z)$ is of dimension $n$ by $m$.

Analytic steady state tracking filter solutions are often investigated for simple second and third order Kalman filters (Ref. 4). Similar expressions are derived below for fading memory filters. In a typical case, range (or range and velocity) measurements are used in estimating range, velocity, and perhaps acceleration. The resulting $\phi$ and $H$, when no velocity measurement is available, are given in Table 1. Without loss of generality, $R$ is assumed to be unity. This can be done since any linear scaling of $R$ causes similar scaling for $M_n$, but has no impact on the filter gain. Thus, the state update equation is independent of scaling of $R$.

Table 2 presents the input-to-output transfer function components for these filters, assuming steady state filter gains. It is interesting to notice that the transfer functions have all their poles at $z = -\alpha$, within the unit circle.

Fading memory filters, described by Eqs. (14) - (16), are being investigated for the DSN Advanced Receiver carrier PLL loop filter (Ref. 5), where phase, frequency, and frequency rate correspond to range, velocity, and acceleration. It is expected that these filters, in conjunction with a predictor, will reduce the effect of loop transport lag.

Similar filters were also successfully used in the Mobile Automated Field Instrumentation System (MAFIS) Position Location Demonstration[1,2] and for the High Dynamics GPS Receiver Validation Demonstration (Ref. 6), both performed at JPL.

## IV. Conclusions

The fading memory filter and Kalman filter are presented as special cases of a general least squares estimator problem. It is shown that both filters can be implemented by the same set of recursion equations, with an appropriate choice of parameters. A simple recursive solution for a class of fading memory tracking filters is presented. Filter implementation for this class is computationally efficient, and exhibits good stability performance. It is proposed as part of the loop filter for the DSN Advanced Receiver carrier PLL.

[1] Hurd, W. J., *MAFIS Position Location Feasibility Demonstration Final Report* (JPL Internal Document 7011-22), Vol II B.2, March 1982.

[2] Wallis, D. E., private communications.

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**Table 1. Matrices for second and third order filters**

| Order | $\phi$ | $H$ | $R$ |
|---|---|---|---|
| 2 | $\begin{Bmatrix} 1 & T \\ 0 & 1 \end{Bmatrix}$ | (1, 0) | (1) |
| 3 | $\begin{Bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{Bmatrix}$ | (1, 0, 0) | (1) |

**Table 2. Closed form transfer functions from input range to output parameter**

| Output parameter | Second order filter | Third order filter |
|---|---|---|
| Range | $\dfrac{(1-\alpha^2)\, z \left(z - 2\, \dfrac{\alpha}{1+\alpha}\right)}{(z-\alpha)^2}$ | $\dfrac{(1-\alpha)\, z\, ((\alpha^2+\alpha+1)\, z^2 - 3\alpha\, (1+\alpha)\, z + 3\alpha^2)}{(z-\alpha)^3}$ |
| Velocity | $\dfrac{(1-\alpha)^2\, z\, (z-1)}{T(z-\alpha)^2}$ | $\dfrac{(1-\alpha)\, z\, (z-1)\, ((3-3\alpha^2)\, z + 5\alpha^2 - 4\alpha - 1)}{2T\, (z-\alpha)^3}$ |
| Acceleration | — | $\dfrac{(1-\alpha)^3\, z\, (z-1)^2}{T^2\, (z-\alpha)^3}$ |

# Appendix A
# Derivation of Simplified Recursive Formulas

This appendix derives simplified fading memory equations, Eqs. (9) through (11), from Sorensen's recursive solution, given in Eqs. (5) through (8). The simplification is accomplished in three steps. First Eqs. (7) and (8) are combined. Then, a specific form of $Q_n$ is assumed, resulting in a simple equation for the update of state estimate covariance matrix. Finally, a new state estimate update equation is derived. The first step uses a matrix inversion lemma (Ref. 7):

$$(B + UV)^{-1} = B^{-1} - B^{-1} U (I + VB^{-1} U)^{-1} VB^{-1} \quad \text{(A-1)}$$

Applying Eq. (A-1) to Eq. (8), results in:

$$\begin{aligned} P_{n,n}^{-1} = {} & P_{n,n-1}^{-1} - P_{n,n-1}^{-1} (-K_n) \\ & \times (I + H_n P_{n,n-1} P_{n,n-1}^{-1} (-K_n))^{-1} \\ & \times H_n P_{n,n-1} P_{n,n-1}^{-1} \end{aligned} \quad \text{(A-2)}$$

or

$$P_{n,n}^{-1} = P_{n,n-1}^{-1} + P_{n,n-1}^{-1} K_n (I - H_n K_n)^{-1} H_n \quad \text{(A-3)}$$

but:

$$\begin{aligned} I - H_n K_n & = I - H_n P_{n,n-1} H_n^T \\ & \quad \times (H_n P_{n,n-1} H_n^T + R_n)^{-1} \\ & = I - (H_n P_{n,n-1} H_n^T + R_n - R_n) \\ & \quad \times (H_n P_{n,n-1} H_n^T + R_n)^{-1} \\ & = R_n (H_n P_{n,n-1} H_n^T + R_n)^{-1} \end{aligned} \quad \text{(A-4)}$$

combining Eqs. (A-3) and (A-4) results in:

$$\begin{aligned} P_{n,n}^{-1} & = P_{n,n-1}^{-1} + H_n^T (H_n P_{n,n-1} H_n^T + R_n)^{-1} \\ & \quad \times (H_n P_{n,n-1} H_n^T + R_n) R_n^{-1} H_n \\ & = P_{n,n-1}^{-1} + H_n^T R_n^{-1} H_n \end{aligned} \quad \text{(A-5)}$$

Equation (A-5) has a form found commonly in literature (Ref. 4). Next, let us assume that the matrix $Q_{n-1}$ has a special form:

$$Q_{n-1} = \beta_n \phi_{n,n-1} P_{n-1,n-1} \phi_{n,n-1}^T \qquad \beta_n \geqslant 0, \quad \text{(A-6)}$$

The rationale for this assumption is discussed at the end of the appendix. With this assumption, Eq. (6) becomes:

$$P_{n,n-1} = (\beta + e^{c_{n-1}}) \phi_{n,n-1} P_{n-1,n-1} \phi_{n,n-1}^T \quad \text{(A-7)}$$

Inserting Eq. (A-7) into (A-5), and assuming that $P_{n,n-1}$ and $\phi_{n,n-1}$ are invertible, we get:

$$\begin{aligned} P_{n,n}^{-1} = {} & H_n^T R_n^{-1} H_n \\ & + \frac{1}{\beta + e^{c_{n-1}}} \phi_{n,n-1}^{-T} P_{n-1,n-1}^{-1} \phi_{n,n-1}^{-1} \end{aligned} \quad \text{(A-8)}$$

Following the notation in the body of the paper, Eq. (A-8) can be represented as:

$$M_n = H_n^T R_n^{-1} H_n + \alpha_n \phi_{n,n-1}^{-T} M_{n-1} \phi_{n,n-1}^{-1} \quad \text{(A-9)}$$

where

| | | | |
|---|---|---|---|
| $M_n$ | = | $P_{n,n}^{-1}$ | Inverse of state estimate covariance matrix |
| $\alpha_n$ | = | $\dfrac{1}{\beta_n + e^{c_{n-1}}}$ | Filter decay factor |

Note that $e^{c_{n-1}} > 1$ and $\beta_n \geqslant 0$, thus $1 > \alpha_n > 0$. This completes the derivation of the simplified covariance matrix update equation. To complete the proof, the expression from Eq. (15) is now derived for the gain $K_n$. From Eq. (A-5):

$$M_n = P_{n,n-1}^{-1} + H_n^T R_n^{-1} H_n \quad \text{(A-10)}$$

Applying Eq. (A-1) to Eq. (A-10):

$$M_n^{-1} = P_{n,n-1} - P_{n,n-1} H_n^T$$

$$\times (I + R_n^{-1} H_n P_{n,n-1} H_n^T)^{-1} R_n^{-1} H_n^T p_{n,n-1}$$

$$= P_{n,n-1} - P_{n,n-1} H_n^T$$

$$\times (R_n + H_n P_{n,n-1} H_n^T)^{-1} H_n^T P_{n,n-1} \tag{A-11}$$

After multiplying both sides of this equation by $H_n^T R_n^{-1}$, and some tedious arithmetic, Eq. (A-11) becomes:

$$M_n^{-1} H_n^T R_n^{-1} = P_{n,n-1} H_n^T (H_n P_{n,n-1} H_n^T + R_n)^{-1} = K_n \tag{A-12}$$

This completes the derivation.

The selection of $Q_n$, as defined by Eq. (A-6), is of particular interest. In most Kalman filter applications, $Q_n$ serves a dual function. First, it represents the modeled process noise, which is its declared objective. Then, it also reduces the risk of numerical instability by establishing a minimal value to the state estimate covariance matrix. The interested reader can evaluate Eqs. (5) through (8) for simple cases, using $Q_n = 0$, and observe that $P_{n,n}$ approaches zero as $n$ goes to infinity. Since, for non-trivial Kalman filters, $P_{n,n}$ must be positive definite, small numerical problems can make the filter unstable. For this reason, Kalman filter designers tend to inflate $Q_n$ above its modeled level.

The fading memory filter does not suffer from a corresponding problem. Even when $Q_n$ is zero, $P_{n,n}$ reaches a finite nonzero value. Conceptually, a Kalman filter $P_{n,n}$ is affected by a larger set of measurements than a fading memory filter $P_{n,n}$ and thus tends to be smaller. Since $Q_n$ is not required for numerical stability, a reasonable choice is for $Q_n$ to be proportional to the state estimate covariance matrix, or to a related quantity, as defined by Eq. (A-6). It can be viewed as an adaptive definition of $Q_n$. Also, if $Q_n$ is chosen as 0, the decay factor $e^{c_{n-1}}$ defined in this appendix, and the filter decay factor $\alpha$, are reciprocals of each other.

Anderson and Moore (Ref. 8) suggest that a fading memory filter can be viewed as a Kalman filter with exponential inflation applied to past $Q_n$ and $R_n$. They also emphasize the inherent stability associated with such a filter. Their approach, though insightful, does not simplify filter mechanization.

In summary, the particular form of $Q_n$ used in this appendix is not derived from an independent statistical model of process noise, thus resulting in a sub-optimal solution. In many real life applications, the benefit associated with numerical stability, reduced sensitivity to mismodeling, and reduced computational load may far outweigh this loss in optimality.