Cramer-Rao Bounds for Signal-to-Noise Ratio and Combiner Weight Estimation

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Cramer-Rao lower bounds on estimator variance are calculated for arbitrary unbiased estimates of signal-to-noise ratio and combiner weight parameters. Estimates are assumed to be based on a discrete set of observables obtained by matched filtering of a biphase modulated signal. The bounds are developed first for a problem model based on one observable per channel symbol period, and then extended to a more general problem in which subperiod observables are also available.

I. Introduction

This article calculates the Cramer-Rao bounds on the performance of arbitrary unbiased estimates of signal-to-noise ratio (SNR) and combiner weight parameters. Estimates are assumed to be based on a discrete set of observables obtained by matched filtering of a biphase modulated signal. Initially, we assume in Section II that exactly one observable or "sample" is available per channel symbol period. Later, in Section III, we consider a more general problem in which multiple observables or "subinterval samples" are obtained per symbol period by filtering over equal-length subintervals of each symbol period.

Estimates of signal-to-noise ratio and combiner weight are of interest in a variety of applications, such as symbol stream combining. In this article, Cramer-Rao bounds are determined directly for these parameters of interest, rather than for the underlying signal mean and noise variance parameters. This approach also reduces the mathematical complexity, because many expressions are separable functions of signal-to-noise ratio and combiner weight. The result is an almost-closed-form solution in which only one easily characterizable function of a single variable (SNR) is not explicitly determined.

Reference 1 provides additional background information on the significance of the parameters being estimated, and on the origin of our probabilistic model for the symbol period observables. Our model for the subinterval observables is a straightforward generalization, and it has been discussed previously as the basis for analyzing so-called "split-symbol" estimators (e.g., see Ref. 2) or "generalized" maximum likelihood estimators.[1]

II. Estimation with Symbol Period Sampling

We first consider estimation based on one sample per symbol period. Under this model, there are N discrete observables

[1]Vilnrotter, V. A., "A Generalized Class of Maximum Likelihood Estimators," IOM 331-86.5-82, Jet Propulsion Laboratory, Pasadena, Calif., January 13, 1986 (JPL Internal Document).

x_i, $i = 1, \cdots, N$, which can be represented in the form (see Eq. (7) of Ref. 1):

$$x_i = D_i m + n_i \sigma \tag{1}$$

where $\{D_i\}$ is a data modulation sequence corresponding to the transmitted channel symbols, and $\{n_i\}$ is a noise sequence. As in Ref. 1, we assume that the $\{n_i\}$ are independent unit normal random variables, and that the $\{D_i\}$ are independent and take on the values +1 and -1 with equal probability. The unknown parameters m and σ represent the magnitudes of the "signal" and "noise" components of the observables $\{x_i\}$.

Our estimation problem is to estimate a signal-to-noise ratio parameter ρ and a combiner weight parameter α, based on the vector of observables $\mathbf{x} = (x_1, \cdots, x_N)$. The parameters ρ and α are defined in terms of the underlying signal and noise parameters m and σ as

$$\left.\begin{aligned} \rho &= \frac{m^2}{\sigma^2} \\ \alpha &= \frac{m}{\sigma^2} \end{aligned}\right\} \tag{2}$$

We note from Eq. (12) of Ref. 1 that the actual signal-to-noise ratio at the receiver is only one-half ρ, but we prefer the definition in Eq. (2) for mathematical convenience.

The log-likelihood function for this problem is taken from Eq. (20) of Ref. 1:

$$\frac{1}{N} \ln p(\mathbf{x}|m, \sigma) = -\frac{1}{2} \ln (2\pi) - \ln \sigma - \frac{\langle x^2 \rangle_N}{2\sigma^2} - \frac{m^2}{2\sigma^2} + \left\langle \ln \cosh \frac{mx}{\sigma^2} \right\rangle_N \tag{3}$$

where the notation $\langle \cdot \rangle_N$ represents a sample average value: for any function F applied to the N samples $x_1, \cdots, x_N$,

$$\langle F(x) \rangle_N \triangleq \frac{1}{N} \sum_{i=1}^{N} F(x_i) \tag{4}$$

The log-likelihood function may also be written directly in terms of the signal-to-noise ratio and combiner weight parameters,

$$\frac{1}{N} \ln p(\mathbf{x}|\rho, \alpha) = -\frac{1}{2} \ln (2\pi) - \frac{1}{2} \ln \rho + \ln \alpha - \frac{\langle x^2 \rangle_N \alpha^2}{2\rho} - \frac{1}{2} \rho + \langle \ln \cosh \alpha x \rangle_N \tag{5}$$

The Cramer-Rao bound requires computation of the Fisher information matrix $\mathbf{J}$,

$$\mathbf{J} = \begin{bmatrix} -E\left\{\dfrac{\partial^2 \ln p(\mathbf{x}|\rho, \alpha)}{\partial \rho^2}\right\} & -E\left\{\dfrac{\partial^2 \ln p(\mathbf{x}|\rho, \alpha)}{\partial \rho \partial \alpha}\right\} \\ -E\left\{\dfrac{\partial^2 \ln p(\mathbf{x}|\rho, \alpha)}{\partial \alpha \partial \rho}\right\} & -E\left\{\dfrac{\partial^2 \ln p(\mathbf{x}|\rho, \alpha)}{\partial \alpha^2}\right\} \end{bmatrix}$$

$$\triangleq \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{6}$$

After evaluating the indicated derivatives, we find that

$$\left.\begin{aligned} \frac{1}{N} \rho^2 J_{11} &= \frac{E\{\langle \alpha^2 x^2 \rangle_N\}}{\rho} - \frac{1}{2} \\ \frac{1}{N} \rho\alpha J_{12} = \frac{1}{N} \rho\alpha J_{21} &= \frac{-E\{\langle \alpha^2 x^2 \rangle_N\}}{\rho} \\ \frac{1}{N} \alpha^2 J_{22} &= 1 + \frac{E\{\langle \alpha^2 x^2 \rangle_N\}}{\rho} - E\{\langle \alpha^2 x^2 \operatorname{sech}^2 \alpha x \rangle\}_N \end{aligned}\right\} \tag{7}$$

The expectations in Eq. (7) may be evaluated by substituting Gaussian random variables $\{u_i\}$ for the non-Gaussian random variables $\{x_i\}$. Defining

$$u_i = \alpha D_i x_i \tag{8}$$

where α is the unknown combiner weight and $D_i = \pm 1$ is the random data modulation embedded in x_i, we see that u_i is Gaussian with mean and variance both equal to ρ:

$$\left.\begin{aligned} E\{u_i\} &= \alpha m = m^2/\sigma^2 = \rho \\ E\{u_i^2\} &= \alpha^2(m^2+\sigma^2) = \rho^2+\rho \\ \text{var}\{u_i\} &= E\{u_i^2\} - [E\{u_i\}]^2 = \rho \end{aligned}\right\} \quad (9)$$

Furthermore, because $D_i^2 = 1$ and because sech $(\cdot)$ is an even function of its argument,

$$\left.\begin{aligned} E\{\langle\alpha^2 x^2\rangle_N\} &= E\{u_i^2\} = \rho^2+\rho \\ E\{\langle\alpha^2 x^2 \operatorname{sech}^2 \alpha x\rangle_N\} &= E\{u_i^2 \operatorname{sech}^2 u_i\} \triangleq E_2(\rho) \end{aligned}\right\} \quad (10)$$

The second expectation is written as $E_2(\rho)$, which is not determined in closed form. However, it is important to note that $E_2(\rho)$ is a function of ρ only, because the statistics of u_i are a function of ρ only. Inserting these results into the expressions for J_{ij}, we obtain

$$\left.\begin{aligned} \frac{1}{N}\rho^2 J_{11} &= \rho + \frac{1}{2} \\ \frac{1}{N}\rho\alpha J_{12} &= \frac{1}{N}\rho\alpha \mathbf{J}_{21} = -(\rho+1) \\ \frac{1}{N}\alpha^2 J_{22} &= 2+\rho-E_2(\rho) \end{aligned}\right\} \quad (11)$$

In terms of $\mathbf{J}$, the Cramer-Rao bound (Ref. 3) states that for any *unbiased* estimates $\hat{\rho}$, $\hat{\alpha}$ of the unknown parameters ρ, α,

$$\left.\begin{aligned} \frac{\text{var}(\hat{\rho})}{\rho^2} &\geqslant \frac{(\mathbf{J}^{-1})_{11}}{\rho^2} \\ \frac{\text{var}(\hat{\alpha})}{\alpha^2} &\geqslant \frac{(\mathbf{J}^{-1})_{22}}{\alpha^2} \end{aligned}\right\} \quad (12)$$

Calculating $\mathbf{J}^{-1}$ from Eq. (11), we obtain

$$\left.\begin{aligned} \frac{\text{var}(\hat{\rho})}{\rho^2} &\geqslant \frac{2}{N}\,\frac{2+\rho-E_2(\rho)}{\rho-(2\rho+1)E_2(\rho)} \\ \frac{\text{var}(\hat{\alpha})}{\alpha^2} &\geqslant \frac{2}{N}\,\frac{\rho+\frac{1}{2}}{\rho-(2\rho+1)E_2(\rho)} \end{aligned}\right\} \quad \text{general result for arbitrary } N, \rho, \alpha \quad (13)$$

Note that both fractional variance bounds are functions of N and ρ but not α. These are exact expressions so far, but further analysis requires characterization of the function $E_2(\rho)$. This function is easy to evaluate numerically, but first we consider its limiting behavior for large and small ρ. The general case is discussed and plotted at the end of Section III of this article.

A. High SNR Case

For large ρ, it can be shown that the function $E_2(\rho)$ is exponentially small,

$$E_2(\rho) \sim \rho^{-1/2} e^{-\rho/2} \qquad \rho \gg 1 \quad (14)$$

Thus, the fractional variance bounds can be written

$$\left.\begin{aligned} \frac{\text{var}(\hat{\rho})}{\rho^2} &\gtrsim \frac{2}{N}\left[1+\frac{2}{\rho}\right] \\ \frac{\text{var}(\hat{\alpha})}{\alpha^2} &\gtrsim \frac{2}{N}\left[1+\frac{1}{2\rho}\right] \end{aligned}\right\} \quad \rho \gg 1 \quad (15)$$

Both of these expressions are accurate within terms that are exponentially small in ρ (i.e., there are no $1/\rho^n$ terms for $n > 1$).

B. Low SNR Case

For small ρ, we make use of the Taylor series expansion for $u^2 \operatorname{sech}^2 u$ around $u = 0$,

$$u^2 \operatorname{sech}^2 u = u^2 - u^4 + \frac{2}{3}u^6 - \frac{17}{45}u^8 + \cdots \quad (16)$$

and apply the formula for the moments of a Gaussian random variable (Ref. 4) with mean and variance both equal to ρ,

$$\left.\begin{aligned} E\{u^2\} &= \rho^2+\rho \\ E\{u^4\} &= \rho^4+6\rho^3+3\rho^2 \\ E\{u^6\} &= \rho^6+15\rho^5+45\rho^4+15\rho^3 \\ E\{u^8\} &= \rho^8+28\rho^7+210\rho^6+420\rho^5+105\rho^4 \end{aligned}\right\} \quad (17)$$

This leads to a Taylor series expansion of the function $E_2(\rho)$ around $\rho = 0$:

$$E_2(\rho) = \rho - 2\rho^2 + 4\rho^3 - \frac{32}{3}\rho^4 + \cdots \quad (18)$$

When this expression is inserted back into Eq. (13), all of the ρ, ρ^2, and ρ^3 terms in the denominator cancel, i.e.,

$$\rho - (2\rho + 1) E_2(\rho) = \frac{8}{3}\rho^4 + \text{higher order terms} \qquad (19)$$

This deep singularity at $\rho = 0$ causes the bounds on the fractional variance to be very large for low SNR.

$$\left.\begin{aligned} \frac{\text{var}(\hat{\rho})}{\rho^2} &\gtrsim \frac{3}{2N\rho^4} \\ \frac{\text{var}(\hat{\alpha})}{\alpha^2} &\gtrsim \frac{3}{8N\rho^4} \end{aligned}\right\} \quad \rho \ll 1 \qquad (20)$$

III. Estimation with Subinterval Sampling

Now we consider the same type of bound for a more general model in which multiple subinterval samples are taken before their mean value has a chance to change sign. We assume M independent and equally spaced subinterval samples X_{ij}, $j = 1, \cdots, M$, for each of the N symbol intervals. The subinterval samples are modeled as

$$X_{ij} = D_i m_0 + n_{ij}\sigma_0 \qquad (21)$$

where $D_i = \pm 1$ is the same data modulation variable defined in Eq. (1), $\{n_{ij}\}$ are independent unit normal random variables, and m_0 and σ_0 denote the subinterval signal and noise parameters. Note that the M subinterval samples X_{ij}, $j = 1, \cdots, M$, are affected by one data modulation variable D_i and M independent noise variables n_{ij}, $j = 1, \cdots, M$.

Our estimation problem is still to estimate the signal-to-noise ratio and combiner weight parameters for the full symbol period. Each block of M subinterval samples X_{ij}, $j = 1, \cdots, M$, sums to form a symbol period sample x_i,

$$x_i = \sum_{j=1}^{M} X_{ij} \qquad (22)$$

This implies that the subinterval signal and noise parameters appearing in Eq. (21) are related to the full symbol period parameters by

$$\left.\begin{aligned} m_0 &= \frac{m}{M} = \frac{\rho}{M\alpha} \\ \sigma_0^2 &= \frac{\sigma^2}{M} = \frac{\rho}{M\alpha^2} \end{aligned}\right\} \qquad (23)$$

The log-likelihood function for the vector of subinterval observables $\mathbf{X} = (X_{11}, \cdots, X_{1M}, \cdots, X_{N1}, \cdots, X_{NM})$ is obtained analogously to Eq. (3) in terms of the subinterval signal and noise parameters as

$$\frac{1}{MN} \ln p(\mathbf{X}|m_0, \sigma_0) = -\frac{1}{2}\ln 2\pi - \ln \sigma_0 - \frac{\langle X^2\rangle_{MN}}{2\sigma_0^2} - \frac{m_0^2}{2\sigma_0^2} + \frac{1}{M}\left\langle \ln\cosh\frac{m_0 x}{\sigma_0^2}\right\rangle_N \qquad (24)$$

or, alternatively, in terms of the symbol period signal-to-noise ratio and combiner weight parameters as

$$\frac{1}{MN} \ln p(\mathbf{X}|\rho, \alpha) = -\frac{1}{2}\ln\left(\frac{2\pi}{M}\right) - \frac{1}{2}\ln\rho + \ln\alpha - \frac{\langle X^2\rangle_{MN}\alpha^2}{2\rho/M} - \frac{1}{2}\frac{\rho}{M} + \frac{1}{M}\langle \ln\cosh\alpha x\rangle_N \qquad (25)$$

In Eqs. (24) and (25), $\langle X^2\rangle_{MN}$ denotes the mean square value of the MN subinterval samples

$$\langle X^2\rangle_{MN} = \frac{1}{MN}\sum_{i=1}^{N}\sum_{j=1}^{M} X_{ij}^2 \qquad (26)$$

and $\langle \ln\cosh\alpha x\rangle_N$ is the same quantity appearing in Eqs. (3) and (5), i.e., an average based on the full symbol period samples x_i,

$$\begin{aligned} \langle \ln\cosh\alpha x\rangle_N &= \frac{1}{N}\sum_{i=1}^{N} \ln\cosh\alpha x_i \\ &= \frac{1}{N}\sum_{i=1}^{N}\ln\cosh\left(\alpha\sum_{j=1}^{M} X_{ij}\right) \end{aligned} \qquad (27)$$

We observe from comparing Eqs. (5) and (25) that the Cramer-Rao bounds for this problem can be obtained trivially from the bounds derived earlier by substituting $ME\{\langle\alpha^2 X^2\rangle_{MN}\}$ for $E\{\langle\alpha^2 x^2\rangle_N\}$ and $M^{-1}\ E\{\langle\alpha^2 x^2\ \text{sech}^2\ \alpha x\rangle_N\}$ for $E\{\langle\alpha^2 x^2\ \text{sech}^2\ \alpha x\rangle_N\}$. We note that

$$
\left.\begin{aligned}
M\,E\,\{\langle\alpha^2X^2\rangle_{MN}\} &= M\alpha^2(m_0^2+\sigma_0^2)\\
&= \rho\left(1+\frac{\rho}{M}\right)\\
\frac{1}{M}\,E\,\{\langle\alpha^2x^2\,\mathrm{sech}^2\,\alpha x\rangle_N\} &= \frac{1}{M}\,E_2(\rho)
\end{aligned}\right\}\qquad(28)
$$

where $E_2(\rho)$ is the same function defined earlier. The Fisher information matrix elements can immediately be evaluated from Eqs. (7) and (28):

$$
\left.\begin{aligned}
\frac{1}{MN}\,\rho^2J_{11} &= \frac{\rho}{M}+\frac{1}{2}\\
\frac{1}{MN}\,\rho\alpha J_{12} &= \frac{1}{MN}\,\rho\alpha J_{21} = -\left(\frac{\rho}{M}+1\right)\\
\frac{1}{MN}\,\alpha^2J_{22} &= 2+\frac{\rho}{M}-\frac{1}{M}\,E_2(\rho)
\end{aligned}\right\}\qquad(29)
$$

Inversion of this matrix produces the bounds on the fractional variance of the estimates $\widehat{\rho}, \widehat{\alpha}$:

$$
\left.\begin{aligned}
\frac{\mathrm{var}\,(\widehat{\rho})}{\rho^2} &\geqslant \frac{2}{N}\,\frac{2M+\rho-E_2(\rho)}{M\rho-(2\rho+M)E_2(\rho)}\\
\frac{\mathrm{var}\,(\widehat{\alpha})}{\alpha^2} &\geqslant \frac{2}{N}\,\frac{\rho+M/2}{M\rho-(2\rho+M)E_2(\rho)}
\end{aligned}\right\}\ \text{general result for arbitrary } N, M, \rho, \alpha\qquad(30)
$$

A. High SNR Case

For high values of ρ, the bounds reduce to

$$
\left.\begin{aligned}
\frac{\mathrm{var}\,(\widehat{\rho})}{\rho^2} &\gtrsim \frac{2}{MN}\left(1+\frac{2M}{\rho}\right)\\
\frac{\mathrm{var}\,(\widehat{\alpha})}{\alpha^2} &\gtrsim \frac{2}{MN}\left(1+\frac{M}{2\rho}\right)
\end{aligned}\right\}\quad \rho \gg 1\qquad(31)
$$

As before, these bounds are accurate within terms that are exponentially decreasing with ρ. We see that the performance bound improves with the total number of samples MN, regardless of whether they are subinterval samples or symbol period samples. However, this conclusion is not correct if the number of *subinterval* samples gets arbitrarily large. If M is increased beyond the value of ρ, the bounds eventually saturate at

$$
\left.\begin{aligned}
\frac{\mathrm{var}\,(\widehat{\rho})}{\rho^2} &\gtrsim \frac{4}{N\rho}\\
\frac{\mathrm{var}\,(\widehat{\alpha})}{\alpha^2} &\gtrsim \frac{1}{N\rho}
\end{aligned}\right\}\quad 1 \ll \rho \ll M\qquad(32)
$$

B. Low SNR Case

For low SNR, the deep singularity at $\rho = 0$ in the denominator of the accuracy bounds is partially relaxed for $M > 1$, i.e.,

$$
M\rho-(2\rho+M)E_2(\rho) = 2(M-1)\,\rho^2 + \text{higher order terms}\qquad \text{for } M > 1\qquad(33)
$$

The "higher order terms" in Eq. (33) are small with respect to $(M-1)\,\rho^2$ as ρ gets small, no matter how large M is. The accuracy bounds are approximately

$$
\left.\begin{aligned}
\frac{\mathrm{var}\,(\widehat{\rho})}{\rho^2} &\gtrsim \frac{2}{N\rho^2}\,\frac{M}{M-1}\\
\frac{\mathrm{var}\,(\widehat{\alpha})}{\alpha^2} &\gtrsim \frac{1}{2N\rho^2}\,\frac{M}{M-1}
\end{aligned}\right\}\quad \rho \ll 1 < M\qquad(34)
$$

We see from Eqs. (20) and (34) that the performance bounds for small ρ improve by a large factor $3/(8\rho^2)$ in going from $M = 1$ to $M = 2$, and then by only an additional factor of 2 from $M = 2$ to $M = \infty$.

C. Large Number of Subinterval Samples Case

We have seen that the performance bound saturates at a nonzero limit for both the low SNR and high SNR cases, as the number of subinterval samples M goes to infinity. This saturation value can be calculated from Eq. (30) for all SNR values, in terms of the function $E_2(\rho)$.

$$
\left.\begin{aligned}
\frac{\mathrm{var}\,(\widehat{\rho})}{\rho^2} &\gtrsim \frac{4}{N\,[\rho-E_2(\rho)]}\\
\frac{\mathrm{var}\,(\widehat{\alpha})}{\alpha^2} &\gtrsim \frac{1}{N\,[\rho-E_2(\rho)]}
\end{aligned}\right\}\quad M \gg \max\,(\rho, 1)\qquad(35)
$$

D. General Case

The Cramer-Rao bounds for the general case are plotted in Figs. 1 and 2 for the signal-to-noise ratio and combiner weight estimates, respectively. Each curve shows the lower bound on the fractional estimator variance times the number of symbol period samples N, as a function of signal-to-noise ratio ρ. Curves are drawn for various numbers of subinterval samples M, including the case $M = 1$, which is equivalent to the case of full symbol period sampling considered in Section II of this article.

The ordinate in these plots may be interpreted as a lower bound on the number of symbol period samples N required to achieve a fractional estimator variance of 100%. If a smaller fractional estimator variance is desired, say ϵ, the bound on the required number of samples is simply increased by the factor $1/\epsilon$.

IV. Conclusions

Figures 1 and 2 present a strong case for taking subinterval samples. In the low SNR region, split-symbol estimators can potentially reduce the number of required samples by orders of magnitude relative to estimates based entirely on full symbol period samples. The bulk of this reduction results from splitting the symbol period in half ($M = 2$), and additional improvement is limited to 3 dB as the number of subinterval samples is increased further. In the high SNR region, it pays to keep increasing the number of subinterval samples, but rapidly diminishing returns are encountered when the number of subinterval samples is increased beyond the true value of the signal-to-noise ratio. Of course, if the true SNR is extremely high, the practical limit on the number of worthwhile subinterval samples may be set by bandwidth constraints rather than SNR constraints.

A caveat must be attached to all of the analysis, and hence the conclusions, in this article. Performance bounds derived here apply only to unbiased estimators. Perhaps the requirement that the full symbol period estimator be perfectly unbiased is too tight a constraint to impose in the low SNR region. Further work should investigate the possible trade-offs between estimator bias and estimator variance, especially for full symbol period estimators at low SNR.

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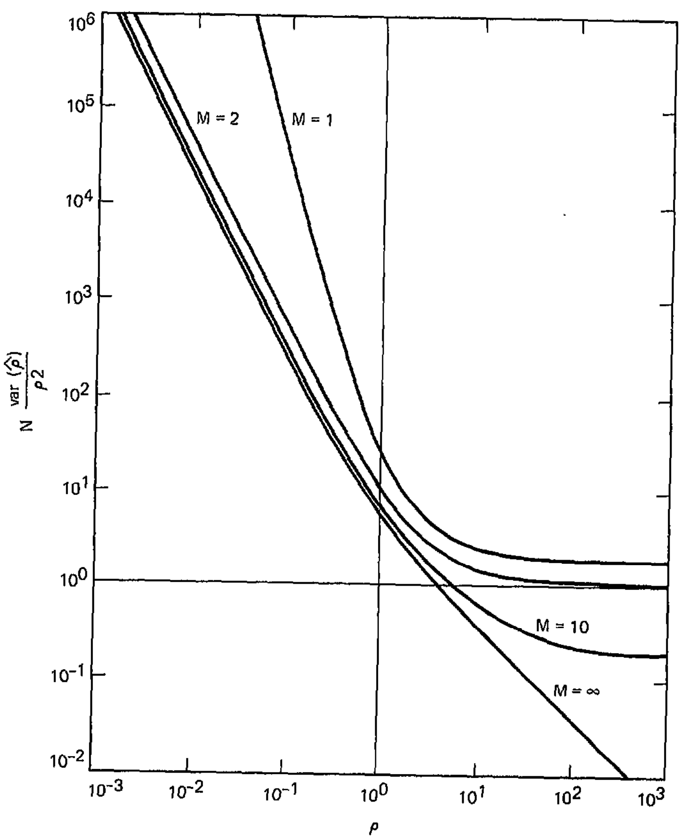

Fig. 1. Cramer-Rao lower bound on unbiased signal-to-noise ratio estimator performance

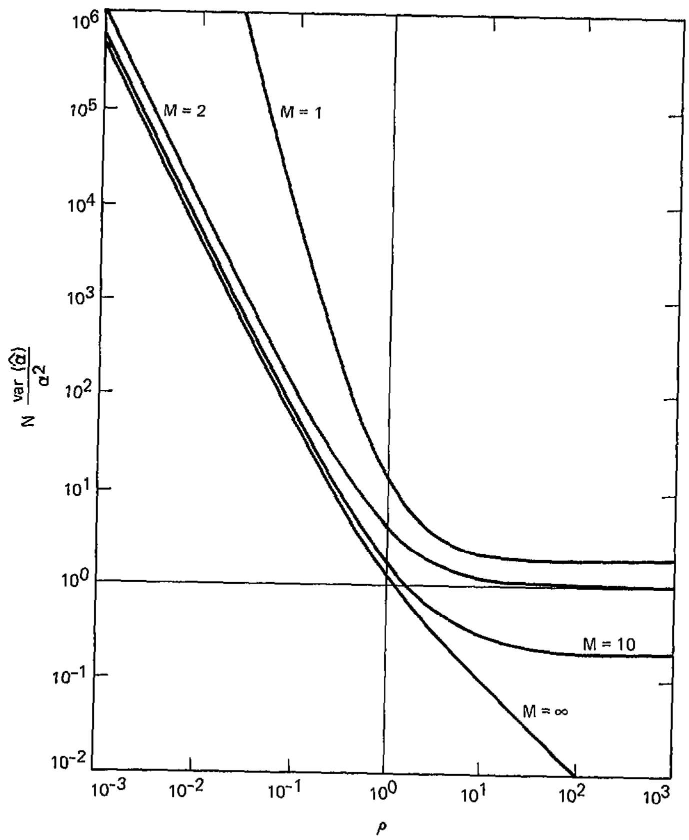

Fig. 2. Cramer-Rao lower bound on unbiased combiner weight estimator performance