Detection of Signals by Weighted Integrate-and-Dump Filter

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A Weighted Integrate-and-Dump Filter (WIDF) is presented that results in reducing those losses in telemetry symbol SNR which occur in digital Integrate-and-Dump Filters (IDFs) when the samples are not phase locked to the input data symbol clock. The Minimum Mean Square Error (MMSE) criterion is used to derive a set of weights for approximating the analog integrate-and-dump filter, which is the matched filter for detection of signals in additive white Gaussian noise. This new digital matched filter results in considerable performance improvement compared to unweighted digital matched filters. An example is presented for a sampling rate of four times the symbol rate. As the sampling offset (or phase) varies with respect to the data symbol boundaries, the output SNR varies 1 dB for an unweighted IDF, but only 0.3 dB for the optimum WIDF, averaged over random data patterns. This improvement in performance relative to unweighted IDF means that significantly lower sampling and processing rates can be used for given telemetry symbol rates, resulting in reduced system cost.

I. Introduction

The effect of "offset sampling" for the unweighted digital Integrate-and-Dump Filter (IDF) was considered in [1]. A set of practical guidelines is outlined in [1] that can be used to determine the appropriate sampling period and the filter bandwidth for the digital IDF. In addition, the effect of offset sampling was comprehensively studied, and the degradation due to approximating the analog IDF with digital IDF was analyzed. By "offset sampling," we mean that the sampling clock is not phase locked to the telemetry symbol clock.

The IDF is the optimum matched filter for detection of signals in Additive White Gaussian Noise (AWGN). In this article, a new class of digital matched filters is considered which decreases the degradation due to approximating the analog IDF with the digital IDF.

The problem is formulated in the context of waveform tracking. The waveform which is tracked by the linear estimator is the sampled output of the analog IDF. The mean square error criterion is used to derive the digital matched filter. The observed signal for derivation of the digital matched filter is the sampled sequence of the received signal during a single symbol time of T seconds. In a sampled data system, normally an anti-aliasing low-pass (or bandpass) filter is used to filter the analog source. The effect of this filter is specifically considered.

In Section II, the underlying system is described. In Section III, the new digital matched filter is formulated. In Section IV, a linear system is proposed that generates the necessary autocorrelation functions for computation of the optimal weight sequence. In Section V, the average signal response expression is derived for the weighted integrate-and-dump filter in the presence of offset sampling. In Section VI, the noise response of the system is considered. In Section VII, the definition of SNR loss due to the approximation of the analog IDF with the digital IDF is stated. In Section VIII, results of the previous sections are used to find the optimum weighted IDF for a special case when an ideal filter is used prior to sampling the observed signal and the transmitted signal is a sequence of rectangular pulses. The relationship of our approach to linear equalizers, Wiener filtering, and decoding for intersymbol interference channels is also discussed in this section. In Section IX, the performance of the system is evaluated. A glossary of terms appears at the end of the article.

II. System Description

The received signal plus noise is denoted by $r(t) = s(t - \tau_0) + n(t)$, where $s(t)$ is the signal, $n(t)$ is AWGN and τ_0 is the delay from the transmitter to the receiver. The transmitted signal $s(t)$ is

$$s(t) = \sum_k a_k \, p(t - kT) \tag{1}$$

a sequence of pulses with a pulse-shaping waveform $p(t)$. The input alphabet U is a finite alphabet with $a_i \epsilon U = \{\pm 1, \pm 2, \ldots, \}$.

The analog IDF is shown in Fig. 1(a). The analog IDF is an ideal matched filter when $p(t)$ is a rectangular pulse from $t = 0$ to $t = T$. It detects the kth symbol by integrating over time $kT + \tau_0$ to $(k+1)T + \tau_0$.

The digital IDF is depicted in Fig. 1(b). In the digital implementation a low-pass anti-aliasing filter is used for filtering the input signal. In this article the one-sided bandwidth of this filter is denoted by W(Hz). The filter output is sampled, with the ith sample occurring at time $iT_s + \tau_1$. The digital IDF detects the kth symbol by summing all the samples from $t = kT + \tau_0$ to $t = (k+1)T + \tau_0$.

We assume that there is perfect symbol synchronization at the receiver, in the sense that the beginning and end times of each symbol are known. For the kth symbol the "Sampling Offset" is defined by the length of time after the start of the symbol to when the first sample in the symbol occurs. This time is $(iT_s + \tau_1) - (kT + \tau_0)$ for the smallest i such that the expression is nonnegative. The first sample of each symbol may occur anywhere between 0 and T_s seconds after the beginning of the symbol. A typical symbol waveform and the sampling points are shown in Fig. 2.

To illustrate the effect of offset in sampling, Fig. 3 depicts one pulse of the sampled waveform for an alternating rectangular data pattern of length 21, when the anti-aliasing filter is an ideal low-pass filter. The sampled waveform for the 11th symbol, a -1 pulse, is shown in Fig. 3 for $WT = 2$, and for $T = 4T_s$. The filtered waveform is not rectangular due to the finite bandwidth of the anti-aliasing filter. In Fig. 3, for every sampling offset value, with increments $T_s * 0.05$, a unique English letter (a through t) is used to indicate the point at which the sample occurs. Every letter occurs four times, corresponding to the four samples per symbol.

In an earlier article [1], we considered the effect of offset in the digital IDF. It was shown that the loss due to offset in sampling is significant when the number of samples per symbol is low ($T/T_s < 8$). The loss depends on the bandwidth W, the sampling rate, and the relative phase of the samples and symbols. If the signal is sampled at the optimum sampling time the loss is relatively small. This loss is due to bandwidth limiting of the input signal. For example, when four samples per symbol are used in the digital IDF, i.e., $T/T_s = 4$ and $WT = 2$, the worst case loss is approximately 1.2 dB averaged over random data patterns. This occurs when the offset is zero, indicated by the letter a in Fig. 3. The minimum loss is 0.35 dB when the offset is $T_s/2$, indicated by the letter j in Fig. 3. Thus a variation of 0.8 dB in the loss occurs due to the phase of the offset in sampling. To decrease this variation and, as a result, to reduce system sensitivity due to the offset, we are led to consider the Weighted Integrate-and-Dump Filter (WIDF), which is the main subject of this article.

III. Derivation of the WIDF Using MSE

In Fig. 1 the IDF is shown for both the analog and digital implementations. The sampled output of the analog IDF is $A(kT)$, denoted as simply A_k. In this section, the minimum mean square error criterion is used to estimate the sequence A_k from the digital samples.

We formulate the problem in the context of Fig. 4(a). In this figure the digital IDF filter is denoted by $f(\cdot)$. The operator $f(\cdot)$ maps the observation vector $\mathbf{y} = (y_1, y_2, \cdots, y_N)$ in the kth symbol onto $\hat{A}(kT)$, an estimate of $A(kT)$.

We seek to find $f(\cdot)$ such that the minimum mean square error criterion is minimized, i.e., we minimize

$$E\,[(A_k - \hat{A}_k)^2 \mid \mathbf{y}] \tag{2}$$

where $\hat{A}_k = f(\mathbf{y})$, and $E[\cdot]$ denotes the expectation operator. Note here that the estimate of A_k is based only on the observation vector during a single symbol time. In Section III, we briefly discuss the case when this restriction is relaxed, when the relationship of WIDF to linear equalizers is pointed out.

It can be shown [2]-[4] that the optimal $f(\mathbf{y})$ is the conditional expectation of A_k conditioned on the observed vector $\mathbf{y}$:

$$f(\mathbf{y}) = E[A_k \mid \mathbf{y}] \tag{3}$$

Since $r(t) = s(t) + n(t)$, and $n(t)$ is AWGN, the conditional probability density function of $r(t)$ conditioned on the input data sequence a is Gaussian. However, the conditional probability density function of A_k conditioned on $\mathbf{y}$ is not Gaussian due to Inter Symbol Interference (ISI), and it is almost impossible to explicitly evaluate this probability density function. We assume this density function is Gaussian, and hence the conditional expectation is a linear function of the observed vector $\mathbf{y}$. Thus, under this assumption,

$$E[A_k \mid \mathbf{y}] = \sum_{i=1}^{N} w_i y_i \tag{4}$$

We shall not state the complete derivation of the Linear Minimum Mean Square Error (LMMSE) criterion. Interested readers could refer to [2], [3] to obtain the complete derivation of the following result.

The optimum weight sequence $\mathbf{w} = (w_1, w_2, \ldots, w_N)$ may be expressed in terms of the second order statistics of the observed vector $\mathbf{y}$ as

$$\mathbf{w} = \mathbf{R}_{\mathbf{yy}}^{-1} \mathbf{R}_{\mathbf{yA}} \tag{5}$$

where the matrix $\mathbf{R}_{\mathbf{yy}}$ is the autocorrelation matrix (assuming $\mathbf{R}_{\mathbf{yy}}$ is nonsingular) with elements $E[y_j y_i]$, and $\mathbf{R}_{\mathbf{yA}}$ is the cross-correlation vector between y_i and A_k, with elements $E[y_i A_k]$, where $E[\cdot]$ denotes the expectation operator.

$$\mathbf{R}_{\mathbf{yy}} = \begin{bmatrix} E[y_1 y_1] & E[y_1 y_2] & \cdots & E[y_1 y_N] \\ E[y_2 y_1] & E[y_2 y_2] & & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ E[y_N y_1] & & & E[y_N y_N] \end{bmatrix} \tag{6}$$

and

$$\mathbf{R}_{\mathbf{yA}} = \begin{bmatrix} E[y_1 A_k] \\ E[y_2 A_k] \\ \cdot \\ \cdot \\ \cdot \\ E[y_N A_k] \end{bmatrix} \tag{7}$$

In order to evaluate the matrix $\mathbf{R}_{\mathbf{yy}}$ and the vector $\mathbf{R}_{\mathbf{yA}}$, in the following section a linear system is specified which generates the autocorrelation functions $R_{yy}(\tau)$ and $R_{yA}(\tau)$. The matrix $\mathbf{R}_{\mathbf{yy}}$ and the vector $\mathbf{R}_{\mathbf{yA}}$ are obtained by sampling the autocorrelation functions at time $t = iT_s + \delta$, for $i \epsilon [1,N]$, where δ is the offset.

IV. Evaluation of Matrix $\mathbf{R}_{\mathbf{yy}}$ and Vector $\mathbf{R}_{\mathbf{yA}}$

The following results are a direct consequence of the application of second order statistics of a stationary stochastic process to the input-output relations of a linear system [3], [5]. Throughout this article on-line "$*$" denotes convolution, and superscript "$*$" denotes the complex conjugate.

To compute the autocorrelation function $R_{yy}(\tau)$, note that

$$y(t) = s(t) * h(t) + n(t) * h(t) \tag{8}$$

let $x(t) = s(t) * h(t)$ represent the filtered signal component, and $z(t) = n(t) * h(t)$ represent the filtered noise component of $y(t)$. The autocorrelation function $R_{yy}(\tau)$ may be expressed in terms of the cross-correlation of $R_{ys}(\tau)$ and $R_{yn}(\tau)$ as

$$R_{yy}(\tau) = R_{ys}(\tau) * h(\tau) + R_{yn}(\tau) * h(\tau) \tag{9}$$

The two cross-correlation functions $R_{ys}(\tau)$ and $R_{yn}(\tau)$ are

$$\begin{aligned} R_{ys}(\tau) &= R_{ss}(\tau) * h^*(-\tau) \\ R_{yn}(\tau) &= R_{nn}(\tau) * h^*(-\tau) \end{aligned} \tag{10}$$

To compute $R_{yA}(\tau)$, note that

$$A_0 = \int_0^T s(\xi) d\xi \tag{11}$$

The cross-correlation function $R_{yA}(\tau)$ may be expressed as

$$R_{yA}(\tau) = E[y(\tau)A(T)]$$

$$= -\int_{\tau}^{\tau-T} \left(R_{ys}(\xi) + R_{ny}(\xi)\right) d\xi \qquad (12)$$

for a fixed T, where the pair $R_{ys}(\tau)$ and $R_{yn}(\tau)$ are given by (10).

Figure 5 depicts a linear system which can be used to evaluate the two autocorrelation functions R_{yy} and R_{yA}. In Fig. 5 the input to the system is the autocorrelation function of the received signal, and the outputs are the desired autocorrelation functions $R_{yy}(\tau)$ and $R_{yA}(\tau)$. The sampled sequence of $R_{yy}(\tau)$ and $R_{yA}(\tau)$ generates the corresponding matrices $\mathbf{R_{yy}}$ and $\mathbf{R_{yA}}$. This illustrates a method to obtain the correlation necessary to calculate the optimal weight sequence w, from (5).

V. Average Signal Response

We now seek to determine the average signal response for the output of the WIDF.

The response of the low pass filter to the observed signal $r(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(t-\xi)\, s(\xi - \tau_0)\, d\xi$$

$$+ \int_{-\infty}^{\infty} h(t-\xi)\, n(\xi)\, d\xi \qquad (13)$$

Using (1) for $s(t)$ we have

$$y(t) = \sum_{k'=-\infty}^{\infty} \int_{-\infty}^{\infty} a_{k'}\, h(t-\xi)\, p(\xi - k'T - \tau_0)\, d\xi$$

$$+ \int_{-\infty}^{\infty} h(t-\xi)\, n(\xi)\, d\xi \qquad (14)$$

The signal $y(t)$ is sampled each T_s sec, at time $iT_s + \tau_1$. We denote $y(iT_s + \tau_1)$ as y_i. Taking the expectation of (14) conditioned on a given data sequence **a** and noting that the noise $n(t)$ is assumed to have zero mean, the conditional expectation of y_i is

$$E[y_i \mid \mathbf{a}] = \sum_{k'} a_{k'} \int_{-\infty}^{\infty} h(iT_s + \tau_1 - \xi)\, p(\xi - kT - \tau_0) d\xi \qquad (15)$$

With a change of variable (15) can be written as

$$E[y_i \mid \mathbf{a}] = \sum_{k'} a_{k'} \int_{-\infty}^{\infty} h(iT_s - kT + \delta - x)\, p(x) dx \qquad (16)$$

where $\delta = \tau_1 - \tau_0$. Let

$$q_i(k, \delta) = \int_{-\infty}^{\infty} h(iT_s - kT + \delta - x)\, p(x) dx \qquad (17)$$

represent the signal response of the filter at time $iT_s + \tau_1$ due to a single pulse at time $kT + \tau_0$. For simplicity we denote $q_i(k, \delta)$ as simply $q_i(k)$. The total average signal response from (16), for a given fixed δ, may be expressed as

$$E[y_i \mid \mathbf{a}, \delta] = \sum_{k'} a_{k'}\, q_i(k') \qquad (18)$$

Let I^k be the set of all i such that the ith sample falls in the kth symbol time, i.e.,

$$I^k = \{i\!: \; kT \leqslant iT_s + \delta < (k+1)T\} \qquad (19)$$

The WIDF output for the kth symbol, denoted by A_k, is

$$\hat{A}_k = \sum_{i \in I^k} y_i \mathrm{w}_i \qquad (20)$$

The expectation of $\hat{A}_k$ over the noise, conditioned on **a** and δ, is

$$E[\hat{\mathrm{A}}_k \mid \mathbf{a}, \delta] = \sum_{i \in I^k} \sum_{k'} a_{k'}\, q_i(k')\, w_i \qquad (21)$$

To further simplify this expression, define the event indicator function which is 1 if and only if $i \epsilon I^k$, i.e.,

$$\ell_i(\delta, k) = \begin{cases} 1 & \text{when } kT \leqslant iT_s + \delta < (k+1)T \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Thus from (21) we have

$$E[\hat{A}_k \mid \mathbf{a}; \delta] = \sum_i \sum_{k'} a_{k'} \, \ell_i(\delta, k) \, w_i \, q_i(k') \tag{23}$$

VI. Noise Response

Now we consider the noise response of WIDF in order to compute the total SNR at the output of the WIDF. Let z_i denote the sampled noise response of the filter at time $iT_s + \tau_1$.

$$z_i = \int_{-\infty}^{\infty} n(\xi) h(iT_s + \tau_1 - \xi) \, d\xi \tag{24}$$

Since the WIDF is a linear system, the variance of $\hat{A}_k$ conditioned on **a** is equal to the variance of the response of the kth symbol due to noise alone, i.e., it is independent of $s(t)$. The variance of $\hat{A}_k$ is

$$\mathrm{var}[\hat{A}_k \mid \delta, \mathbf{a}] = \sum_{i \epsilon I^k} \sum_{j \epsilon I^k} E[z_i z_j] \, w_i w_j \tag{25}$$

Note that this variance does depend on δ and k, since the number of samples occurring in the kth symbol depends on δ. Using (22) and noting that $E[n(t)n(\tau)] = N_0/2 \, \delta_0(t - \tau)$ (δ_0 here is the Dirac delta function), we have

$$\mathrm{var}[\hat{A}_k \mid \delta, \mathbf{a}] = \sum_i \sum_j \ell_i(k, \delta) \, \ell_j(k, \delta) \, w_i w_j R_z\big((i-j)T_s\big)$$

$$R_z[(i-j)T_s] = \frac{N_0}{2} \int_{-\infty}^{\infty} h\big((i-j)T_s - \xi\big) h(\xi) d\xi \tag{26}$$

where $R_z(\cdot)$ is the autocorrelation of z_i.

VII. Definition of SNR Loss

In this section, we define a measure to evaluate the degradation which results in using the WIDF as opposed to analog IDF. The analog IDF of Fig. 1(a) is the optimum matched filter when $p(t) = 1$ for $0 < t < T$ and zero otherwise. We define SNR at the IDF output as the ratio of the square of the mean to the variance. Denoting SNR_A for the analog IDF, it is well known [6] that

$$\mathrm{SNR}_A = \frac{2A^2 T}{N_0} \tag{27}$$

We assume with no loss of generality that the signal amplitude $A = 1$. Denoting SNR_D as the SNR at the output of the WIDF, we compare the SNR_D with the analog IDF by considering the ratio

$$\gamma \triangleq \frac{\mathrm{SNR}_D}{\mathrm{SNR}_A} \tag{28}$$

Define

$$\mathrm{SNR}_D \triangleq \frac{(E[\hat{A}_k \mid \mathbf{a}, \delta])^2}{\mathrm{var}[\hat{A}_k \mid \mathbf{a}, \delta]}$$

and then we have

$$\gamma = \frac{N_0}{2T} \mathrm{SNR}_D \tag{29}$$

In the remaining sections $\gamma_{\mathrm{dB}} = 10^* \log_{10}(\gamma)$ (dB) is computed for various filters and data patterns. Normally $\gamma < 1$, because the digital IDF has a loss with respect to the analog IDF. The loss in dB is $-\gamma_{\mathrm{dB}}$. The minimum loss corresponds to the maximum γ which typically approaches one ($\gamma_{\mathrm{dB}} = 0$ dB). Maximum loss is unbounded and corresponds to infinity (in dB).

VIII. WIDF for Rectangular Pulse and Ideal Filter

In general, the pulse shape $p(t)$ may be chosen to take numerous shapes (e.g., raised root-cosine). In some cases, it is chosen to extend over more than one symbol duration, such as for partial response signaling (sometimes referred to as correlated coding or controlled intersymbol interference). For bandwidth-limited channels, the pulse shape and duration are selected to increase the bandwidth efficiency of the communication system.

The motivation to consider the ideal low pass filter is to eliminate aliasing in an ideal manner. The use of a realizable filter such as Butherworth or Chebyshev [7] does not greatly influence the results, since the realizable filter can be considered as an approximation to the unrealizable filter with finite group delay [7].

We consider only non-overlapping rectangular pulses throughout the rest of this article, since this pulse shape has traditionally been used for NASA's deep space missions.

In the case of the rectangular pulse we simply have

$$p(t) = \begin{cases} 1 & t \in [0, T] \\ 0 & \text{otherwise} \end{cases}$$

Then from (17), $q_i(k)$ is

$$q_i(k) = \int_0^T h(iT_s - kT + \delta - x)\,dx \tag{30}$$

and from (21), the average signal response is

$$E[\hat{A}_k \mid \mathbf{a}, \delta] = \sum_{k'} \sum_{i \in I^k} a_{k'} q_i(k') w_i \tag{31}$$

The ideal low pass filter with unit gain and low pass bandwidth W - Hz is noncausal with impulse response

$$h(t) = 2W \frac{\sin 2\pi W t}{2\pi W t} = 2W \sin c(2\pi W t) \tag{32}$$

The expression for the signal response in (30) does not evaluate to a closed form in this case, but is

$$q_i(k) = \frac{1}{\pi} \int_0^T \frac{\sin 2\pi W(iT_s - kt + \delta - x)}{(iT_s - kT + \delta - x)}\,dx \tag{33}$$

It is possible to express (33) in terms of

$$Si(x) = \int_0^x \frac{\sin u}{u}\,du$$

as

$$q_i(k) = \frac{1}{\pi}\,[Si(2\pi W(iT_s - (k+1)T + \delta)) - Si(2\pi W(iT_s - kT + \delta))] \tag{34}$$

Inserting (34) into (31) yields the average signal response. To find the noise variance, it suffices to note that the noise spectral density at the output of the filter is

$$S_n(f) = \begin{cases} \dfrac{N_0}{2} & |f| < W \\ 0 & \text{otherwise} \end{cases} \tag{35}$$

and thus the autocorrelation function is

$$R_n(\tau) = \frac{N_0 W}{2} \frac{\sin 2\pi W \tau}{2\pi W \tau} \tag{36}$$

Thus, the noise variance at the output of WIDF can be expressed from (26) and (36) as

$$\mathrm{var}[\hat{A}_k \mid \delta, \mathbf{a}] = \frac{N_0 W}{2} \sum_i \sum_j w_i \ell_i(k, \delta) w_j \ell_j(k, \delta) \sin c(2\pi(i - j)2WT_s) \tag{37}$$

Thus from (29) and (37), γ can be evaluated for arbitrary $\mathbf{w}$ and rectangular pulse shapes and ideal filters as

$$\gamma = \frac{\left[\frac{1}{\pi} \sum_{i \in I^k} \sum_{k'} a_{k'} w_i \left(Si(2\pi W(iT_s - (k+1)T + \delta)) - Si(2\pi W(iT_s - kT + \delta))\right)\right]^2}{WT \sum_{i \in I^k} \sum_{j \in I^k} w_i w_j \sin|c(2\pi W(i - j)T_s)} \tag{38}$$

The next step is to compute the optimum weights according to (5). This requires the evaluation of R_{yy} and R_{yA} given by (9) and (12). We consider the special case in this section where the filter $h(t)$ is an ideal rectangular filter and $p(t)$ is a rectangular pulse. Referring to Fig. 5, and using the $h * (-\tau) = h(\tau)$, one can verify that for an ideal filter

$$R_{yy}(\tau) = R_{ss}(\tau) * h(\tau) + R_{nn}(\tau) * h(\tau) \tag{39}$$

The signal autocorrelation function $R_{ss}(\tau)$ for a random binary waveform [7] is

$$R_{ss}(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & |\tau| < T \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

The ideal filter impulse response as in (32) is

$$h(t) = 2W \sin c(2\pi W t) \tag{41}$$

For simplicity, let $R_x(\tau) = R_{ss}(\tau) * h(\tau)$ and $R_z(\tau) = R_{nn}(\tau) * h(\tau)$. Thus $R_x(\tau)$ is

$$R_x(\tau) = \int_{-\infty}^{\infty} R_s(\beta)\, h(\tau - \beta) d\beta \tag{42}$$

Since the input signal autocorrelation function (40) is nonzero only in the interval $[-T, T]$, (42) is

$$R_x(\tau) = \int_{-T}^{T} R_{ss}(\beta)\, h(\tau - \beta) d\beta \tag{43}$$

This integral can be explicitly evaluated for $h(t)$ in (41) by decomposing it into two successive integrals

$$R_x(\tau) = \int_{-T}^{0} R_{ss}(\beta)\, h(\tau - \beta) d\beta + \int_{0}^{T} R_{ss}(\beta)\, h(\tau - \beta) d\beta \tag{44}$$

After some manipulation (44) can be explicitly evaluated in terms of the $Si(\cdot)$ function, and it yields

$$\begin{aligned} R_x(\tau) = \frac{1}{\pi}\Bigg\{ &\frac{\tau}{T}\,[Si(\mathrm{B}(\tau - T)) + Si(\mathrm{B}(\tau + T)) - 2\,Si(\mathrm{B}\tau)] \\ &+ Si(\mathrm{B}(\tau - T)) - Si(\mathrm{B}(\tau - T)) \\ &+ \frac{1}{\mathrm{B}T}\,[\cos(\mathrm{B}(\tau + T)) + \cos(\mathrm{B}(\tau - T)) - 2\cos(\mathrm{B}\tau)]\Bigg\} \end{aligned} \tag{45}$$

where $\mathrm{B} = 2\pi W$.

To evaluate $R_z(\tau)$, consider

$$R_z(\tau) = \int_{-\infty}^{\infty} R_n(\xi)\, h(\tau - \xi)\, d\xi \tag{46}$$

which is simply

$$R_z(\tau) = \frac{W N_0}{2} \sin c(2\pi W \tau) \tag{47}$$

To evaluate $R_{yA}(\tau)$ we need to integrate (45) and (47) over $[t - T, t]$. The expression in (45) does not evaluate to a closed form expression, but integrating (47) over this interval yields

$$\int_{t-T}^{t} R_z(\tau) = \frac{N_0 W}{2}\,(-Si(\mathrm{B}(t - T)) + Si(\mathrm{B}\tau)) \tag{48}$$

Thus, we have

$$R_{yA} = \int_{\tau-T}^{\tau} R_x(t) dt + \frac{N_0 W}{2}\,(Si(\mathrm{B}\tau)) - Si(\mathrm{B}(\tau - T)) \tag{49}$$

and $R_{yy}(\tau)$ is

$$R_{yy}(\tau) = R_x(\tau) + R_z(\tau) \tag{50}$$

where $R_x(\tau)$ is defined in (45) and $R_z(\tau)$ is defined in (47). The optimum weights are calculated using (49) and (50) in (5).

A. Relationship to Linear Equalizer

In general, the signal processing algorithm that is designed to compensate for the ISI of the communication channel is referred to as an "equalizer." The most common method for equalization is a transversal filter [8], which is designed such that its coefficients optimize the performance of a system according to criteria selected by the designer.

When the MSE criterion is used to obtain the tap weight coefficients of the equalizer, the equalizer is equivalent to the WIDF when N, the length of the observation vector y in (2), exceeds the number samples in a single symbol time, i.e., $N > T/T_s$. It is pointed out that all our results will hold in this case, and our analysis for derivation of the WIDF can be effectively used for designing LMSE equalizers.

The optimal decoding algorithm for channels with ISI uses the maximum likelihood sequence estimation. Viterbi and Omura [9] discuss optimal decoding for ISI channels using the maximum likelihood sequence estimation, and they formulate the application of the Viterbi algorithm for estimating the data sequence, which results in a nonlinear estimator.

B. Relationship to Wiener and Kalman Filtering

If the length of the observation vector y in (2) is infinite ($N = \infty$), it is well known that the optimal matched filter is the discrete time Wiener filter [2]. The Wiener–Hopf method requires the factorization of a spectral density matrix. Analytical solutions for this method are very difficult to derive, and even when they do exist, it is an arduous task to physically realize such filters.

For lumped processes [2] which result by passing the received signal through a realizable filter, it is possible to model the observation process using a state space model. In this case Kalman filtering [2]-[4] can be applied to both vector observation (finite N) and time varying state space models. That subject is beyond the scope of this article.

IX. Performance Analysis

In this section we compute the set of weight coefficients for the case when $N = 4$, evaluate the degradation of the WIDF, and compare its loss with the digital IDF.

The software simulation programs explicitly compute (38) for arbitrary input signal sequences **a**, when an ideal filter is used and the input pulse shape $p(t)$ is a rectangular pulse.

In Table 1, the optimum set of weight coefficients for the case when $T/T_s = 4$ and $WT = 2$ is shown. These weights were computed using (5) and computing R_{yy} and R_{yA} using (49) and (50).

The output of the ideal low pass filter depends on both the past and future inputs. To approximate this, we consider a 21-symbol block, and the 11th symbol is analyzed for each data pattern. A block of 21 symbols was found to be sufficiently long to analyze the IDF [1].

Figure 6 shows the performance when the input data pattern **a** is a sequence of alternating +1, -1 sequences, i.e., $\mathbf{a} = (+1, -1, +1, -1, \ldots)$. The degradation for the 11th symbol is shown for both the IDF and WIDF. The WIDF is less lossy than the IDF for all values of offset. For the best offset, 0.5, the loss is 0.4 dB for the WIDF and 0.46 dB for the IDF, a minimal difference. This loss is mainly due to the bandwidth limiting. For the worst case offset, 0.0, the WIDF is more than 1.0 dB better than the IDF, with a loss of 0.77 dB for the WIDF and 1.81 dB for the IDF. The variation in the degradation due to offset decreases from 1.35 dB to only 0.33 dB for the WIDF.

In Fig. 7, the average loss is shown for a random binary vector of 4640 symbols, using the same set of weighing coefficients. For each offset, the average is computed by breaking the 4640 symbol vector into 220 blocks 21 symbols long and computing the loss of the 11th symbol for each block, and finally computing the average loss. For the best offset, the losses for the WIDF and IDF are again similar, and they are slightly less than for the alternating data pattern. The worst case losses, averaged over the data patterns, are 1.26 dB for the IDF but only 0.68 dB for the WIDF, an improvement of 0.58 dB. For the WIDF, the variation in average performance over offset is less than 0.3 dB.

X. Conclusion

In this article, based on the MSE criterion, a new class of digital matched filters was derived, which approximates the analog IDF. A linear system was outlined that generates the autocorrelation functions necessary to evaluate the WIDF. The SNR loss due to using WIDF was formulated.

The WIDF weighting coefficients which are optimum in the mean square sense were computed for the case when an ideal filter and rectangular pulse shape are used. The performance for this case was evaluated for the case of four samples per symbol. It was shown that the variation due to offset was reduced to under 0.3 dB for the WIDF, from almost 1 dB for the IDF, averaged over a random pattern. Compared to a system with the samples phase locked to the symbol clock, a WIDF with offset sampling suffers a worst case offset loss of less than 0.3 dB, and an average loss of less than 0.15 dB.

This improved performance means that lower sampling and processing rates can be used for a given symbol rate, resulting in reduced system cost. Alternately, for a fixed bandwidth and sampling rate, higher telemetry rates are enabled. Telemetry symbol rates of one-half the bandwidth and one-fourth of the sampling rate can be realized with a loss due to bandwidth limiting, sampling and filtering of less than 0.6 dB. An unweighted IDF [1] would require approximately twice the bandwidth and twice the sampling rate for the same performance.

Glossary of Terms

T_s	Sampling time in sec
T	Symbol time in sec
$2W$	Filter bandwidth in hertz
a_i	Transmitted symbol
$p(t)$	Pulse shaping waveform
A_k	The sampled output of Integrate-and-Dump Filter at time k
y_i	The sampled output of the prefilter
$s(t)$	The transmitted signal
$n(t)$	Additive White Gaussian Noise with flat spectral density $N_0/2$
$r(t)$	The received signal
$y(t)$	The output of the low-pass prefilter
$h(t)$	The transfer function of the low-pass prefilter
τ_0	Transport lag from transmitter to receiver in sec
τ_1	Sampling offset in sec
$R_x(\tau)$	Autocorrelation function of signal $x(t)$

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Table 1. Weight coefficients for WIDF N = 4, WT = 2

Offset	w_1	w_2	w_3	w_4
0	0.5	1.24	0.9	1.24
0.05 T_s	0.5	1.22	0.9	1.22
0.1 T_s	0.6	1.20	0.9	1.20
0.15 T_s	0.6	1.18	0.9	1.18
0.2 T_s	0.7	1.16	0.9	1.16
0.25 T_s	0.7	1.13	0.9	1.13
0.3 T_s	0.8	1.12	0.9	1.10
0.35 T_s	0.8	1.09	0.9	1.07
0.4 T_s	0.8	1.07	1.01	1.03
0.45 T_s	0.9	1.05	1.03	1.01
0.5 T_s	0.9	1.03	1.03	0.9
0.55 T_s	1.01	1.03	1.05	0.9
0.6 T_s	1.03	1.01	1.07	0.8
0.65 T_s	1.07	0.9	1.09	0.8
0.7 T_s	1.10	0.9	1.12	0.8
0.75 T_s	1.13	0.9	1.13	0.7
0.8 T_s	1.16	0.9	1.16	0.7
0.85 T_s	1.18	0.9	1.18	0.6
0.9 T_s	1.20	0.9	1.20	0.6
0.95 T_s	1.22	0.9	1.22	0.5
T_s	1.24	0.9	1.24	0.5

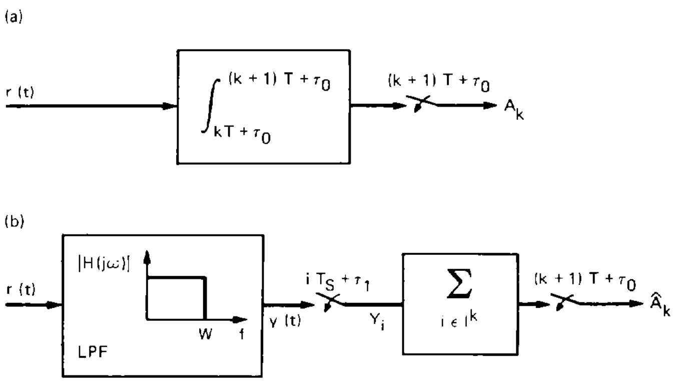

Fig. 1. Integrate-and-dump filters: (a) analog; (b) digital

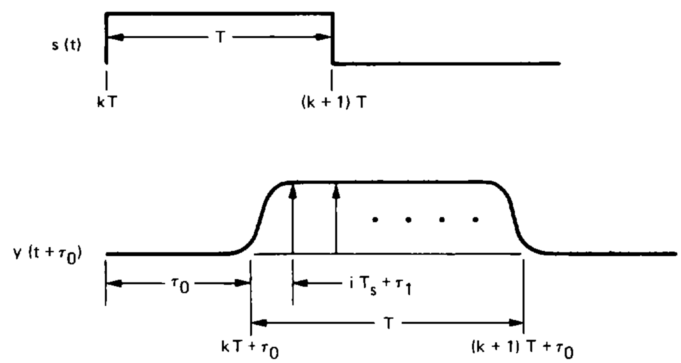

Fig. 2. Offset in sampling

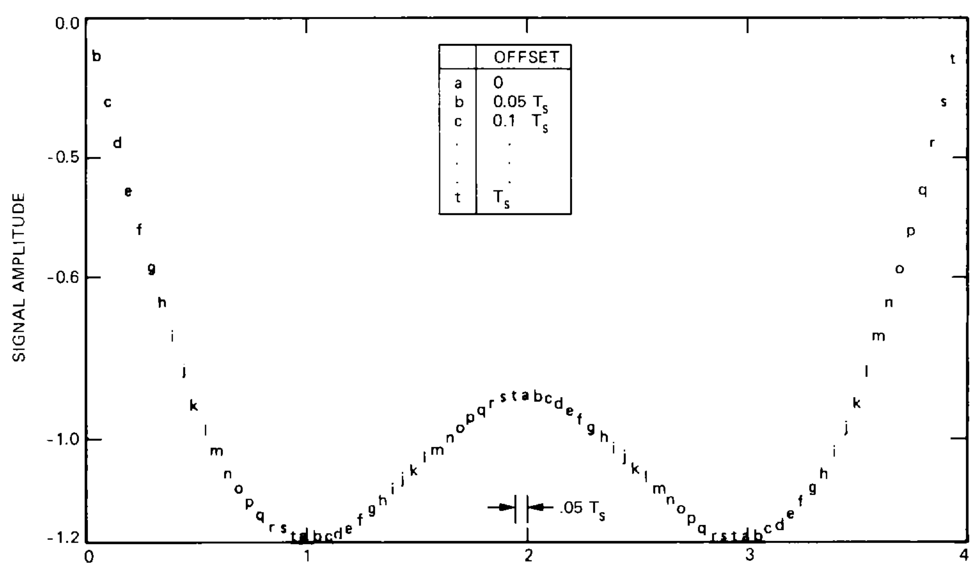

Fig. 3. Sampled waveform ($WT = 2$), alternating data pattern with $T/T_S = 4$

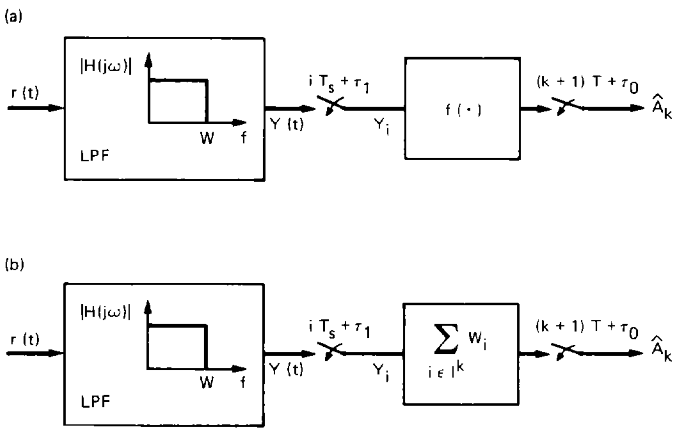

Fig. 4. Digital matched filtering: (a) optimum digital matched filter; (b) weighted integrate-and-dump filter

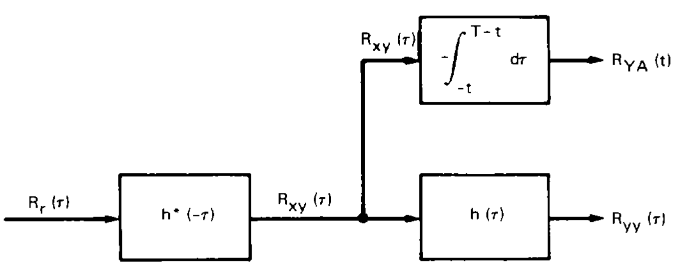

Fig. 5. Linear system to generate R_{yy} (τ), $R_{yA}(\tau)$

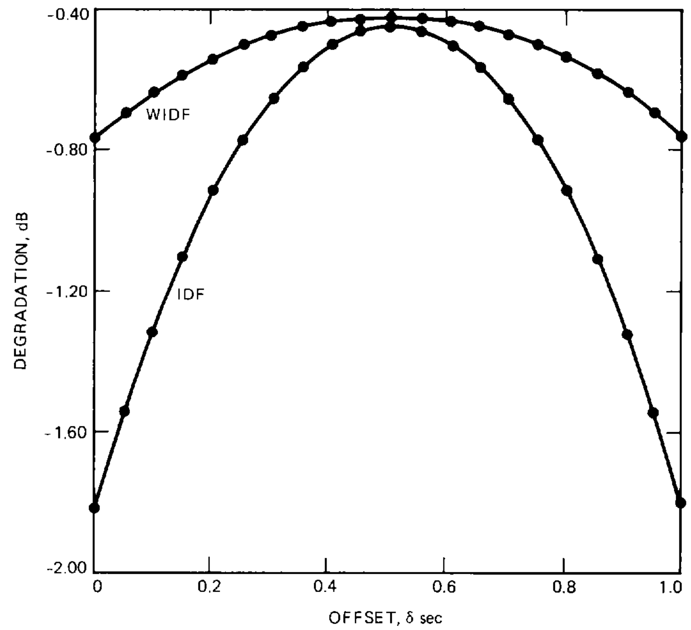

Fig. 6. Comparison of λ_{dB} for IDF and WIDF for alternating data pattern

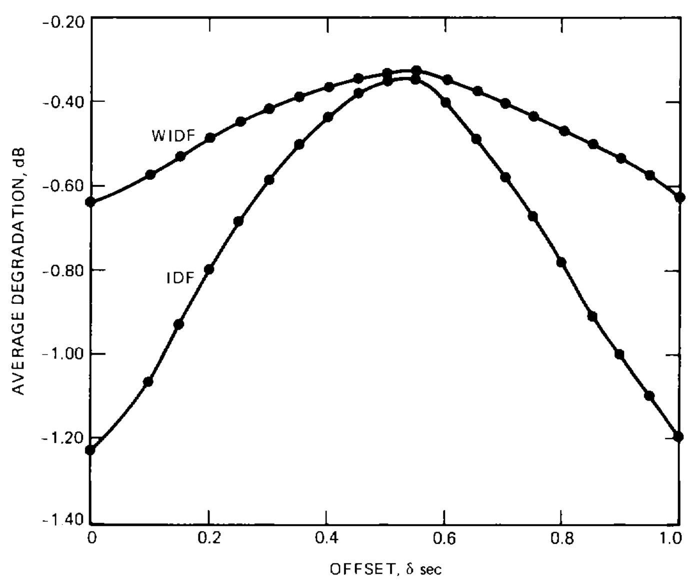

Fig. 7. Comparison of λ_{dB} for IDF and WIDF for random data pattern