The Use of Interleaving for Reducing Radio Loss in Convolutionally Coded Systems

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In this article the use of interleaving after convolutional coding and deinterleaving before Viterbi decoding is proposed. This effectively reduces radio loss at low-loop SNRs by several decibels and at high-loop SNRs by a few tenths of a decibel. Performance of the coded system can further be enhanced if the modulation index is optimized for this system. This will correspond to a reduction of bit SNR at a certain bit error rate for the overall system. The introduction of interleaving/deinterleaving into communication systems designed for future deep space missions does not substantially complicate their hardware design or increase their system cost.

I. Introduction

In analyzing the performance of coherent receivers of convolutionally encoded phase modulation, one often makes the simplifying assumption that the reference signal used for demodulation is perfectly phase synchronized to the transmitted signal, i.e., one assumes ideal coherent detection [1].

In a practical receiver, such as that used by NASA's Deep Space Network, the coherent demodulation reference is derived from a carrier synchronization subsystem, e.g., a type of phase-locked loop or Costas loop, resulting in a performance degradation due to the phase error between the received signal and the locally generated reference. Since this subsystem forms its demodulation reference from a noise-perturbed version of the transmitted signal, the phase error is a random process. The manner in which this process degrades the system error probability performance depends on the ratio of data rate to loop bandwidth, i.e., the rate of variation of the phase error over the data symbol interval [2, 3].

In this article, the above degradation in signal-to-noise ratio (SNR) performance (often referred to as *radio* or *noisy reference loss*) is discussed, and it is shown how it can be reduced by employing interleaving/deinterleaving. Specific closed form results are derived for both discrete and suppressed carrier systems and the differences between the two are discussed and numerically illustrated.

II. Upper Bound on the Average Bit Error Probability Performance

A. Perfect Carrier Phase Synchronization

For a convolutionally encoded binary phase-shift-keyed (BPSK) modulation transmitted over a perfectly phase-synchronized additive white Gaussian noise (AWGN) channel (Fig. 1), an upper bound on the average bit error probability is given as [1]

$$P_b \leq \sum_{\underline{x}, \hat{\underline{x}} \in C} \sum a(\underline{x}, \hat{\underline{x}}) p(\underline{x}) P(\underline{x} \rightarrow \hat{\underline{x}}) \tag{1}$$

where $a(\underline{x}, \hat{\underline{x}})$ is the number of bit errors that occurs when the sequence $\underline{x}$ is transmitted and the sequence $\hat{\underline{x}} \neq \underline{x}$ is chosen by the decoder,[1] $p(\underline{x})$ is the a priori probability of transmitting $\underline{x}$, and C is the set of all coded sequences. Also, in Eq. (1), $P(\underline{x} \rightarrow \hat{\underline{x}})$ represents the pairwise error probability, i.e., the probability that the decoder chooses $\hat{\underline{x}}$ when indeed $\underline{x}$ was transmitted. The upper bound of Eq. (1) is efficiently evaluated using the transfer function bound approach [1].

In general, evaluation of the pairwise error probability depends on the proposed decoding metric, the presence or absence of channel state information (CSI), and the type of detection used, i.e., coherent versus differentially coherent. For the case of interest here, namely, coherent detection with no CSI and a Gaussian metric (optimum for the AWGN channel), it is well known [1] that the pairwise error probability is given by

$$P(\underline{x} \rightarrow \hat{\underline{x}}) \leq \exp\left\{-\frac{E_s}{N_0} d_H(\underline{x}, \hat{\underline{x}})\right\} \tag{2}$$

where

$$d_H(\underline{x}, \hat{\underline{x}}) = \frac{1}{4} \sum_n \left| x_n - \hat{x}_n \right|^2 \tag{3}$$

represents the Hamming distance between $\underline{x}$ and $\hat{\underline{x}}$, i.e., the number of symbols in which the sequences $\underline{x}$ (the correct one) and $\hat{\underline{x}}$ (the incorrect one) differ. In Eq. (2), E_s is the energy-per-output coded symbol and N_0 is the single-sided noise spectral density.

B. Imperfect Carrier Phase Synchronization

1. Discrete Carrier (No Interleaving). When a carrier phase error $\phi(t)$ exists between the received signal and the locally generated demodulation reference, then the result in Eq. (2) is modified as follows.

Assuming the case in which the data symbol rate $1/T_s$ is high compared to the loop bandwidth B_L, then $\phi(t)$ can be assumed constant (independent of time) over a number of symbols on the order of $1/B_L T_s$. In this case, let $\phi(t) = \phi$. Since the decoder has no knowledge of ϕ, the decoding metric can make no use of this information and as such is *mismatched* to the channel. Under these conditions, it can be shown (see the Appendix) that using the maximum-likelihood metric for a perfectly phase-synchronized AWGN, one obtains

$$P(\underline{x} \rightarrow \hat{\underline{x}} \mid \phi; \lambda) \leq \begin{cases} \frac{1}{2} \exp\left\{-\frac{E_s}{N_0} 4\lambda(\cos\phi - \lambda)\, d_H(\underline{x}, \hat{\underline{x}})\right\}; & 0 \leq |\phi| \leq \frac{\pi}{2} \\ 1; \quad \frac{\pi}{2} \leq |\phi| \leq \pi \end{cases} \tag{4}$$

where $\lambda \geq 0$ is a parameter to be optimized. Note that for $\phi = 0$, the expression $4\lambda(1 - \lambda)$ is maximized by the value $\lambda = 1/2$, which when substituted in Eq. (4) yields Eq. (2) as it should.

Letting $p(\phi)$ denote the probability density function (p.d.f.) of the phase error ϕ, the average bit error probability is upper bounded by[2]

$$P_b \leq \sum_{\underline{x}, \hat{\underline{x}} \in C} \sum a(\underline{x}, \hat{\underline{x}}) p(\underline{x}) \min_{\lambda} E_\phi \left\{ P(\underline{x} \rightarrow \hat{\underline{x}} \mid \phi; \lambda) \right\} \tag{5}$$

where $E_\phi\{\cdot\}$ denotes statistical averaging over the p.d.f. $p(\phi)$. Using Eq. (4), the statistical average required in Eq. (5) becomes

[1] For simplicity of notation, the sequences $\underline{x}$ and $\hat{\underline{x}}$ are assumed to be normalized such that their elements take on values ± 1.

[2] Later on a tighter bound for this case is presented by optimizing on λ prior to performing the expectation over ϕ.

$$E_\phi\left\{P(\underline{x}\to\hat{\underline{x}}\mid\phi;\lambda)\right\} = \int_{-\pi/2}^{\pi/2}\frac{1}{2}\exp\left\{-\frac{E_s}{N_0}4\lambda(\cos\phi-\lambda)\,d_H(\underline{x},\hat{\underline{x}})\right\}p(\phi)\,d\phi + 2\int_{\pi/2}^{\pi}p(\phi)\,d\phi \tag{6}$$

2. **Discrete Carrier (With Interleaving).** Ordinarily, one thinks of using interleaving/deinterleaving to break up the effects of error bursts in coded communication systems. One can gain an intuitive notion of how it may be applied in systems with noisy carrier phase reference by considering the $\cos\phi$ degradation factor as an "amplitude fade" whose duration is on the order of $1/B_L T_s$ symbols. Thus, if we break up this "fade" by interleaving to a depth on the order of $1/B_L T_s$, then, after deinterleaving, the degradation due to $\cos\phi$ will be essentially *independent* from symbol to symbol. From a mathematical standpoint, this is equivalent to replacing Eq. (4) by[3]

$$P(\underline{x}\to\hat{\underline{x}}\mid\underline{\phi};\lambda) \leqslant \begin{cases}\dfrac{1}{2}\exp\left\{-\dfrac{E_s}{N_0}\displaystyle\sum_{n=1}^{d_H}4\lambda(\cos\phi_n-\lambda)\right\}; & \displaystyle\sum_{n=1}^{d_H}\cos\phi_n > 0\\ 1; & \displaystyle\sum_{n=1}^{d_H}\cos\phi_n \leqslant 0\end{cases} \tag{7}$$

where the ϕ_n variables are independent identically distributed (i.i.d.) random variables with p.d.f. $p(\phi)$, and $\underline{\phi}$ refers to the vector whose components are ϕ_ns. The derivation of Eq. (7) is given in the Appendix. The expectation required in Eq. (5) now involves computation of multidimensional integrals over regions of $\underline{\phi}$ corresponding to the inequalities in Eq. (7). In these regions, since the intervals of integration per dimension are *dependent* on one another, the expectation required in Eq. (5) is extremely difficult to compute.

Thus, instead we turn to a looser upper bound on conditional pairwise error probability, which has the advantage of not having to separate the multidimensional integration required in Eq. (5) into two disjoint regions. Indeed, it is straightforward to see that the right-hand side of Eq. (7) is upper bounded by the exponential in its first line (without the factor of 1/2) over the *entire* domain of $\underline{\phi}$, i.e., $\{\phi_n \in (-\pi,\pi); n \in \eta\}$. Hence,

$$P(\underline{x}\to\hat{\underline{x}}\mid\underline{\phi};\lambda) \leqslant \exp\left\{-\frac{E_s}{N_0}\sum_{n=1}^{d_H}4\lambda(\cos\phi_n-\lambda)\right\} = \prod_{n=1}^{d_H}\exp\left\{-\frac{E_s}{N_0}4\lambda(\cos\phi_n-\lambda)\right\} \tag{8}$$

which is identically equal to the Chernoff bound. Now, substituting Eq. (8) into Eq. (5) gives the much simpler result

$$E_\phi\left\{P(\underline{x}\to\hat{\underline{x}}\mid\phi;\lambda)\right\} = \prod_{n=1}^{d_H}\int_{-\pi}^{\pi}\exp\left\{-\frac{E_s}{N_0}4\lambda(\cos\phi_n-\lambda)\right\}\times p(\phi_n)\,d\phi_n \tag{9}$$

3. **Suppressed Carrier (No Interleaving).** When the carrier synchronization loop used to track the input phase is of the suppressed carrier type (e.g., a Costas loop), then the results of Section B.1 have to be somewhat modified since the appropriate domain for ϕ is no longer $(-\pi,\pi)$. In fact, for suppressed carrier tracking of BPSK with a Costas-type loop, and assuming perfect phase ambiguity resolution, ϕ takes on values only in the interval $(-\pi/2,\pi/2)$ [2]. Thus, the interval of integration for the first integral in Eq. (6) becomes $(-\pi/2,\pi/2)$ and the second integral in Eq. (6) disappears, i.e.,

$$E_\phi\left\{P(\underline{x}\to\hat{\underline{x}}\mid\phi;\lambda)\right\} = \int_{-\pi/M}^{\pi/M}\frac{1}{2}\exp\left\{-\frac{E_s}{N_0}4\lambda(\cos\phi-\lambda)\,d_H(\underline{x},\hat{\underline{x}})\right\}p(\phi)\,d\phi \tag{10}$$

The significance of the second integral in Eq. (6) being equal to zero will be mentioned shortly relative to a discussion of *irreducible error probability*.

4. **Suppressed Carrier (With Interleaving).** Once again assuming suppressed carrier tracking of BPSK with a Costas-type

[3] Herein, for simplicity of notation, we drop the dependence of d_H on $\underline{x}$ and $\hat{\underline{x}}$.

loop, and perfect phase ambiguity resolution, one obtains, analogous to Eq. (9),[4]

$$E_\phi\left\{P(\underline{x} \to \hat{\underline{x}}\,|\phi;\lambda)\right\} =$$

$$\frac{1}{2}\prod_{n=1}^{d_H}\int_{-\pi/2}^{\pi/2}\exp\left\{-\frac{E_s}{N_0}4\lambda(\cos\phi_n-\lambda)\right\}p(\phi_n)\,d\phi_n \tag{11}$$

III. Carrier Synchronization Loop Statistical Model and Average Pairwise Error Probability Evaluation

To evaluate Eq. (5), using Eqs. (6), (9), (10), or (11), one must specify the functional form of the probability density function (p.d.f.) $p(\phi)$ of the modulo 2π reduced phase error ϕ. For a discrete carrier synchronization loop of the phase-locked type, $p(\phi)$ is given by the Tikhonov p.d.f. [2]

$$p(\phi) = \begin{cases} \dfrac{\exp(\rho\cos\phi)}{2\pi I_0(\rho)}; & |\phi| \leqslant \pi \\ 0; & \text{otherwise} \end{cases} \tag{12}$$

where ρ is the SNR in the loop bandwidth.

In order to allow evaluation of Eq. (5) in closed form, one must recognize that for the case of no interleaving, Eq. (6) can be further upper bounded by using $(-\pi, \pi)$ instead of $(-\pi/2, \pi/2)$ in the first integral. Then, making this replacement

$$\min_\lambda E_\phi\left\{P(\underline{x} \to \hat{\underline{x}}\,|\underline{\phi};\lambda)\right\} \leqslant$$

$$\min_\lambda\left(\frac{1}{2}\exp\left\{4d_H\lambda^2\frac{E_s}{N_0}\right\}\frac{I_0(\rho_A)}{I_0(\rho)} + I\right) \tag{13}$$

$$\rho_A \triangleq \rho - 4d_H\lambda\frac{E_s}{N_0}$$

$$I = \frac{1}{\pi I_0(\rho)}\left[\int_{\pi/2}^{\pi}\exp(\rho\cos\phi)\,d\phi\right]$$

[4]Note that the factor of 1/2 can be included here since for $0 \leqslant |\phi_n| \leqslant \pi/2$; $n \in \eta$, the condition on the first line of Eq. (7) is always satisfied and thus we need not use the looser upper bound of Eq. (8).

When Eq. (13) is substituted into Eq. (5), the term I will contribute an *irreducible error probability*, i.e., the system will exhibit a finite error probability when ρ is held fixed and E_s/N_0 approaches infinity. An example of such a system is one which apportions a fixed amount of the total available input power to a discrete carrier component for the purpose of deriving a coherent carrier reference at the receiver.

When interleaving is employed, Eq. (9) (minimized over λ) together with Eq. (12) become

$$\min_\lambda E_\phi\left\{P(\underline{x} \to \hat{\underline{x}}\,|\underline{\phi};\lambda)\right\} \leqslant$$

$$\min_\lambda\left\{\prod_{n\in\eta}\exp\left\{4\lambda^2\frac{E_s}{N_0}\right\}\frac{I_0(\rho_B)}{I_0(\rho)}\right\}$$

$$= \min_\lambda\left\{\left(\exp\left\{4\lambda^2\frac{E_s}{N_0}\right\}\frac{I_0(\rho_B)}{I_0(\rho)}\right)^{d_H}\right\} \tag{14}$$

$$\rho_B \triangleq \rho - 4\lambda\frac{E_s}{N_0}$$

For suppressed carrier tracking with a biphase Costas loop, $p(\phi)$ again has a Tikhonov-type p.d.f., which is given by [2] as

$$p(\phi) = \begin{cases} \dfrac{\exp(\rho\cos 2\phi)}{\pi I_0(\rho)}; & |\phi| \leqslant \dfrac{\pi}{2} \\ 0; & \text{otherwise} \end{cases} \tag{15}$$

Here ρ is the "effective" loop SNR which includes the effects of $S \times S$, $S \times N$, and $N \times N$ degradations commonly referred to as "squaring loss." Since suppressed carrier systems of this type derive their carrier demodulation reference from the data-bearing signal, the loop SNR, ρ, is directly proportional to E_s/N_0; thus there can be no irreducible error probability since $\rho \to \infty$ when $E_s/N_0 \to \infty$. Furthermore, for perfect phase ambiguity resolution, it has been shown previously that for no interleaving, the term I is identically zero since the p.d.f. is zero in the region $(\pi/2, \pi)$. Thus, the average pairwise error probability results become

$$\min_\lambda E_\phi\left\{P(\underline{x} \to \hat{\underline{x}}\,|\underline{\phi};\lambda)\right\} \leqslant \min_\lambda\left\{\frac{1}{2}\exp\left\{4d_H\lambda^2\frac{E_s}{N_0}\right\}\frac{f_A(\rho)}{I_0(\rho)}\right\}$$

$$f_A(\rho) \triangleq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\exp\left\{\rho\cos 2\phi - 4d_H\lambda\frac{E_s}{N_0}\cos\phi\right\}d\phi \tag{16}$$

for no interleaving and

$$\min_{\lambda} E_{\phi}\{P(\underline{x} \rightarrow \hat{\underline{x}} \,|\, \underline{\phi};\lambda)\} \leqslant \min_{\lambda}\left\{\prod_{n\in\eta} \exp\left\{4\lambda^2 \frac{E_s}{N_0}\right\}\frac{f_B(\rho)}{I_0(\rho)}\right\}$$

$$= \min_{\lambda}\left\{\left(\exp\left\{4\lambda^2 \frac{E_s}{N_0}\right\}\frac{f_B(\rho)}{I_0(\rho)}\right)^{d_H}\right\}$$

$$f_B(\rho) \triangleq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \exp\left\{\rho\cos 2\phi - 4\lambda\frac{E_s}{N_0}\cos\phi\right\} d\phi \tag{17}$$

for the case of interleaving.

In arriving at Eqs. (13), (14), (16), and (17), we have assumed the "same type" of Chernoff bound in the sense that in all cases, the minimization over λ was performed *after* the averaging over ϕ. The principal reason for doing this is to allow comparison of performance with and without interleaving using bounds with "similar degrees of looseness." For the case of no interleaving, one can actually achieve a tighter bound than that given above by performing the minimization over λ on the conditional pairwise probability in Eq. (4). When this is done, one obtains

$$\lambda_{\text{opt}} = \frac{1}{2}\cos\phi \tag{18}$$

and Eq. (4) becomes

$$P(\underline{x} \rightarrow \hat{\underline{x}} \,|\, \phi) \leqslant \begin{cases} \dfrac{1}{2}\exp\left\{-\dfrac{E_s}{4N_0} d_H \cos^2\phi\right\}; & 0 \leqslant |\phi| \leqslant \dfrac{\pi}{2} \\ 1; & \dfrac{\pi}{2} \leqslant |\phi| \leqslant \pi \end{cases} \tag{19}$$

Unfortunately, the integral of Eq. (19) over the p.d.f.s of Eqs. (12) and (15) cannot be obtained in closed form. Defining the integral

$$L(J) = \int_{-\pi/J}^{\pi/J} \exp\left\{-\frac{E_s d_H}{N_0}\cos^2\phi\right\} \frac{\exp(\rho\cos J\phi)}{\left(\frac{2\pi}{J}\right) I_0(\rho)} d\phi \tag{20}$$

then, the average pairwise error probabilities are now as follows:

Discrete Carrier

$$E_{\phi}\left\{\min_{\lambda} P(\underline{x} \rightarrow \hat{\underline{x}} \,|\, \underline{\phi};\lambda)\right\} \leqslant \frac{1}{2}L(1) + I \tag{21}$$

where I is defined in Eq. (13).

Suppressed Carrier

$$E_{\phi}\left\{\min_{\lambda} P(\underline{x} \rightarrow \hat{\underline{x}} \,|\, \underline{\phi};\lambda)\right\} \leqslant \frac{1}{2}L(2) \tag{22}$$

Using Eqs. (21) and (22) (rather than Eqs. 13 and 16) will result in a smaller improvement in performance due to interleaving/deinterleaving since Eqs. (21) and (22) result in a tighter bound on P_b (no interleaving).

IV. Evaluation of Bit Error Probability

A. Discrete Carrier Tracking

To evaluate the upper bound on bit error probability, e.g., Eq. (5), we use the transfer function bound approach [1], which, for the ideal case of perfect carrier synchronization, gives

$$P_b \leqslant \frac{1}{2}\frac{d}{dz} T(D,z)\bigg|_{z=1} \tag{23}$$

where $T(D, z)$ is the transfer function of the pair-state diagram associated with the trellis diagram of the code. When noisy carrier synchronization references are present, the appropriate upper bound on bit error probability for the case of no interleaving becomes

$$P_b \leqslant \frac{1}{2}\left[2\int_0^{\pi/2} \frac{d}{dz} T(D(\phi), z)\bigg|_{z=1} p(\phi)\,d\phi\right.$$

$$\left. + 2\int_{\pi/2}^{\pi} p(\phi)\,d\phi\right]$$

$$= \int_0^{\pi/2} \frac{d}{dz} T(D(\phi), z)\bigg|_{z=1} p(\phi)\,d\phi + I \tag{24}$$

where I is defined in Eq. (13) and from Eq. (19), the equivalent Bhattacharyya parameter becomes

$$D(\phi) = \exp\left\{-\frac{E_s}{N_0}\cos^2\phi\right\}; \qquad 0 \leqslant |\phi| \leqslant \frac{\pi}{2} \tag{25}$$

For the case of interleaving, we use Eq. (23) without the factor 1/2 and with D defined in accordance with Eq. (14), namely,

$$D = \min_{\lambda}\left(\exp\left\{4\lambda^2 \frac{E_s}{N_0}\right\}\frac{I_0(\rho_B)}{I_0(\rho)}\right) \tag{26}$$

In arriving at Eq. (26), we have made use of the fact that, for any d,

$$\min_{\lambda}\left\{\left(\exp\left\{4\lambda^2 \frac{E_s}{N_0}\right\}\frac{I_0(\rho_B)}{I_0(\rho)}\right)^d\right\} = \left\{\min_{\lambda}\left(\exp\left\{4\lambda^2 \frac{E_s}{N_0}\right\}\frac{I_0(\rho_B)}{I_0(\rho)}\right)\right\}^d \tag{27}$$

Figures 2 through 10 illustrate the upper bound on bit error probability versus bit energy-to-noise ratio E_b/N_0 (E_b is related to E_s by $E_s = rE_b$ where r is the code rate) for the rate 1/2, constraint length 7 Voyager code, the rate 1/4, constraint length 15 Galileo experimental code [4], the rate 1/6, constraint length 15 "2 dB" code [5], and various values of loop SNR, ρ. In these curves we have assumed carrier synchronization with a PLL, i.e., the discrete carrier case.[5] On each figure are plotted the results for the case of no interleaving, the case of interleaving, and the case of no radio loss, i.e., ideal carrier synchronization. The transfer function bounds (truncated to 15 terms) on P_b for the above three codes are given in Table 1.

One may observe from the results in Figs. 2 through 10 that even for large values of ρ, e.g., 13 dB, a substantial reduction of bit error rate is possible by using interleaving. Also, since the tighter bound was used for the no-interleaving case and the looser bound for the interleaving case, the performance improvement illustrated is, as previously mentioned, somewhat pessimistic, i.e., in reality, one will do even better than shown.

[5] Also, in the computation of Eq. (24), we have set the value of one-half the derivative of the transfer function evaluated at $z = 1$ to one whenever it would normally exceed one. This is allowable since the conditional error probability cannot exceed one. Doing so results in a tighter bound.

B. Suppressed Carrier Tracking

The upper bound on bit error probability for the no-interleaving case is analogous to Eq. (23), and is given by

$$P_b \leqslant \int_0^{\pi/2} \frac{d}{dz} T(D(\phi), z)\bigg|_{z=1} p(\phi)\, d\phi \tag{28}$$

with $p(\phi)$ as in Eq. (15) and $D(\phi)$ as in Eq. (25). For the case of interleaving and the tighter upper bound, we again use Eq. (23) with, however, D now defined analogous to Eq. (26) by

$$D = \min_{\lambda}\left(\exp\left\{4\lambda^2 \frac{E_s}{N_0}\right\}\frac{f_B(\rho)}{I_0(\rho)}\right) \tag{29}$$

where $f_B(\rho)$ is given in Eq. (17).

Assuming a Costas loop with integrate-and-dump arm filters (matched filters), the equivalent loop SNR is given by [2, 3]

$$\rho = \frac{1}{4}\left(\frac{S}{N_0 B_L}\right)S_L = \frac{1}{4}\left(\frac{E_b}{N_0}\right)\left(\frac{1}{B_L T_b}\right)S_L; \qquad S_L = \frac{2\dfrac{E_s}{N_0}}{1 + 2\dfrac{E_s}{N_0}} \tag{30}$$

where $\mathbf{S}_L$ denotes the "squaring loss" associated with BPSK modulation. Figures 11 through 18 illustrate results analogous to Figs. 2 through 10 for the case of suppressed carrier tracking and various values of $B_L T_b$. Since, as already mentioned, in suppressed carrier systems there is no irreducible error (since, from Eq. (30), $\rho \to \infty$ as $E_b/N_0 \to \infty$), the noisy reference losses are much smaller to begin with (i.e., no interleaving) than for the discrete carrier case. Thus, for sufficiently small $B_L T_b$, interleaving does not provide significant improvement. We also observe from the above figures that the noisy reference losses are much larger for the rate 1/4 and rate 1/6 codes than for the rate 1/2 code. The reason for this is that for a given E_b/N_0, the value of E_s/N_0 is 3 dB smaller for the rate 1/4 code and 4.77 dB smaller for the rate 1/6 code than for the rate 1/2 code and thus, from Eq. (30), for the same value of $B_L T_b$, the equivalent loop SNR is smaller because of the increased squaring loss.

V. Concluding Remarks

It has been shown that by interleaving the transmitted coded bits in convolutionally coded systems the radio loss can be significantly reduced. The amount of this reduction depends on the particular convolutional code used and the region of operation of the system as characterized by such parameters as bit error rate and loop SNR. In some cases, the performance of the interleaved system is close to that of the ideal coherent detection case.

Acknowledgment

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Table 1. Transfer function bounds of r = 1/2, K = 7, r = 1/4, K = 15, and r = 1/6, K = 15 convolutional codes

$$P_b \leqslant \frac{1}{2}\frac{d}{dz} T(D,z)\Big|_{z=1} = \frac{1}{2}\sum_{d=d_f}^{\infty} \beta_d D^d$$

$r = 1/2, K = 7$		$r = 1/4, K = 15$[a]		$r = 1/6, K = 15$	
d	β_d	d	β_d	d	β_d
10	36	35	6	56	2
11	0	36	2	57	15
12	211	37	16	58	2
13	0	38	8	59	0
14	1404	39	11	60	12
15	0	40	20	61	25
16	11633	41	24	62	56
17	0	42	76	63	43
18	77433	43	126	64	24
19	0	44	180	65	44
20	502690	45	255	66	62
21	0	46	416	67	48
22	3322763	47	628	68	62
23	0	48	850	69	167
24	21292910	49	1313	70	162

[a]This code has been incorporated as an experiment for the Galileo mission. It is a good code for a concatenated coding scheme but not the optimum code from the standpoint of maximizing d_f.

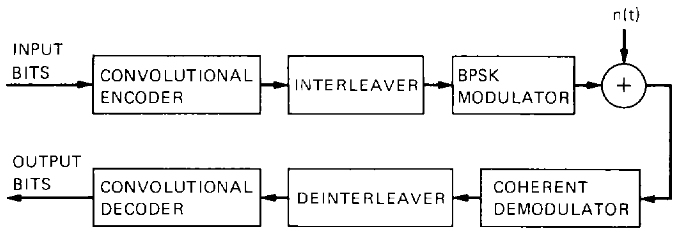

Fig. 1. System block diagram.

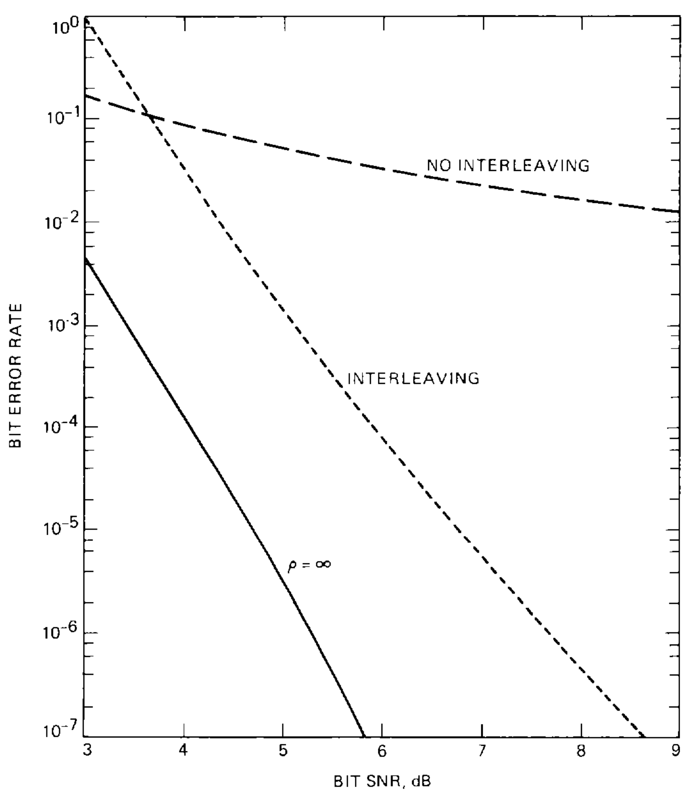

Fig. 2. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, constraint length 7 convolutional code; loop SNR = 7 dB; discrete carrier.

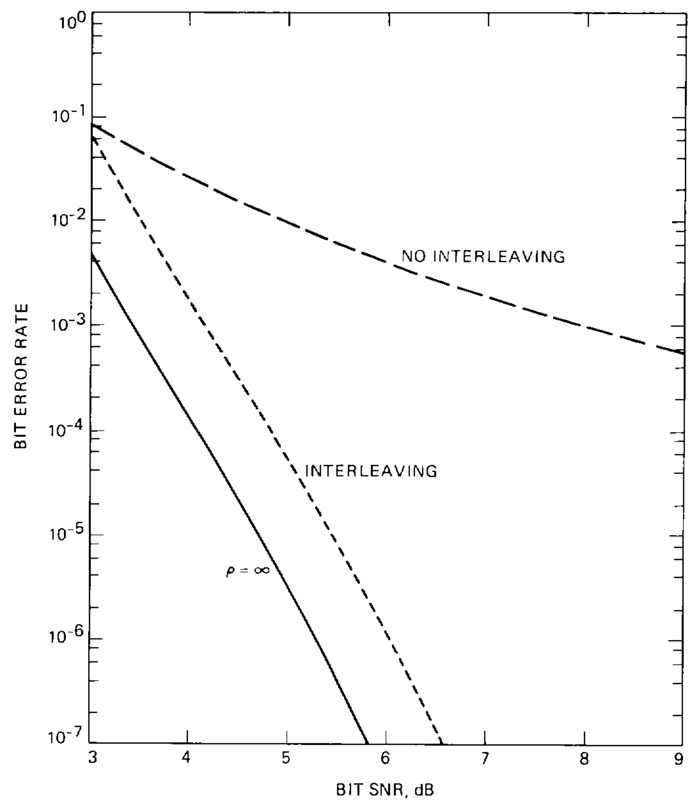

Fig. 3. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, constraint length 7 convolutional code; loop SNR = 10 dB; discrete carrier.

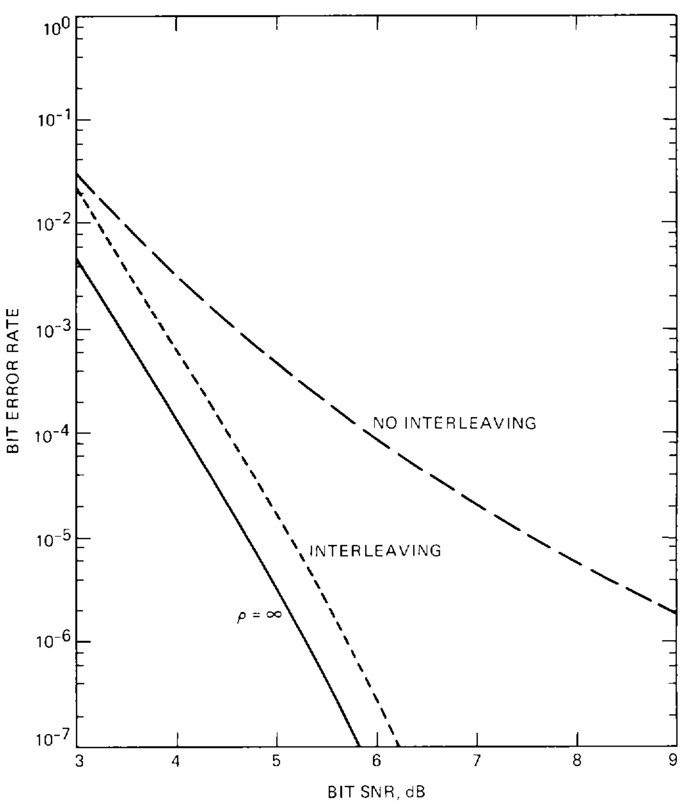

Fig. 4. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, constraint length 7 convolutional code; loop SNR = 13 dB; discrete carrier.

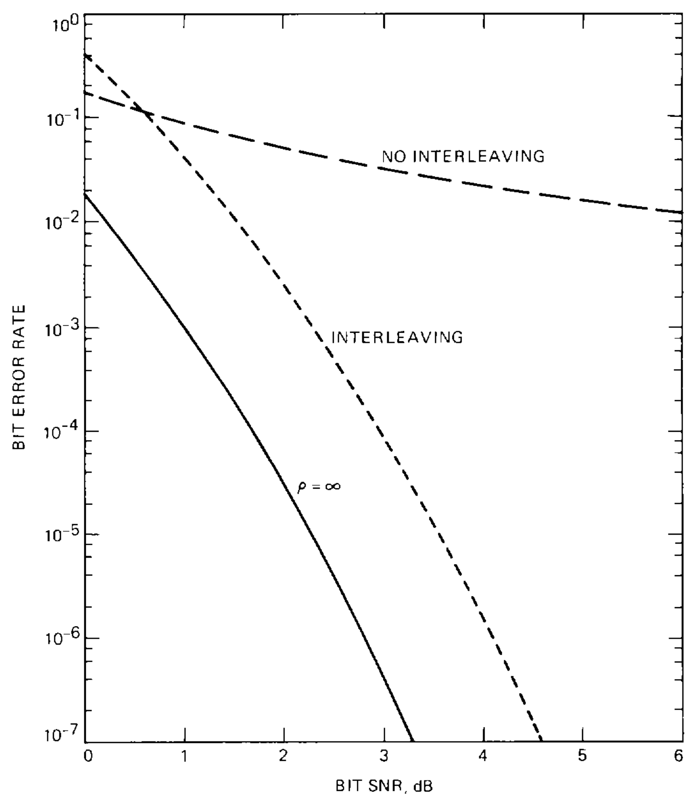

Fig. 5. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/4, constraint length 15 convolutional code; loop SNR = 7 dB; discrete carrier.

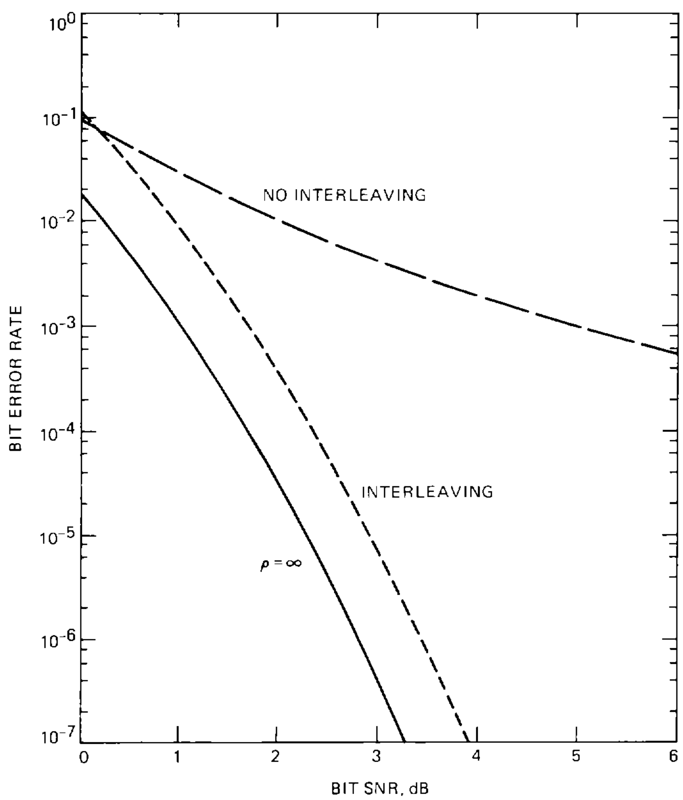

Fig. 6. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/4, constraint length 15 convolutional code; loop SNR = 10 dB; discrete carrier.

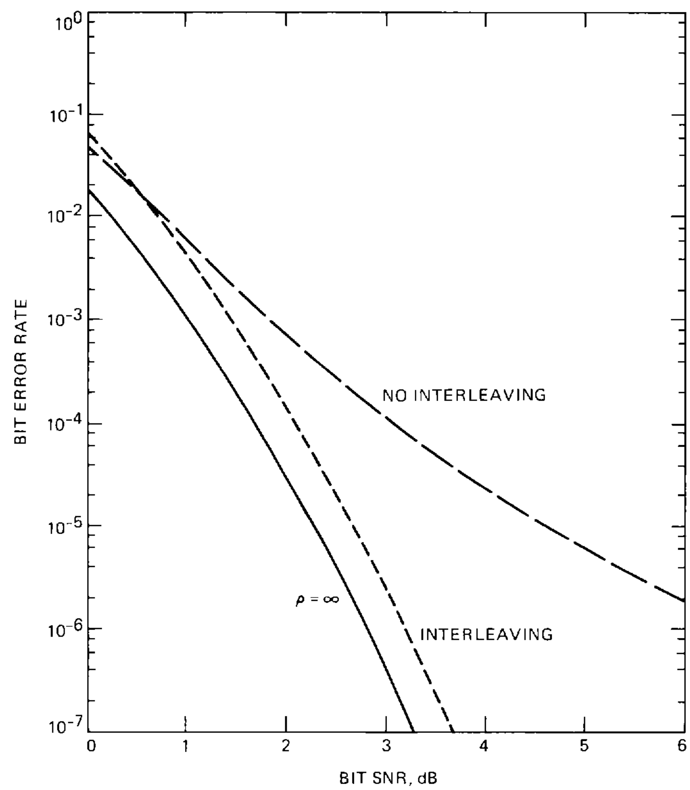

Fig. 7. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/4, constraint length 15 convolutional code; loop SNR = 13 dB; discrete carrier.

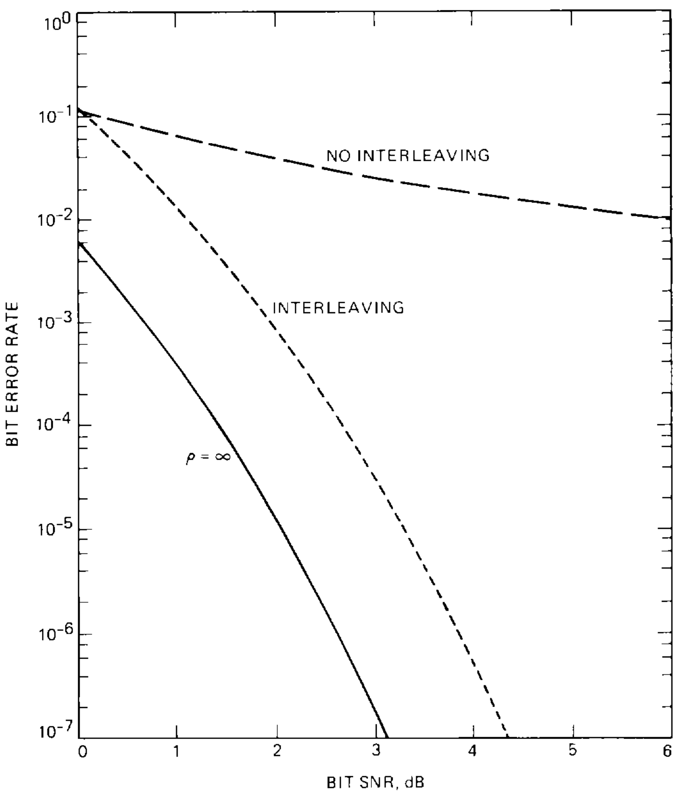

Fig. 8. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/6, constraint length 15 convolutional code; loop SNR = 7 dB; discrete carrier.

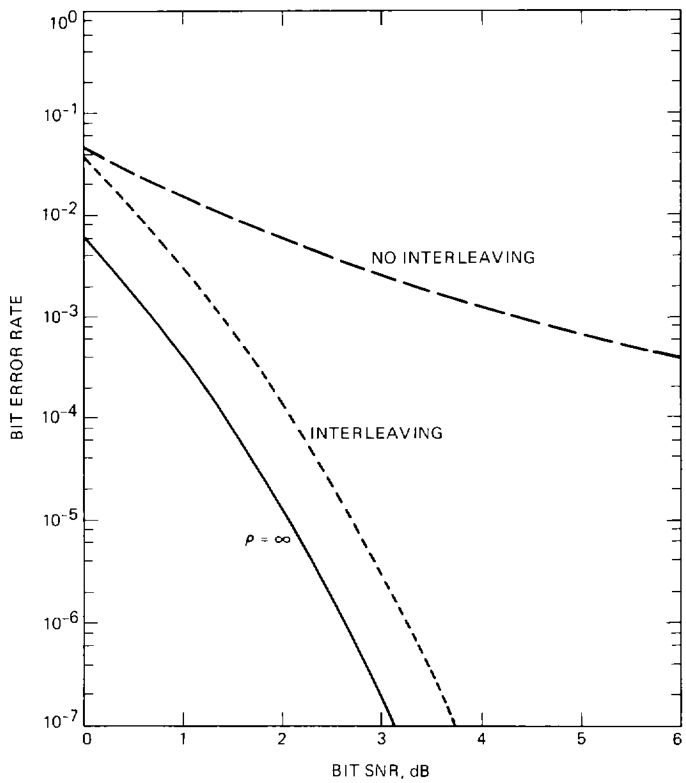

Fig. 9. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/6, constraint length 15 convolutional code; loop SNR = 10 dB; discrete carrier.

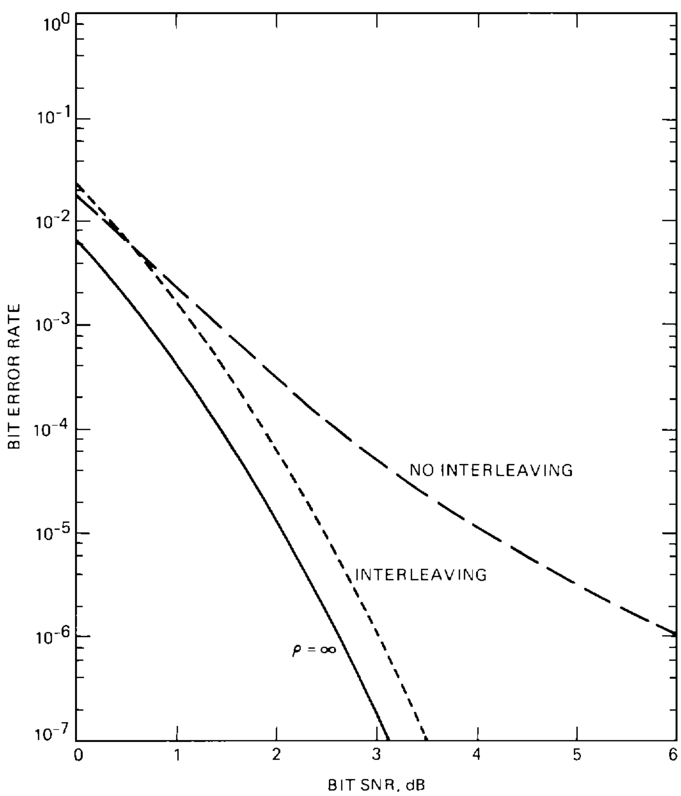

Fig. 10. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/6, constraint length 15 convolutional code; loop SNR = 13 dB; discrete carrier.

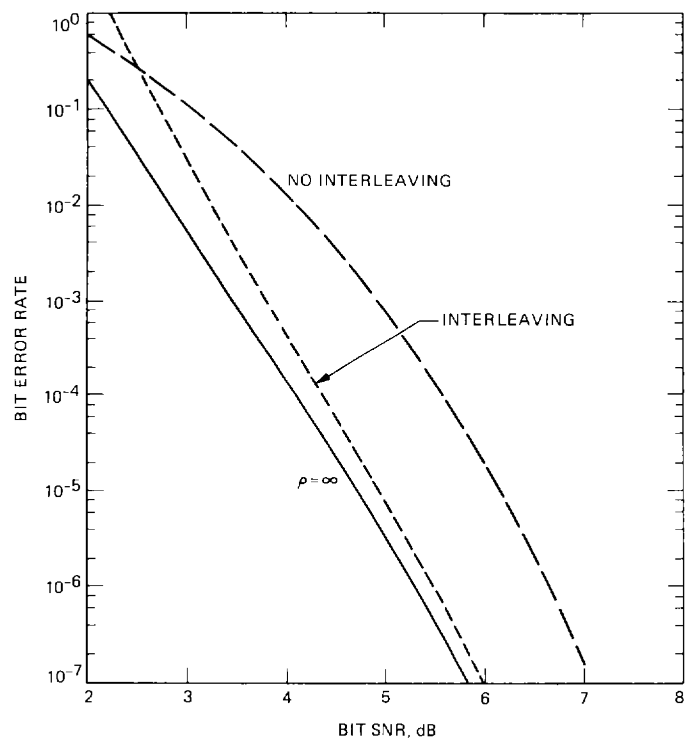

Fig. 11. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, constraint length 7 convolutional code; suppressed carrier; $1/B_L T_b = 10$.

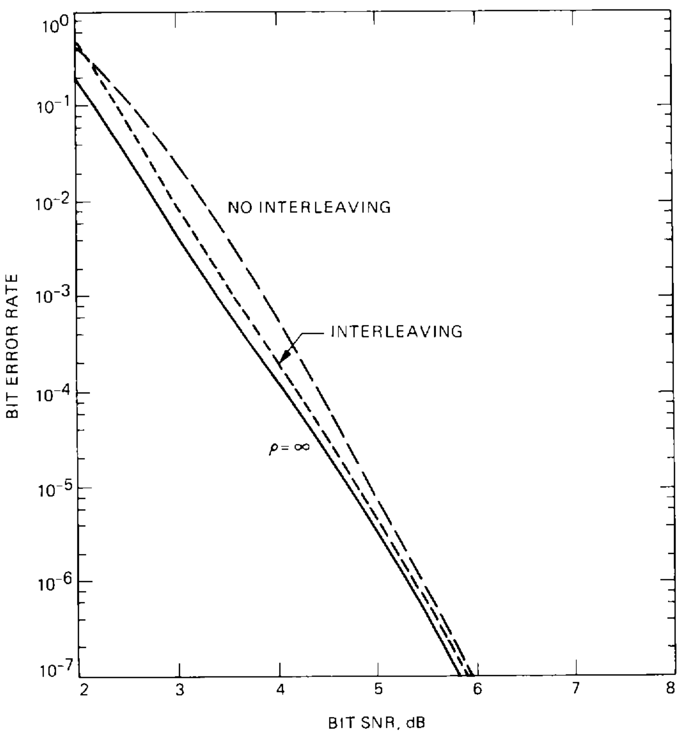

Fig. 12. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/2, constraint length 7 convolutional code; suppressed carrier; $1/B_L T_b = 20$.

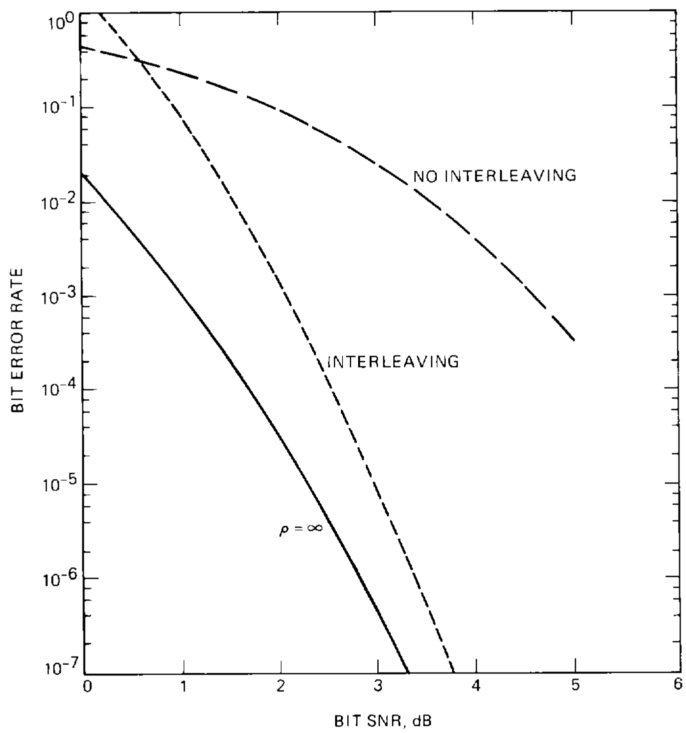

Fig. 13. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/4, constraint length 15 convolutional code; suppressed carrier; $1/B_L T_b = 10$.

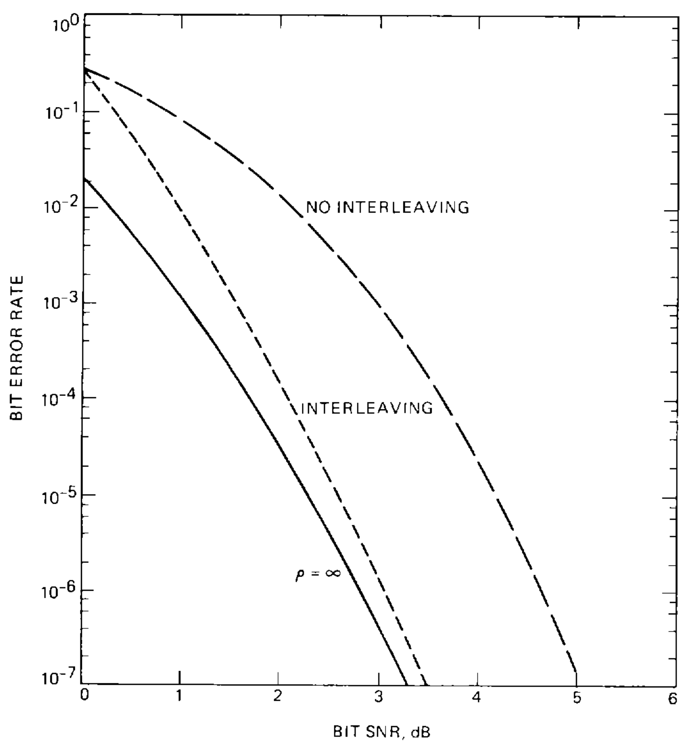

Fig. 14. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/4, constraint length 15 convolutional code; suppressed carrier; $1/B_L T_b = 20$.

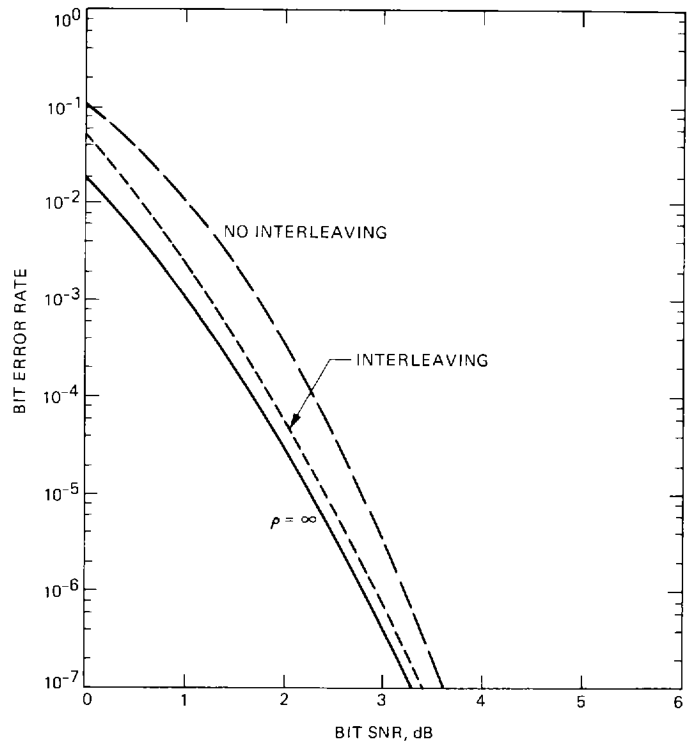

Fig. 15. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/4, constraint length 15 convolutional code; suppressed carrier; $1/B_L T_b = 40$.

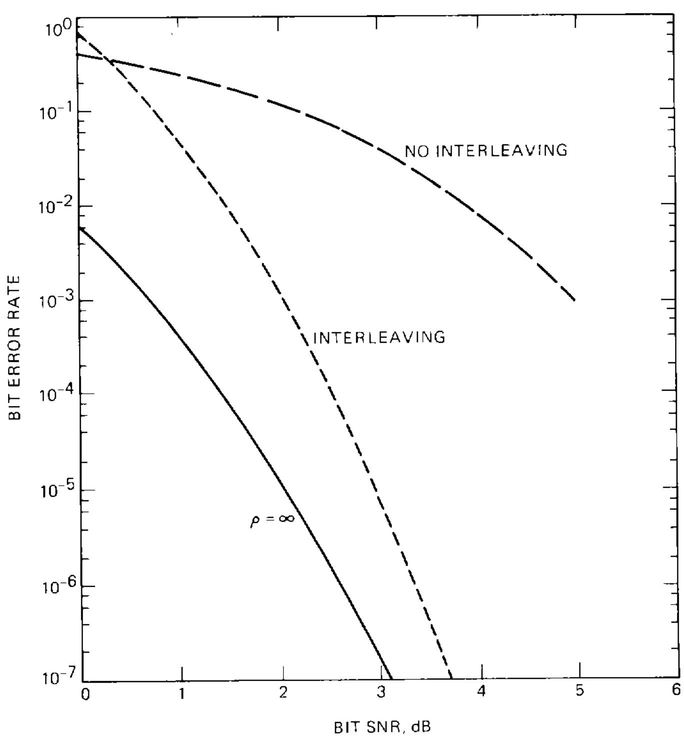

Fig. 16. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/6, constraint length 15 convolutional code; suppressed carrier; $1/B_L T_b$ = 10.

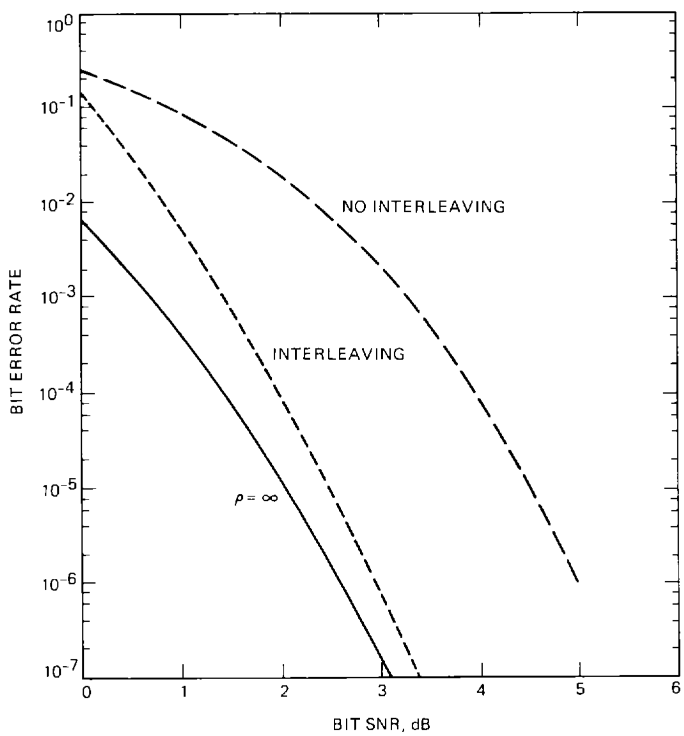

Fig. 17. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/6, constraint length 15 convolutional code; suppressed carrier; $1/B_L T_b$ = 20.

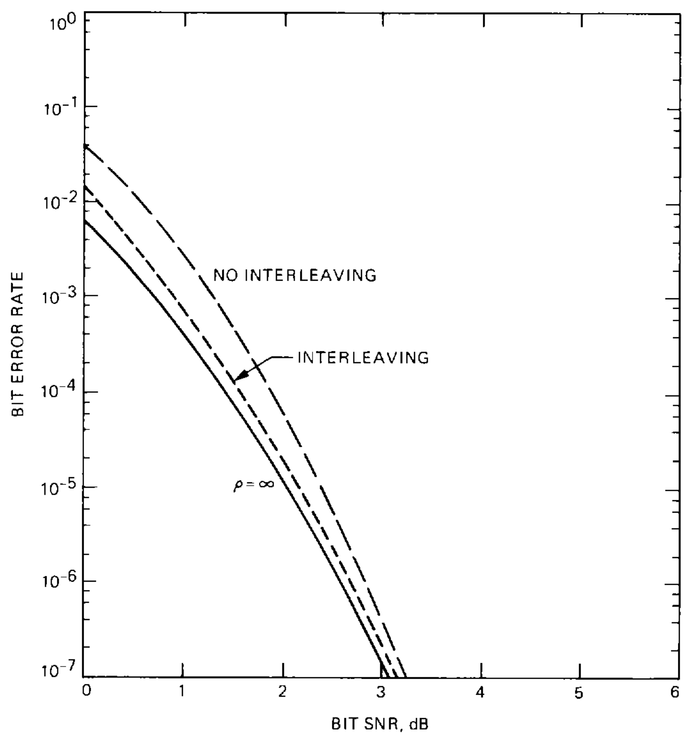

Fig. 18. Upper bound on average bit error probability versus bit energy-to-noise ratio for rate 1/6, constraint length 15 convolutional code; suppressed carrier; $1/B_L T_b = 60$.

Appendix

Derivation of an Upper Bound on the Pairwise Error Probability for Convolutionally Coded BPSK With Imperfect Carrier Phase Reference

Let $\underline{y} = (y_1, y_2, \ldots, y_N)$ denote the received sequence when the normalized (to unit power) sequence of MPSK symbols $\underline{x} = (x_1, x_2, \ldots, x_N)$ is transmitted. A *pairwise error* occurs if $\underline{\hat{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N) \neq \underline{x}$ is chosen by the receiver, which, if the receiver uses a distance metric to make this decision, implies $\underline{y}$ is closer to $\underline{\hat{x}}$ than to $\underline{x}$. Assuming that distance metric which is maximum-likelihood for ideal coherent detection (perfect carrier phase reference), then such an error occurs whenever

$$\sum_{n=1}^{N} \left| y_n - \hat{x}_n \right|^2 < \sum_{n=1}^{N} \left| y_n - x_n \right|^2 \tag{A-1}$$

Since BPSK is a constant envelope signaling set, we have $|x_n|^2 = |\hat{x}_n|^2 = 1$ (assuming a normalized signal) and Eq. (A-1) reduces to

$$\sum_{n=1}^{N} \mathrm{Re}\left\{ y_n \hat{x}_n^* \right\} > \sum_{n=1}^{N} \mathrm{Re}\left\{ y_n x_n^* \right\} \tag{A-2}$$

Letting n_n represent the additive noise in the nth signaling interval, and ϕ_n the phase shift introduced by imperfect carrier demodulation in that same interval, then y_n and x_n are related by

$$y_n = x_n e^{j\phi_n} + n_n; \qquad n = 1, 2, \ldots, N \tag{A-3}$$

Substituting Eq. (A-3) into Eq. (A-2) and simplifying gives

$$\mathrm{Re}\left\{ \sum_{n\in\eta} (\hat{x}_n - x_n)^* n_n \right\} > \mathrm{Re}\left\{ \sum_{n\in\eta} x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right\} \tag{A-4}$$

where η is the set of all n such that $x_n \neq \hat{x}_n$.

Since for an AWGN channel, n_n is a complex Gaussian random variable whose real and imaginary components have variance

$$E\left\{ [\mathrm{Re}(n_n)]^2 \right\} = E\left\{ [\mathrm{Im}(n_n)]^2 \right\} \triangleq \sigma^2 \tag{A-5}$$

then

$$\mathrm{var}\left\{ \mathrm{Re}\left\{ \sum_{n\in\eta} (\hat{x}_n - x_n)^* n_n \right\} \right\} = \sigma^2 \sum_{n\in\eta} \left| x_n - \hat{x}_n \right|^2 \tag{A-6}$$

and the conditional pairwise error probability $P(\underline{x} \to \underline{\hat{x}} \,|\, \underline{\phi})$ is given by

$$\begin{aligned} P(\underline{x} \to \underline{\hat{x}} \,|\, \underline{\phi}) &= \Pr\left\{ \mathrm{Re}\left\{ \sum_{n\in\eta} (\hat{x}_n - x_n)^* n_n \right\} \right. \\ &\qquad \left. > \mathrm{Re}\left\{ \sum_{n\in\eta} x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right\} \right\} \\ &= Q\left\{ \frac{\mathrm{Re}\left\{ \sum_{n\in\eta} x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right\}}{\sigma \sqrt{\sum_{n\in\eta} \left| x_n - \hat{x}_n \right|^2}} \right\} \end{aligned} \tag{A-7}$$

where $\underline{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$ is the sequence of carrier phase errors and $Q(x)$ is the Gaussian integral defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{y^2}{2} \right) dy \tag{A-8}$$

To simplify Eq. (A-7), proceed as follows. For BPSK modulation

$$\left| x_n - \hat{x}_n \right|^2 = 4; \qquad n \in \eta \tag{A-9}$$

and

$$\sum_{n\in\eta} \mathrm{Re}\left\{ x_n (x_n - \hat{x}_n)^* e^{j\phi_n} \right\} = 2 \sum_{n\in\eta} \cos\phi_n \tag{A-10}$$

Thus, Eq. (A-7) can be written as

$$P(\underline{x} \rightarrow \underline{\hat{x}} \,|\, \underline{\phi}) = Q\left(\frac{\sum_{n\in\eta} \cos \phi_n}{\sqrt{\sigma^2\, d_H(\underline{x}, \underline{\hat{x}})}}\right) \tag{A-11}$$

where $d_H(\underline{x}, \underline{\hat{x}})$ is the Hamming distance between $\underline{x}$ and $\underline{\hat{x}}$, i.e., the number of elements in the set η (see Eq. 3).

The argument of the Gaussian integral in Eq. (A-11) is in the form $a/\sqrt{b}$. For $a > 0$, we can upper bound this integral by[1]

$$Q\left(\frac{a}{\sqrt{b}}\right) \leqslant \frac{1}{2}\, e^{-a^2/2b} \tag{A-12}$$

Since for any λ, we have $(a - 2\lambda b)^2 > 0$, rearranging this inequality gives the equivalent form

$$\frac{a^2}{b} \geqslant 4\lambda a - 4\lambda^2 b \tag{A-13}$$

[1]Note that for perfect carrier demodulation, i.e., $\phi = 0$, we always have $a > 0$.

Thus, for $a > 0$,

$$Q\left(\frac{a}{\sqrt{b}}\right) \leqslant \frac{1}{2} \exp\{-2\lambda\,[a - \lambda b]\} \tag{A-14}$$

For $a < 0$, the loose upper bound must be used

$$Q\left(\frac{a}{\sqrt{b}}\right) = Q\left(-\frac{|a|}{\sqrt{b}}\right) = 1 - Q\left(\frac{|a|}{\sqrt{b}}\right) \leqslant 1 \tag{A-15}$$

Finally, using Eqs. (A-14) and (A-15) in Eq. (A-11) gives the desired upper bound on pairwise error probability as

$$P(\underline{x} \rightarrow \underline{\hat{x}} \,|\, \underline{\phi}; \lambda) \leqslant$$

$$\begin{cases} \frac{1}{2} \exp\left\{-\frac{E_s}{N_0} \sum_{n\in\eta} 4\lambda(\cos \phi_n - \lambda)\right\}; & \sum_{n\in\eta} \cos \phi_n > 0 \\ 1; & \sum_{n\in\eta} \cos \phi_n \leqslant 0 \end{cases} \tag{A-16}$$

In Eq. (A-16), use is made of the fact that for the unnormalized system, $1/2\sigma^2 = E_s/N_0$ where E_s is the symbol energy and N_0 the noise spectral density, and λ is replaced by the normalized quantity $\lambda\sigma^2$. Also, note that if Eq. (A-16) is minimized over λ, then it is identically in the form of a Chernoff bound.