Orthogonal Sets of Data Windows Constructed From Trigonometric Polynomials

C. A. Greenhall
Communications Systems Research Section

This article gives suboptimal, easily computable substitutes for the discrete prolate-spheroidal windows used by Thomson for spectral estimation. Trigonometric coefficients and energy leakages of the window polynomials are tabulated.

I. Motivation

In a paper [1] on precise time and time interval (PTTI) applications, D. Percival argued as follows for the increased use of frequency domain methods for studying PTTI data:

> In contrast to common practise in many other physical sciences, the statistical analysis of PTTI data is often based directly on time domain techniques rather than on frequency domain (spectral analysis) techniques. The predominant analysis technique in the PTTI community, namely, the two-sample (or Allan) variance, is often used to indirectly infer frequency domain properties under the assumption of a power-law spectrum. Here we argue that direct use and estimation of the spectrum of PTTI data have a number of potential advantages. First, spectral estimators are typically scaled independent chi-square random variables with a known number of degrees of freedom. These properties allow easy computation of the variance of estimators of various quantities that are direct functions of the spectrum. Second, the effect of detrending data can be quantified more easily in the frequency domain than in the time domain. Third, the variance of estimators of the two-sample variance can be expressed in terms of readily estimated spectral density functions. This allows one to generate confidence intervals for the two-sample variance without explicitly assuming a statistical model. Fourth, there exist tractable statistical techniques for estimating the spectrum from data sampled on an unequally spaced grid or from data corrupted by a small proportion of additive outliers. The two-sample variance cannot be readily generalized to these situations.

In the course of testing frequency standards and distribution equipment for the Deep Space Network, the JPL Frequency Standards Laboratory generates computer files containing records of phase residuals of pairs of frequency sources. The statistical properties of these time series are routinely summarized by estimating the two-sample variance [2]. This article is a step toward the goal of supplementing this current ability with the ability to generate estimates of power spectral density of phase for Fourier frequencies f below the Nyquist frequency of these records, typically 0.5 Hz. These estimates, together with measurements of spectral density for higher f provided by commercial real-time spectrum analyzers, could characterize the phase noise of frequency sources over many decades of f extending from microhertz to kilohertz. Periodic disturbances, which are now detected haphazardly by visual inspection of the residuals or by noticing steps or oscillations in the plots of two-sample variance, could be detected unambigously by statistical procedures based on spectral estimators or periodograms.

In [1] quoted above, Percival recommends a new method of spectral estimation, due to Thomson [3], that is especially suited to situations in which the range of spectral densities to be estimated is large. The method uses multiple orthogonal data windows (also called weights or tapers), the approximate computation of which is the main subject here.

II. The Thomson Spectral Estimation Method

In the Thomson method for estimating the power spectrum of a stationary time series $x[n]$ given N samples $x[0], \ldots,$ $x[N-1]$, a frequency band $[f_0 - W, f_0 + W]$ is chosen, and an estimate for the spectral density value $S(f_0)$ is computed as an average of the windowed periodograms, namely,

$$\hat{S}(f_0) = \frac{1}{K} \sum_{k=0}^{K-1} |y_k(f_0)|^2 \tag{1}$$

where

$$y_k(f) = \sum_{n=0}^{N-1} x[n]\, v_k[n; N, W]\, e^{-i2\pi fn} \tag{2}$$

The window sequences $v_0, \ldots, v_{N-1}$ are the discrete prolate-spheroidal sequences (DPSS) of Slepian [4]. They are orthonormal and are *leakage-optimal* over the space of sequences index-limited to $0, \ldots, N-1$, in the sense that

(1) v_0 has the smallest leakage of all nonzero elements

(2) for $k > 0$, v_k has the smallest leakage of all nonzero elements orthogonal to $v_0, \ldots, v_{k-1}$

For a given bandwidth W, the *leakage* $L(g, W)$ of a function g of discrete or continuous time is defined here as the fraction of its total energy contained in frequencies outside $[-W, W]$. The leakage $L(v_k, W)$ increases with k and decreases with W. By virtue of the orthogonality of the v_k, the estimate of Eq. (1) has approximately $2K$ degrees of freedom if x is Gaussian, $S(f)$ is nearly constant for $|f - f_0| \leq W$, and the leakage of v_{k-1} is small. Thus, by adjusting W and K, one can achieve a tradeoff in Eq. (1) among variance, resolution, and the influence of frequencies outside $[f_0 - W, f_0 + W]$ (the usual meaning of "leakage").

Since the computation of the DPSS requires the solution of the eigensystem of an $N \times N$ matrix (or of an integral operator if N is large), the design of easily computable suboptimal substitutes for them may be of value. In view of Nuttall's constructions of windows from cosine polynomials of low degree [5], one might expect trigonometric polynomials with both sines and cosines to make attractive materials for construction of DPSS substitutes. In fact, this idea has already been realized by Bronez ([6], p. 1869) in his recent extension of the Thomson method to the more general situation of unevenly spaced and multidimensional data. The coefficients of his polynomials and their leakages are respectively the eigenvectors and eigenvalues of a certain matrix whose coefficients depend on N, the number of data. The aim of this article is to simplify the situation further for evenly spaced one-dimensional data by deriving the coefficients and leakages of an orthonormal set of continuous-time trigonometric polynomials that do not depend on N. They are converted to discrete-time data windows by sampling them at N properly chosen points.

III. Continuous-Time Windows

In this article, w is used to denote bandwidth in terms of the fundamental frequency unit, which is $1/N$ for windows on $0, \ldots, N-1$, and 1 for windows on $[-1/2, 1/2]$, as constructed below. It is assumed that w is an integer (for the author's convenience only). Consider a time-limited trigonometric polynomial

$$\phi(x) = \sum_{\nu=-M}^{M} c[\nu]\, e^{i2\pi\nu x} \qquad |x| \leq 1/2$$

$$= 0 \qquad |x| > 1/2$$

of degree $\leq M$. Its Fourier transform is

$$\Phi(y) = \sum_{\nu=-M}^{M} c[\nu]\, s(y-\nu) \tag{3}$$

where

$$s(y) = \frac{\sin \pi y}{\pi y} \tag{4}$$

The polynomials sought can be defined immediately: their coefficient arrays are normalized eigenvectors of the positive-definite matrix

$$A[i,j] = \int_{|y|>w} s(y-i)\, s(y-j)\, dy, \qquad i,j = -M \text{ to } M$$

and their leakages are the eigenvalues. Denote the resulting polynomials by $\phi_k(x;w,M)$, $k = 0$ to $2M$, and their coefficients by $c_k[\nu]$, $\nu = -M$ to M, where the leakages $L(\phi_k, w)$ are taken in increasing order. The polynomials are orthonormal and leakage-optimal over the space of polynomials of degree $\leqslant M$. They are called *trig prolates* for short, because they can be regarded as finite-dimensional analogs of the prolate-spheroidal wave functions (PSWF) [7]. The symmetry of A about its reverse diagonal forces the eigenfunctions to be either even or odd (the odd ones are multiplied by $\pm i$ to make them real), and empirically the above indexing gives ϕ_k the parity of k. The trig prolates share with the PSWF the property of double orthogonality: their transforms $\Phi_k(y;w,M)$ are orthogonal over $[-w, w]$ as well as over $(-\infty, \infty)$.

The entries of A were computed as linear combinations of the integrals

$$\int_0^1 \left[\frac{\sin \pi y}{\pi(n+y)}\right]^2 dy, \int_0^1 \left[\frac{\sin^2 \pi y}{\pi^2(n+y)}\right] dy, \quad n = 0 \text{ to } w + M - 1$$

which were computed by Romberg quadrature. The eigenvalues and eigenvectors were computed by EISPACK routines [8], [9]. Although the leakages decrease if M increases, setting $M = w$ gives adequate performance (Section V).

Table 1 gives the coefficients and leakages of the trig prolates for $w = 2$ to $5, M = w$, and for all k such that $L(\phi_k, w) < 0.01$. Figure 1 shows the frequency response $|\Phi_k(y;4,4)|^2$ for $k = 0$ and 4. Comparing these with Thomson's graphs of the frequency responses of the DPSS for large N ([3], Fig. 2), one can see that the maximum sidelobes of the trig prolates are at most 2.5 dB above those of the corresponding DPSS, although the sidelobe structure of the trig prolates is less regular.

IV. Discrete-Time Windows

An orthogonal set of windows for data on $0, \ldots, N - 1$ and bandwidth $W = w/N$ is constructed by sampling the ϕ_k as follows:

$$u_k[n;N,W,M] = \phi_k\left(\frac{n - (N-1)/2}{N}, w, M\right), \qquad n = 0 \text{ to } N-1, k = 0 \text{ to } 2M \tag{5}$$

Notice that the denominator is N instead of $N - 1$. This has two beneficial effects:

(1) the basis functions $e^{i2\pi\nu x}$ remain orthogonal when so sampled

(2) their discrete-time transforms are more closely related to their continuous-time transforms (see below)

The discrete-time windows u_k are called *sampled trig prolates*. Orthogonality is preserved, namely,

$$\sum_{n=0}^{N-1} u_i[n]\, u_j[n] = N\delta_{ij}$$

Their discrete-time Fourier transforms are

$$U_k(f;N,W,M) = e^{-i\pi(N-1)f} \sum_{\nu=-M}^{M} c_k[\nu]\, s(Nf - \nu;N)$$

where

$$s(y;N) = \frac{\sin \pi y}{\sin(\pi y/N)}$$

(compare with Eqs. (3) and (4)). A spectral estimate of Thomson type is obtained by using $(1/\sqrt{N})\, u_k[n;N,W,M]$ in place of $v_k[n;N,W]$ in Eqs. (1) and (2).

V. Comparison with Optimal Windows

How much leakage performance is lost by the use of these suboptimal windows? Let $L(\phi_k, w)$, $L(u_k, N, W)$, and $1 - \lambda_k(N, W)$, where $NW = w = M$, be the leakages of the trig prolates, the sampled trig prolates, and the optimal DPSS respectively. Evaluating $L(u_k)$ by means of the quadratic form in the numerator of Eq. (32) of [6], it is found that $L(u_k)$ is between 0 dB and 1.2 dB less than $L(\phi_k)$ for the instances of w and k given in Table 1 and for $N = 8w$. For $N = 16w$, replace 1.2 dB by 0.6 dB. Thus, the sampled trig prolates have slightly *less* leakage than the trig prolates. Table 2 gives the ratio of sampled trig prolate leakage to DPSS leakage, which was computed by solving the eigensystem given by Eq. (2.9) of [3]. For $N = 8w$, the leakages of the trig prolates are 1.2 dB to 5.4 dB greater than those of the optimal DPSS; for $N = 16w$, the range is 1.2 dB to 2.6 dB. The leakages of the corresponding Bronez discrete polynomial windows, which form the leakage-optimal set of discrete-time polynomials of degree $\leqslant M$, necessarily lie between those of the sampled trig prolates and those of the DPSS.

VI. Conclusions

This article has described several orthonormal systems of data windows, called the sampled trig prolates, that can be used in the Thomson method of spectral estimation. For $w =$

$NW = 2$ to 5, and $4W$ not greater than the Nyquist frequency (i.e., $N \geqslant 16w$), the user of these windows pays a leakage penalty of at most 2.6 dB for not using the optimal DPSS windows. In return, one merely needs to evaluate certain trigonometric polynomials of degree w, with coefficients given in Table 1, at N points according to Eq. (5). By contrast, the evaluation of the DPSS windows requires the solution of an $N \times N$ symmetric Toeplitz matrix eigensystem. If N is large, one can proceed by solving a symmetric $J \times J$ eigensystem obtained from the approximation of a certain integral operator by Gaussian quadrature, in which the required number of knots J depends on the details of floating-point hardware and mathematical software ([3], pp. 1090–1091). The prospective user of the Thomson method might regard the 2.6-dB penalty as an attractive tradeoff for avoiding these complexities.

References

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Table 1. Sine-cosine coefficients $a_k[\nu]$ and leakage $L(\phi_k, w)$ for trig prolate $\phi_k(x;w,w) = a_k[0] + 2\sum_{\nu=1}^{M} a_k[\nu] \cos 2\pi\nu x$ (k even) or $2\sum_{\nu=1}^{M} a_k[\nu] \sin 2\pi\nu x$ (k odd)

$w = M = 2$

k	0	1
L	0.8901E-04	0.3254E-02
ν	$a_k[\nu]$	
0	0.8202108	0.0
1	0.4041691	0.7007932
2	0.0165649	0.0942808

$w = M = 3$

k	0	1	2	3
L	0.2113E-06	0.1520E-04	0.4394E-03	0.6810E-02
ν		$a_k[\nu]$		
0	0.7499700	0.0	0.4969513	0.0
1	0.4596063	0.6507499	−0.3050683	0.2731233
2	0.0867984	0.2765560	−0.5312499	−0.6397174
3	0.0007513	0.0064282	−0.0350227	−0.1271430

$w = M = 4$

k	0	1	2	3	4
L	0.4376E-09	0.4203E-07	0.1965E-05	0.5382E-04	0.9029E-03
ν			$a_k[\nu]$		
0	0.6996910	0.0	0.4783016	0.0	−0.3862293
1	0.4830013	0.5927723	−0.1666510	0.3540569	0.3223025
2	0.1473918	0.3805986	−0.5724443	−0.4929565	−0.0856254
3	0.0141997	0.0613650	−0.1736202	−0.3626279	−0.5584413
4	0.0000368	0.0003329	−0.0022015	−0.0117722	−0.0484379

$w = M = 5$

k	0	1	2	3	4	5	6
L	0.1056E-11	0.1148E-09	0.6265E-08	0.2307E-06	0.6065E-05	0.1112E-03	0.1411E-02
ν				$a_k[\nu]$			
0	0.6632850	0.0	0.4560698	0.0	−0.3821638	0.0	0.3246026
1	0.4915713	0.5401300	−0.0704481	0.3866087	0.2527019	−0.2216043	−0.2957322
2	0.1927963	0.4383060	−0.5519198	−0.3363930	0.1138304	0.3885522	0.1964585
3	0.0347859	0.1266343	−0.2915206	−0.4760267	−0.5457777	−0.3657298	0.0266965
4	0.0019243	0.0105462	−0.0379143	−0.1037856	−0.2286313	−0.4072901	−0.5631039
5	0.0000018	0.0000191	−0.0001319	−0.0007467	−0.0037712	−0.0165910	−0.0588589

Table 2. Ratio (dB) of sampled trig prolate leakage to optimal DPSS leakage[a]

k / w	0	1	2	3	4	5	6
2	2.1, 1.9	1.2, 1.2					
3	2.7, 2.0	2.4, 2.2	2.0, 1.9	1.3, 1.3			
4	3.7, 1.8	2.8, 1.7	2.6, 2.1	2.4, 2.2	1.9, 1.9		
5	5.4, 2.6	4.2, 2.1	3.3, 1.9	2.7, 2.0	2.6, 2.2	2.4, 2.2	1.9, 1.8

[a]The first entry is for $N = 8w$; the second entry is for $N = 16w$.

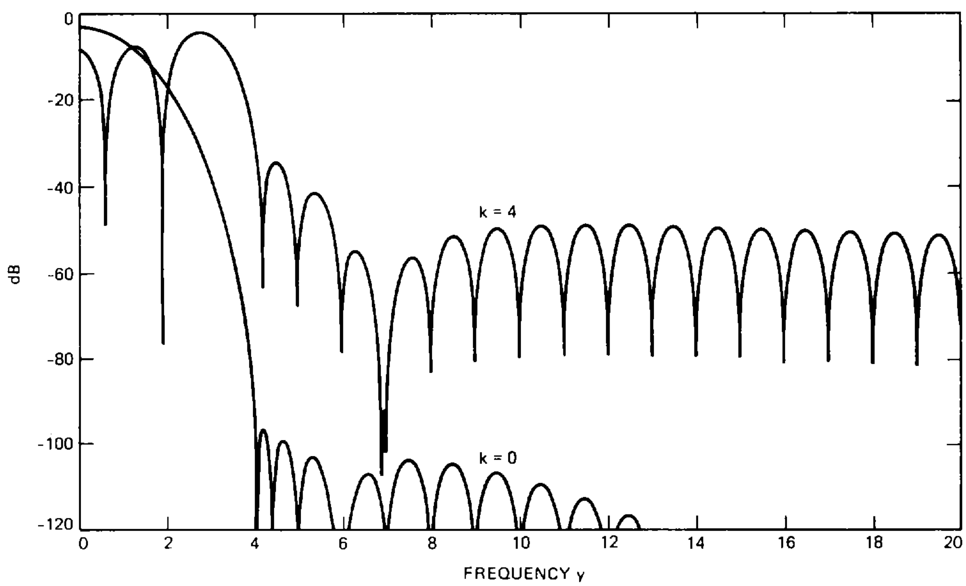

Fig. 1. Frequency response of trig prolate windows for bandwidth $w = 4$, degree $M = 4$. The total energy of each window is 1.