An Analytic Development of Orbit Determination for a Distant, Planetary Orbiter

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With the advent of the Mariner '71 Mission, NASA has been sending spacecraft to orbit various distant bodies within the solar system. At present, there is still no adequate theory describing the inherent state estimation accuracy, based on two-way, coherent range-rate data. It is the purpose of this article to lay the groundwork for a general elliptic theory, and in addition to provide an analytic solution for the special case of circular orbits.

I. Introduction

When one begins to analyze a possible future mission employing an orbiting spacecraft, it is desirable to know the characteristics of the position and velocity errors of that spacecraft, based on a single revolution of range-rate data. To know these errors as a function of orbit shape, size, and orientation with respect to the Earth is the central issue. If range were the data type of interest, the problem would be vastly simpler, since velocities would not be involved. Unfortunately for the analyst, however, range-rate data is central and must be treated. This particular problem has proved to be a most formidable analytic endeavor.

II. Development of the Observable

One of the most important aspects of this orbital estimation problem is to pose it properly. In this regard, the "binary star problem" of classical astronomy can provide insight. In attempting to obtain the orbits of a distant binary star system through the use of spectroscopic Doppler shifts, astronomers discovered that they could determine all the elements of the motion except one. The one element, which turned out to be completely indeterminate, is the longitude of the ascending node relative to the plane-of-the-sky, with the plane-of-the-sky being that plane normal to the line-of-sight from the observer to the binary system.

The reason for this indeterminacy is that the binary star system is effectively "at infinity," meaning that no significant parallactic motion exists between the observer and the observed.

In this problem, the body about which the spacecraft is in orbit is not "at infinity," but is relatively near. This implies that there will be some parallactic motion between the observer and the observed, and hence we should be able to obtain a "reasonable" solution for the longitude of the ascending node relative to the plane-of-the-sky. This parameter will typically be the principal contributor to the spacecraft state error. As such, classical elements relative to the plane-of-the-sky appear to be a sensible "eigen" system in which to solve the problem.

The geometry of a spacecraft in orbit about a distant body is depicted in Fig. 1. Let:

$$\mathbf{r}_p = \begin{pmatrix} x_p \\ y_p \\ z_p \end{pmatrix} \tag{1}$$

$$\dot{\mathbf{r}}_p = \begin{pmatrix} \dot{x}_p \\ \dot{y}_p \\ \dot{z}_p \end{pmatrix} \tag{2}$$

where x_p, y_p, and z_p are the plane-of-the-sky Cartesian position coordinates of the spacecraft and $\dot{x}_p$, $\dot{y}_p$, and $\dot{z}_p$ are the plane-of-the-sky Cartesian velocity coordinates of the spacecraft.

The orbital coordinates are then defined as:

$$\mathbf{r}_o = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \tag{3}$$

$$\dot{\mathbf{r}}_o = \begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{pmatrix} \tag{4}$$

with

$$\xi = a(\cos E - e) \tag{5}$$

$$\eta = a\beta \sin E \tag{6}$$

$$\zeta = 0 \tag{7}$$

$$\dot{\xi} = -\left(\frac{an \sin E}{1 - e\cos E}\right) \tag{8}$$

$$\dot{\eta} = \left(\frac{an\beta \cos E}{1 - e\cos E}\right) \tag{9}$$

$$\dot{\zeta} = 0 \tag{10}$$

$$\beta = \sqrt{1 - e^2} \tag{11}$$

where E is the *eccentric anomaly*, a is the orbit *semi-major axis* (km), e is the orbit *eccentricity* (dimensionless), and n is the orbit *mean motion* (s^{-1}).

The orbital coordinates can be related to the Cartesian, plane-of-the-sky coordinates as follows:

$$\mathbf{r}_p = R^T(\Omega, \omega, i)\mathbf{r}_o \tag{12}$$

where $R^T(\Omega, \omega, i)$ is the transpose of the Euler rotation matrix, defined as

$$R(\Omega,\omega,i) = \begin{pmatrix} \cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega & \sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega & \sin i\sin\omega \\ -\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega & -\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega & \sin i\cos\omega \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{pmatrix} \tag{13}$$

and Ω is the *longitude of ascending node* relative to the plane-of-the-sky, ω is the *argument of periapsis* relative to the plane-of-the-sky, and i is the *inclination* relative to the plane-of-the-sky.

To have any hope of analytically solving this problem, all possible simplifications must be made without, of course, loss of generality. One such simplification will be to define the x-axis of the plane-of-the-sky system to be the line-of-nodes. This implies that

$$\Omega = 0 \tag{14}$$

Thus $R^T(\Omega,\omega,i)$ becomes

$$R^T(\omega,i) = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega\cos i & \cos\omega\cos i & -\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{pmatrix} \tag{15}$$

If it is now assumed that the orbital elements are constant in time,

$$\dot{\mathbf{r}}_p = R^T(\omega,i)\dot{\mathbf{r}}_o \tag{16}$$

From Fig. 1, it is seen that

$$\mathbf{r}_d = \mathbf{r}_p - \mathbf{r}_T \tag{17}$$

$$\mathbf{r}_T = \begin{pmatrix} V_N t\cos\theta \\ V_N t\sin\theta \\ V_R t + \rho_{ET} \end{pmatrix} \tag{18}$$

where:

V_N is the component of the velocity of the target central body relative to the Earth that lies in the plane-of-the-sky; this shall hereinafter be referred to as the "cross-velocity"

V_R is the component of the velocity of the target central body relative to the Earth that lies along the line-of-sight

ρ_{ET} is the distance along the line-of-sight between the center of the Earth and the center of the target central body at the initial epoch of estimation, i.e., at $t = 0$

θ is the angle between the cross-velocity and the line-of-nodes

One can now form the range-rate observable, $\dot{\rho}$, as

$$\dot{\rho} = \left(\frac{\mathbf{r}_d \cdot \dot{\mathbf{r}}_d}{\rho}\right) \tag{19}$$

where

$$\rho = |\mathbf{r}_d| \tag{20}$$

Therefore,

$$\rho\dot{\rho} = \mathbf{r}_p \cdot \dot{\mathbf{r}}_p - \mathbf{r}_p \cdot \dot{\mathbf{r}}_T - \mathbf{r}_T \cdot \dot{\mathbf{r}}_p + \mathbf{r}_T \cdot \dot{\mathbf{r}}_T \tag{21}$$

where

$$\dot{\mathbf{r}}_T = \begin{pmatrix} V_N \cos\theta \\ V_N \sin\theta \\ V_R \end{pmatrix} \tag{22}$$

It can then be shown that

$$\mathbf{r}_T \cdot \dot{\mathbf{r}}_T = V_N^2 t + V_R^2 t + V_R \rho_{ET} \tag{23}$$

$$\mathbf{r}_p \cdot \dot{\mathbf{r}}_p = a^2 ne \sin E \tag{24}$$

$$\mathbf{r}_p \cdot \dot{\mathbf{r}}_T = V_N(\xi\alpha_1 + \eta\alpha_2) + V_R(\xi \sin\omega + \eta\cos\omega)\sin i \tag{25}$$

$$\mathbf{r}_T \cdot \dot{\mathbf{r}}_p = V_N t(\dot{\xi}\alpha_1 + \dot{\eta}\alpha_2) + (V_R t + \rho_{ET})(\dot{\xi}\sin\omega + \dot{\eta}\cos\omega)\sin i \tag{26}$$

where

$$\alpha_1 = \cos\theta\cos\omega + \sin\theta\sin\omega\cos i \tag{27}$$

$$\alpha_2 = -\cos\theta\sin\omega + \sin\theta\cos\omega\cos i \tag{28}$$

Next one has

$$\rho = \left(\mathbf{r}_d \cdot \mathbf{r}_d\right)^{\frac{1}{2}} = \left[\mathbf{r}_p \cdot \mathbf{r}_p + \mathbf{r}_T \cdot \mathbf{r}_T - 2\mathbf{r}_p \cdot \mathbf{r}_T\right]^{\frac{1}{2}} \tag{29}$$

This can be shown to be

$$\rho = \Big[a^2\left(1 - e\cos E\right)^2 + \rho_{ET}^2 + 2\rho_{ET} V_R t + \left(V_N^2 + V_R^2\right)t^2$$
$$-2V_N t(\xi\alpha_1 + \eta\alpha_2) - 2(V_R t + \rho_{ET})(\xi\sin\omega + \eta\cos\omega)\sin i\Big]^{\frac{1}{2}} \tag{30}$$

For large ρ_{ET} this implies

$$\frac{1}{\rho} = \frac{1}{\rho_{ET}}\left[1 + \frac{2V_R t}{\rho_{ET}} - \frac{2\sin i}{\rho_{ET}}\Big\{\xi\sin\omega + \eta\cos\omega\Big\} + O\Big(\frac{1}{\rho_{ET}^2}\Big)\right]^{-\frac{1}{2}} \tag{31}$$

This then becomes

$$\frac{1}{\rho} \approx \frac{1}{\rho_{ET}}\left[1 - \frac{V_R t}{\rho_{ET}} + \frac{\sin i}{\rho_{ET}}\Big\{\xi\sin\omega + \eta\cos\omega\Big\}\right] \tag{32}$$

Combining Eqs. (21), (23)-(26), and (32) yields

$$\begin{aligned}\dot{\rho} \approx \frac{1}{\rho_{ET}}\Big[&\rho_{ET}V_R - \rho_{ET}\Big\{\dot{\xi}\sin\omega + \dot{\eta}\cos\omega\Big\}\sin i + V_N^2 t + a^2 ne\sin E \\ &-V_N\Big\{(\xi+\dot{\xi}t)\alpha_1 + (\eta+\dot{\eta}t)\alpha_2\Big\} - \Big\{\xi\sin\omega+\eta\cos\omega\Big\}\Big\{\dot{\xi}\sin\omega+\dot{\eta}\cos\omega\Big\}\sin^2 i\Big]\end{aligned} \tag{33}$$

From the above, one can observe that

$$\Big\{\xi\sin\omega + \eta\cos\omega\Big\} = a\Big\{(\cos E - e)\sin\omega + \beta\sin E\cos\omega\Big\} \tag{34}$$

$$\Big\{\dot{\xi}\sin\omega + \dot{\eta}\cos\omega\Big\} = \left(\frac{an}{1-e\cos E}\right)\Big\{-\sin E\sin\omega + \beta\cos E\cos\omega\Big\} \tag{35}$$

Combining Eqs. (33)-(35) produces

$$\begin{aligned}\dot{\rho} \approx \Big[&V_R - \left(\frac{an\sin i}{1-e\cos E}\right)\Big\{-\sin E\sin\omega + \beta\cos E\cos\omega\Big\} \\ &+ \frac{1}{\rho_{ET}}\Big\{V_N^2 t - \left(\frac{a^2 n\sin^2 i}{1-e\cos E}\right)\Big\{\sin E\sin\omega - \beta\cos E\cos\omega\Big\}\Big\{(\cos E - e)\sin\omega + \beta\sin E\cos\omega\Big\} \\ &+ a^2 ne\sin E - V_N\Big\{(\xi+\dot{\xi}t)\alpha_1 + (\eta+\dot{\eta}t)\alpha_2\Big\}\Big\}\Big]\end{aligned} \tag{36}$$

Now one proceeds to expand the last term of Eq. (36). To do this, Kepler's Equation is needed:

$$E - e\sin E = n(t - T_P) \tag{37}$$

where T_P is the *time of periapsis passage.*

Without loss of generality, one may assume that $T_P = 0$. Therefore

$$nt = E - e\sin E \tag{38}$$

Combining Eqs. (5), (6), (8), (9), and (38) produces

$$\begin{aligned}&\Big\{(\xi + \dot{\xi}t)\alpha_1 + (\eta + \dot{\eta}t)\alpha_2\Big\} \\ &\quad = a\Big\{\alpha_1\Big(\cos E - e - \frac{(E - e\sin E)\sin E}{1 - e\cos E}\Big) + \alpha_2\beta\Big(\sin E + \frac{(E - e\sin E)\cos E}{1 - e\cos E}\Big)\Big\}\end{aligned} \tag{39}$$

The range-rate approximation then becomes

$$\begin{aligned}\dot{\rho} \approx \Bigg[&V_R - \left(\frac{an\sin i}{1 - e\cos E}\right)\Big\{-\sin E\sin\omega + \beta\cos E\cos\omega\Big\} + \frac{1}{\rho_{ET}}\Big\{V_N^2 t + a^2 ne\sin E \\ &- \left(\frac{a^2 n\sin^2 i}{1 - e\cos E}\right)\Big\{\sin E\sin\omega - \beta\cos E\cos\omega\Big\}\Big\{(\cos E - e)\sin\omega + \beta\sin E\cos\omega\Big\} \\ &- V_N a\Big\{\alpha_1\Big(\cos E - e - \frac{(E - e\sin E)\sin E}{1 - e\cos E}\Big) + \alpha_2\beta\Big(\sin E + \frac{(E - e\sin E)\cos E}{1 - e\cos E}\Big)\Big\}\Big\}\Bigg]\end{aligned} \tag{40}$$

If one wished to compute an approximation to the actual received range-rate signal, ignoring the motion of the receiving station relative to the center of the Earth, Eq. (40) would be the expression to use. In this case, however, the range-rate approximation shall be used for the development of partial derivatives only, and the only partial derivatives of interest are with respect to the spacecraft state (classical orbital elements with respect to the plane-of-the-sky). Hence, the elements of interest are a, e, T_P, ω, i, and the final element is θ. Note that θ by our definition is equivalent to the *longitude of the ascending node*, Ω.

Thus those terms in the range-rate approximation which do not contribute to the appropriate partial derivatives will hereinafter be ignored. Additionally, those terms which contribute negligible partial derivatives will also be ignored. As a result, the range-rate approximation which shall be used for the computation of the partial derivatives is

$$\begin{aligned}\dot{\rho} \approx \Bigg[&\left(\frac{an\sin i}{1 - e\cos E}\right)\Big\{\sin E\sin\omega - \beta\cos E\cos\omega\Big\} \\ &- \left(\frac{V_N a}{\rho_{ET}}\right)\Big\{\alpha_1\Big(\cos E - e - \frac{(E - e\sin E)\sin E}{1 - e\cos E}\Big) + \alpha_2\beta\Big(\sin E + \frac{(E - e\sin E)\cos E}{1 - e\cos E}\Big)\Big\}\Bigg]\end{aligned} \tag{41}$$

The partial derivatives with respect to a, e, T_P, ω, and i shall be computed from the first term in the range-rate approximation (since these dominate partials for the same parameters taken from the second term), whereas the partial derivative with respect to θ will be computed from the second term (this is the only term with θ variation).

III. Obtaining the Partial Derivatives

Rewriting Eq. (41) produces

$$\dot{\rho} \approx \left[\left(\frac{a^{-\frac{1}{2}} \mu^{\frac{1}{2}} \sin i}{1 - e \cos E} \right) \left\{ \sin E \sin \omega - \beta \cos E \cos \omega \right\} \right.$$

$$\left. - \left(\frac{V_N a}{\rho_{ET}} \right) \left\{ \alpha_1 \left(\cos E - e - \frac{(E - e \sin E) \sin E}{1 - e \cos E} \right) + \alpha_2 \beta \left(\sin E + \frac{(E - e \sin E) \cos E}{1 - e \cos E} \right) \right\} \right] \tag{42}$$

To produce the partial of the observables with respect to the classical state of the spacecraft, the partials of the eccentric anomaly, E, with respect to the classical state are needed. They are

$$\left(\frac{\partial E}{\partial a} \right) = -\frac{3(E - e \sin E)}{2a(1 - e \cos E)} \tag{43}$$

$$\left(\frac{\partial E}{\partial e} \right) = \frac{\sin E}{(1 - e \cos E)} \tag{44}$$

$$\left(\frac{\partial E}{\partial T_P} \right) = -\frac{n}{(1 - e \cos E)} \tag{45}$$

$$\left(\frac{\partial E}{\partial \omega} \right) = 0 \tag{46}$$

$$\left(\frac{\partial E}{\partial i} \right) = 0 \tag{47}$$

$$\left(\frac{\partial E}{\partial \theta} \right) = 0 \tag{48}$$

Taking partial derivatives of the observable produces

$$\left(\frac{\partial \dot{\rho}}{\partial a} \right) = \left(\frac{n \sin i}{2} \right) \left[\frac{\{- \sin E \sin \omega + \beta \cos E \cos \omega\}}{(1 - e \cos E)} \right.$$

$$\left. - \frac{3(E - e \sin E) \{(\cos E - e) \sin \omega + \beta \sin E \cos \omega\}}{(1 - e \cos E)^3} \right] \tag{49}$$

$$\left(\frac{\partial \dot{\rho}}{\partial e} \right) = an \sin i \left[\frac{- \cos E \{(\cos E - e) \cos \omega - \beta \sin E \sin \omega\}}{\beta (1 - e \cos E)^2} \right.$$

$$\left. + \frac{\sin E \{(\cos E - e) \sin \omega + \beta \sin E \cos \omega\}}{(1 - e \cos E)^3} \right] \tag{50}$$

$$\left(\frac{\partial\dot{\rho}}{\partial T_P}\right) = -an^2\sin i\left[\frac{(\cos E - e)\sin\omega + \beta\sin E\cos\omega}{(1-e\cos E)^3}\right] \tag{51}$$

$$\left(\frac{\partial\dot{\rho}}{\partial\omega}\right) = an\sin i\left[\frac{\sin E\cos\omega + \beta\cos E\sin\omega}{(1-e\cos E)}\right] \tag{52}$$

$$\left(\frac{\partial\dot{\rho}}{\partial i}\right) = an\cos i\left[\frac{\sin E\sin\omega - \beta\cos E\cos\omega}{(1-e\cos E)}\right] \tag{53}$$

$$\left(\frac{\partial\dot{\rho}}{\partial\theta}\right) = -\left(\frac{V_N a}{\rho_{ET}}\right)\left[\alpha_3\left\{\cos E - e - \frac{(E-e\sin E)\sin E}{1-e\cos E}\right\} + \alpha_4\beta\left\{\sin E + \frac{(E-e\sin E)\cos E}{1-e\cos E}\right\}\right] \tag{54}$$

where

$$\alpha_3 = -\sin\theta\cos\omega + \cos\theta\sin\omega\cos i \tag{55}$$

$$\alpha_4 = \sin\theta\sin\omega + \cos\theta\cos\omega\cos i \tag{56}$$

Again, if one is to get through this problem, one must simplify as much as possible or risk getting lost in the algebra. A careful examination of the partials detailed in Eqs. (49)-(54) reveals that the first term of Eq. (49), other than a constant of proportionality, is the same as Eq. (53). Therefore it may be eliminated from Eq. (49) by a change of variable. Let

$$a' = a + \left(\frac{2a}{\tan i}\right)i = a + (2a\cot i)i \tag{57}$$

which implies that

$$\left(\frac{\partial\dot{\rho}}{\partial a'}\right) = \left(\frac{\partial\dot{\rho}}{\partial a}\right)\left(\frac{\partial a}{\partial a'}\right) + \left(\frac{\partial\dot{\rho}}{\partial i}\right)\left(\frac{\partial i}{\partial a'}\right) \tag{58}$$

From Eq. (57), one has

$$\left(\frac{\partial a}{\partial a'}\right) = 1 \tag{59}$$

$$\left(\frac{\partial i}{\partial a'}\right) = \left(\frac{1}{2a\cot i}\right) = \left(\frac{\tan i}{2a}\right) \tag{60}$$

Therefore,

$$\left(\frac{\partial\dot{\rho}}{\partial a'}\right) = \left(\frac{-3n\sin i}{2}\right)\left[\frac{(E-e\sin E)\{(\cos E - e)\sin\omega + \beta\sin E\cos\omega\}}{(1-e\cos E)^3}\right] \tag{61}$$

To obtain the correct partials in Eq. (58), one must carefully discern which of the variables in Eq. (57) are differentiable parameters and which are not; it is then clear that Eq. (61) represents a considerable simplification of the problem.

IV. Accumulating the Information Array

The normal, Gaussian way by which to obtain the covariance of errors in a set of estimated parameters, Γ, is stated as follows:

$$\Gamma = \left[\tilde{\Gamma}^{-1} + A^T W A\right]^{-1} \tag{62}$$

where $\tilde{\Gamma}$ is the a priori error in the covariance, A is the vector partial derivative of the observable (at the time of observation) with respect to the parameters being estimated, and W is the weighting matrix. If it is assumed that the error in the measurement of the observable at time t_i is uncorrelated with the error in the measurement of the observable at time t_j, for all i and j, then W becomes a diagonal matrix of dimension equal to the number of observables. From minimum-variance concepts, the diagonal elements of the weighting matrix may be shown to be just the inverse of the variance of error in the observable (it is assumed the error process of the observable is stationary and hence that the variance of error in the observable is constant and unchanging for every data point).

If one assumes the existence of no a priori information, then the error covariance sought, assuming discrete observations, may be formed as the inverse of the information array:

$$\Gamma = \left[\frac{1}{\sigma_{\dot\rho}^2} \sum_{n=1}^{N} \left(\frac{\partial \dot\rho_n}{\partial \mathbf{X}}\right)^T \left(\frac{\partial \dot\rho_n}{\partial \mathbf{X}}\right)\right]^{-1} \tag{63}$$

where $\mathbf{X}$ is the vector of estimated parameters (in this case this is the spacecraft state in classical elements relative to the plane-of-the-sky), which shall be ordered in the following way:

$$\mathbf{X} = \left(p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ p_6\right)^T = \left(\theta,\ a',\ i,\ e,\ T_P,\ \omega\right)^T$$

and N is the number of data points (observations).

In this case, to simplify the accumulation, continuous observations are assumed, and hence the sums involved will become integrals. For further simplicity, the accumulation shall be performed over a full orbit period in the time domain P (in the manner in which "real" data is taken, in equal increments of time), beginning at $-P/2$ and ending at $+P/2$. Since the observable and the partials are all formulated in terms of the eccentric anomaly, E, it is logical to perform the accumulation with this parameter as the independent variable. From Eq. (37), one has

$$dt = \frac{(1 - e\cos E)}{n} dE \tag{64}$$

Thus a general information array element, I_{ij}, may be stated as

$$I_{ij} = \frac{1}{(\alpha\sigma_{\dot\rho}^2)} \int_{-\pi}^{\pi} \left(\frac{\partial \dot\rho}{\partial p_i}\right) \left(\frac{\partial \dot\rho}{\partial p_j}\right) \frac{(1 - e\cos E)}{n} dE \tag{65}$$

where α is a scale factor chosen such that the information array, I_{ij}, obtained continuously is the same as that formed discretely.

To ascertain the value of α, let $\mathbf{X}$ consist of only one parameter, with its attendant partial derivative being a constant, K. Then the discrete information is just

$$I_{11} = \left[\frac{K^2 N}{\sigma_{\dot{\rho}}^2}\right] \tag{66}$$

The continuous information, however, is

$$I_{11} = \left[\frac{K^2 P}{\alpha\sigma_{\dot{\rho}}^2}\right] \tag{67}$$

where P is the orbit period (in seconds).

Therefore, for the two informations to be equal, one must have

$$\alpha = \left(\frac{P}{N}\right) \tag{68}$$

This, however, implies that

$$\alpha = \Delta t \tag{69}$$

where Δt is the discrete sample rate (in seconds).

In the standard case of two-way, coherent range rate as the orbital data type, the discrete sample rate is 60 seconds.

Thus the elements of the information array may now be stated using the continuous formulation as

$$I_{ij} = \frac{1}{(\sigma_{\dot{\rho}}^2 \Delta t)} \int_{-\pi}^{\pi} \left(\frac{\partial \dot{\rho}}{\partial p_i}\right) \left(\frac{\partial \dot{\rho}}{\partial p_j}\right) \frac{(1 - e\cos E)}{n} dE \tag{70}$$

For simplicity, scaled information array elements, I'_{ij}, shall be stated where

$$I'_{ij} = \left(\frac{\sigma_{\dot{\rho}}^2 n \Delta t}{\pi}\right) I_{ij} \tag{71}$$

Note that Eq. (71) implies that when the determinant of the full 6×6 information array is computed, to obtain a correct value, the result must be multiplied by

$$\left(\frac{\sigma_{\dot{\rho}}^2 n \Delta t}{\pi}\right)^6$$

Similarly for the cofactors of dimension 5 × 5, the result must be multiplied by

$$\left(\frac{\sigma_\rho^2 n\Delta t}{\pi}\right)^5$$

Since the information array is symmetric, only the upper diagonal portion of the array shall be stated.

If one examines Eq. (42), it can be seen that the range-rate value obtained for a certain value of ω is the negative of that for $\omega + \pi$. The partials will also be of opposite signs. The information array elements will, however, be the same. Hence the information array must be periodic in ω, but go through two full periods as ω goes from zero to 360 degrees.

With the complicated nature of the partials, there is a need for a very specific table of integrals to perform the accumulation. Such a table is provided in the Appendices. With the use of the Appendices, the elements of the information array can be straightforwardly computed.

Due to the complexity of the information array, any further simplification will be most welcome. We are only interested in obtaining the variance (standard deviation) of θ, which is the error in the line-of-nodes relative to the plane-of-the-sky. Any element in the inverted information array (the covariance), C_{ij}, can be stated as

$$C_{ij} = \left[\frac{(-1)^{i+j}\, Co\, D_{ij}}{D}\right] \tag{72}$$

where D is the determinant of the full 6 × 6 information array, I', with the appropriate constant factors included, and CoD_{ij} is the cofactor of the ijth element of the information array, which is just the determinant of the remaining 5 × 5 array obtained by removing row i and column j from the full 6 × 6 information array, with also the appropriate constant factors included.

In this case, we are interested in C_{11} only. This is just

$$C_{11} = \sigma_\theta^2 = \frac{Co\, D_{11}}{D} \tag{73}$$

If it is assumed that there are removable coefficients, a_i, in each of the ith rows and ith columns of I', and similar coefficients, b_i, coming from the 5 × 5 matrix whose determinant is the CoD_{11}, then

$$C_{11} = \sigma_\theta^2 = \left(\frac{Co\, D'_{11}}{D'}\right)\frac{\prod_{j=1}^{5} b_j^2}{\prod_{i=1}^{6} a_i^2} = \frac{1}{a_1^2}\left(\frac{Co\, D'_{11}}{D'}\right) \tag{74}$$

where CoD'_{11} and D' are the corresponding determinants with common factors removed.

One can now state the information array, I', to compute D' and CoD_{11}. The information array, with coefficients removed, is shown below.

$$I'_{11} = \Bigg[\left\{\frac{\pi^2}{3} + 1 - 2e + 2e^2\right\}\left(\alpha_3^2 + \alpha_4^2\right)$$

$$+ \left(\frac{2}{e^2}\right)\left\{\beta S(\gamma) + 2\beta^2 \ln\left\{\frac{1+\beta}{2(1+e)}\right\} - \frac{\pi^2}{6}(2-e^2) - \beta^3 + 3e\beta^2 + e - \frac{1}{2}(3-5\beta^2)\right\}\left(\alpha_4^2 - \alpha_3^2\right)\Bigg] \tag{75}$$

where

$$\left(\alpha_3^2 + \alpha_4^2\right) = \sin^2\theta + \cos^2\theta\cos^2 i \tag{76}$$

$$\left(\alpha_4^2 - \alpha_3^2\right) = 2\sin 2\omega \sin\theta\cos\theta\cos i - \cos 2\omega\left(\sin^2\theta - \cos^2\theta\cos^2 i\right) \tag{77}$$

$$I'_{12} = \Bigg[\left\{\frac{1}{\beta^2}S(\gamma) - \frac{2}{\beta}\ln(1+\gamma) - (1+\beta) + \frac{2\beta}{(1+e)}\right\}\sin\theta$$
$$- \frac{2\beta}{e^2}\left\{(2\ln(1+\gamma) + 3\ln\left\{\frac{1+\beta}{2(1+e)}\right\} + 2e + \frac{1}{2}(1-\beta)(1-3\beta)\right\}$$
$$\times\left(\sin\theta\cos 2\omega - \cos\theta\cos i\sin 2\omega\right)\Bigg] \tag{78}$$

$$I'_{13} = \beta\left[\sin\theta + \left\{\gamma^2 + \frac{2}{e} + \frac{2}{e^2}\ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right\}\left(\sin\theta\cos 2\omega - \cos\theta\cos i\sin 2\omega\right)\right] \tag{79}$$

$$I'_{14} = \Bigg[\left(\frac{-1}{e\beta}\right)\left\{\ln\left\{\frac{1+\beta}{2(1+e)}\right\} + 2 - \beta(1+\beta)\right\}\sin\theta$$

$$+ \left(\frac{-1}{e^3\beta}\right)\left\{(2-e^2)\ln\left\{\frac{1+\beta}{2(1+e)}\right\} + (2e^2-1)(1-\beta) + 2e - e^4\right\}\left(\sin\theta\cos 2\omega - \cos\theta\cos i\sin 2\omega\right)\Bigg] \tag{80}$$

$$I'_{15} = \left[\left(\frac{1+2e}{1+e}\right)\cos\theta\cos i - \left(\frac{2+e-\beta(1+2e)}{(1+e)(1+\beta)}\right)\left(\sin\theta\sin 2\omega + \cos\theta\cos i\cos 2\omega\right)\right] \tag{81}$$

$$I'_{16} = \beta\left[\cos\theta\cos i - \left(\gamma^2 + \frac{2}{e} + \frac{2}{e^2}\ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right)\left(\sin\theta\sin 2\omega + \cos\theta\cos i\cos 2\omega\right)\right] \tag{82}$$

$$I'_{22} = \Bigg[\left\{\frac{(2+e^2)}{2\beta^5}S(\gamma) - \frac{6}{\beta^4}\ln(1+\gamma) + \frac{3}{\beta^4}(1-e) - \frac{1}{2\beta^3}(\beta^2 + 2\beta + 3)\right\}$$

$$-\cos 2\omega\left\{\frac{e^2}{4\beta^5}S(\gamma) + \frac{(2-5e^2)}{3e^2\beta^4}\ln(1+\gamma) + \frac{(1-e)(2+e^2)}{6e^2\beta^4} + \frac{1}{4\beta} - \frac{(1+\beta)(3-\beta)}{12e^2\beta^3}\right\}\Bigg] \tag{83}$$

$$I'_{23} = \left[\frac{-e}{(1+e)} + \left\{\frac{1}{1+\beta)} - \frac{e}{1+e}\right\}\cos 2\omega\right] \tag{84}$$

$$I'_{24} = \left[\frac{1}{2e}\left\{\frac{1}{\beta} - \frac{1}{(1+e)^2}\right\} + \frac{\cos 2\omega}{12e^3}\left\{\frac{1}{\beta}\left(5 - 14\beta + 9\beta^2\right) + \frac{2e^2}{(1+e)^2}\right\}\right] \tag{85}$$

$$I'_{25} = \left(\frac{1}{12e^2\beta^2}\right)\left[1 - 3\beta^2 + \frac{2(1+3e)\beta^3}{(1+e)^3}\right]\sin 2\omega \tag{86}$$

$$I'_{26} = \left[\frac{e}{1+e} - \frac{1}{1+\beta}\right]\sin 2\omega \tag{87}$$

$$I'_{33} = 1 - \left(\frac{1-\beta}{1+\beta}\right)\cos 2\omega \tag{88}$$

$$I'_{34} = -\frac{(1-\beta)}{e\beta(1+\beta)}\cos 2\omega \tag{89}$$

$$I'_{35} = 0 \tag{90}$$

$$I'_{36} = \left(\frac{1-\beta}{1+\beta}\right)\sin 2\omega \tag{91}$$

$$I'_{44} = \frac{1}{4\beta^3(1+\beta)}\left[2(3+\beta) + \frac{(1-\beta)^2}{(1+\beta)}\cos 2\omega\right] \tag{92}$$

$$I'_{45} = \left(\frac{e}{4\beta^4}\right)\sin 2\omega \tag{93}$$

$$I'_{46} = \frac{(1-\beta)}{e\beta(1+\beta)}\sin 2\omega \tag{94}$$

$$I'_{55} = \frac{1}{4\beta^5}\left[2(2+e^2) - e^2\cos 2\omega\right] \tag{95}$$

$$I'_{56} = \frac{1}{\beta^2} \tag{96}$$

$$I'_{66} = 1 + \left(\frac{1-\beta}{1+\beta}\right)\cos 2\omega \tag{97}$$

where the removable coefficients, of course, turn out to be the coefficients of the individual partial derivatives, which are

$$a_1 = \left(\frac{-V_N a}{\rho_{ET}}\right) \tag{98}$$

$$a_2 = \left(\frac{-3n\sin i}{2}\right) \tag{99}$$

$$a_3 = an\cos i \tag{100}$$

$$a_4 = an\sin i \tag{101}$$

$$a_5 = -an^2\sin i \tag{102}$$

$$a_6 = an\sin i \tag{103}$$

From these scale factors and Eq. (71), one can infer that the error in θ may be written as

$$\sigma_\theta = \sigma_{\dot{\rho}} \left[\frac{\rho_{ET} (\mu)^{\frac{1}{4}} (\Delta t)^{\frac{1}{2}}}{V_N (a)^{\frac{7}{4}} (\pi)^{\frac{1}{2}}} \right] \left[\frac{f(e,\omega)}{g(\theta, i, e, \omega)} \right]^{\frac{1}{2}} \tag{104}$$

where $f(e,\omega)$ is the cofactor of D'_{11}, and $g(\theta, i, e, \omega)$ is just the determinant, D'.

V. Inverting the Information Array

Since the interest is in a single element of the inversion, only those computations required in Eq. (74) shall be performed. Before plunging into the general elliptic case, the inversion for the special case of the circular orbit (i.e., $e = 0$) shall be demonstrated.

A. Circular Case

After much work, the elements of the information array (in the limit as e tends to zero) can be shown to be:

$$I'_{11} = \left(\frac{\pi^2}{3} + 1\right)\left(\alpha_3^2 + \alpha_4^2\right) - \frac{1}{2}\left(\alpha_4^2 - \alpha_3^2\right) \tag{105}$$

$$I'_{12} = \frac{\pi^2}{3} \sin\theta \tag{106}$$

$$I'_{13} = \sin\theta + \frac{1}{2}\left(\sin\theta \cos 2\omega - \cos\theta \cos i \sin 2\omega\right) \tag{107}$$

$$I'_{14} = \sin\theta - \frac{1}{3}\left(\sin\theta \cos 2\omega - \cos\theta \cos i \sin 2\omega\right) \tag{108}$$

$$I'_{15} = \cos\theta \cos i - \frac{1}{2}\left(\sin\theta \sin 2\omega + \cos\theta \cos i \cos 2\omega\right) \tag{109}$$

$$I'_{16} = \cos\theta \cos i - \frac{1}{2}\left(\sin\theta \sin 2\omega + \cos\theta \cos i \cos 2\omega\right) \tag{110}$$

$$I'_{22} = \frac{\pi^2}{3} - \frac{1}{2} \cos 2\omega \tag{111}$$

$$I'_{23} = \frac{1}{2} \cos 2\omega \tag{112}$$

$$I'_{24} = 1 - \frac{1}{3} \cos 2\omega \tag{113}$$

$$I'_{25} = -\frac{1}{2} \sin 2\omega \tag{114}$$

$$I'_{26} = -\frac{1}{2} \sin 2\omega \tag{115}$$

$$I'_{33} = I'_{44} = I'_{55} = I'_{56} = I'_{66} = 1 \tag{116}$$

$$I'_{34} = I'_{35} = I'_{36} = I'_{45} = I'_{46} = 0 \tag{117}$$

To perform the inversion in this special, circular case, it is necessary to remove the parameter ω from the estimation since the argument of periapsis is indeterminate. Additionally the parameter T_P is removed from the estimation since the partials are large and the same as those of ω, only with the opposite sign. This rank reduction corresponds to assuming that the true anomaly of the spacecraft at epoch is perfectly known, which is realistic in comparison to the error in θ. With a large amount of algebraic manipulation, the four-parameter inversion yields an error in θ of

$$\sigma_\theta = \sigma_{\dot{\rho}} \left[\frac{\rho_{ET}(\mu)^{\frac{1}{4}}(\Delta t)^{\frac{1}{2}}}{V_N(a)^{\frac{7}{4}}(\pi)^{\frac{1}{2}}} \right] \left[\frac{f(\omega)}{g(\theta, i, \omega)} \right]^{\frac{1}{2}} \tag{118}$$

where

$$f(\omega) = \left[\frac{\pi^2}{3} - 1 + \frac{1}{6}\cos 2\omega - \frac{13}{36}\cos^2 2\omega \right] \tag{119}$$

$$g(\theta, i, \omega) = \left(\frac{\cos^2\theta \cos^2 i}{9} \right) \left[\pi^4 - \pi^2 \cos 2\omega - \frac{13}{12}\pi^2 \cos^2 2\omega - \frac{47}{6} + \frac{61}{8}\cos 2\omega - \frac{31}{6}\cos^2 2\omega \right] \tag{120}$$

It is now apparent from Eqs. (118)–(120) that the use of range-rate data for state estimation of a circular orbit about a distant planet can yield singular results. Singular state errors will result if

$$\theta = 90 \text{ deg},\ 270 \text{ deg},\ \text{and/or} \tag{121}$$

$$i = -90 \text{ deg},\ 90 \text{ deg},\ \text{and/or} \tag{122}$$

$$V_N = 0 \tag{123}$$

This latter condition of singularity may occur when the spacecraft is orbiting a superior planet (at the beginning and end points of retrograde motion). These are important results, which are significantly at variance with orbit determination experience for highly eccentric trajectories, which appear well behaved and certainly not singular. Note that even with zero eccentricity, the results of Eqs. (119) and (120) imply a functional dependence upon ω. This is an artifact of the definition of the coordinate system at time zero. If we assume $\omega = 90$ deg, then the maximum variation in the error in θ for any other value of ω can be shown to be 13.6 percent. Thus the error in θ can be said to be only "weakly" dependent upon ω.

B. Elliptic Case

Many thanks go to R. J. Muellerschoen of the Navigation Systems Section for his help in using *MATHEMATICA* to attempt the 6 x 6 inversion. Even with a large amount of available memory, the problem quickly consumed it all and more. Thus it appears unlikely that a "reasonable" functionality for f and g can be obtained in the general elliptic case. However, in having the information array, we have performed the accumulation in a general manner which will save much time. A large number of orbit-determination studies as a function of geometry relative to the plane-of-the-sky can be made quickly and cheaply.

VI. Results

A. Circular Case

Venus Orbiting Imaging Radar (VOIR), the progenitor of Magellan, was extensively studied many years ago. It was in these studies that the circular orbit singularities were first observed. The nominal orbit just happened to have conditions that were near-singular. As a result, orbit determination errors were obtained that were outrageously large (larger than the nominal orbit). This led to the use of several different computer programs to examine whether there were program errors or not. The two main programs used to investigate the state estimation were the Orbit Determination Program (ODP) and the Sequential Orbit Determination (SOD) Program. Both of these pieces of software yielded the same, poor orbit determination behavior and so confirmed the belief that we had inadvertently stumbled upon singularities in the orbit determination process for circular orbiters. The important parameters defining the nominal VOIR orbit were:

Epoch	March 17, 1982
Semi-major axis, a	6552 km
Eccentricity, e	1.52×10^{-4}
Inclination, i	78.17 deg (relative to the plane-of-the-sky)
Argument of periapsis, ω	170.7 deg (relative to the plane-of-the-sky)
Cross-velocity, V_N	14.413 km/sec
Target range, ρ_{ET}	8.36765×10^7 km
Angle, θ	85 deg
Period, P	1.624 hr

The computer solutions were assumed to have as a data base one full orbit of coherent, two-way range-rate data. A slight problem complicating exact comparison with the theory is that the computer programs employed assumed that when the spacecraft was not visible to the Earth tracking station, there would be no data available. This eliminated about 36 minutes of data (depending on the nominal inclination relative to the plane-of-the-sky) out of a total of 98 minutes available. The theory, however, assumes that a full orbit of data is available no matter what. As a result the displayed errors are not in perfect agreement. The trends as functions of the angle θ and the inclination i, however, are in excellent agreement and as seen in Figs. 2 and 3, clearly demonstrate the singular behavior of the circular orbit. The best place to compare theory and computer results is in Fig. 3 where the value of the inclination relative to the plane-of-the-sky is small. In this configuration, a full orbit of data would be available to both theory and computer results, yielding agreement to about 30 percent. As the inclination increases, the agreement between theory and computer results degrades as more data is occulted. The error parameter displayed in these figures is

$$|\sigma_x| = r\sigma_\theta = a\sigma_\theta \tag{124}$$

B. Elliptic Case

The orbit determination studies for the general elliptic case are yet to be published.[1]

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[1] For additional information on the initial studies which helped to delineate the circular orbit problems, see the following: J. Ellis and R. K. Russell, "Earth-Based Determination of a Near-Circular Orbit About a Distant Planet," JPL Technical Memorandum 391-406 (internal document), March 9, 1973; R. K. Russell, "Improved Orbit Determination of Low-Altitude, Nearly Circular Planetary Orbiters," Engineering Memorandum 391-561 (internal document), April 25, 1974; R. K. Russell, "Analytic Orbit Determination, A General Methodology and a Successful Example," JPL Engineering Memorandum 314-59 (internal document), August 9, 1976; and L. J. Wood "Orbit Determination Singularities in the Doppler Tracking of a Planetary Orbiter," *Journal of Guidance, Control, and Dynamics*, volume 9, number 4, p. 485, July–August 1986.

VII. Conclusions

It has been seen that circular orbits about distant planets may suffer singularities in over-all position error estimation. These singularities are due to orbit inclination, placement of the line-of-nodes, and insignificant cross-velocity at the start and end of retrograde motion when orbiting a superior planet.

Even though these conclusions appear to yield poor state estimation, one should not be unduly alarmed inasmuch as the stated conditions for singularity are not maintained for extended periods during typical mission scenarios. However, mission analysts should be aware of these potential pitfalls and realize that spuriously large results for circular orbiters can be obtained and are not the result of incorrect assumptions or faulty software.

The general elliptic problem appears so involved that analytic inversion at this time is just not feasible, and in any case the resulting expression for the position error would likely be so lengthy that any understanding would be lost in the maze.

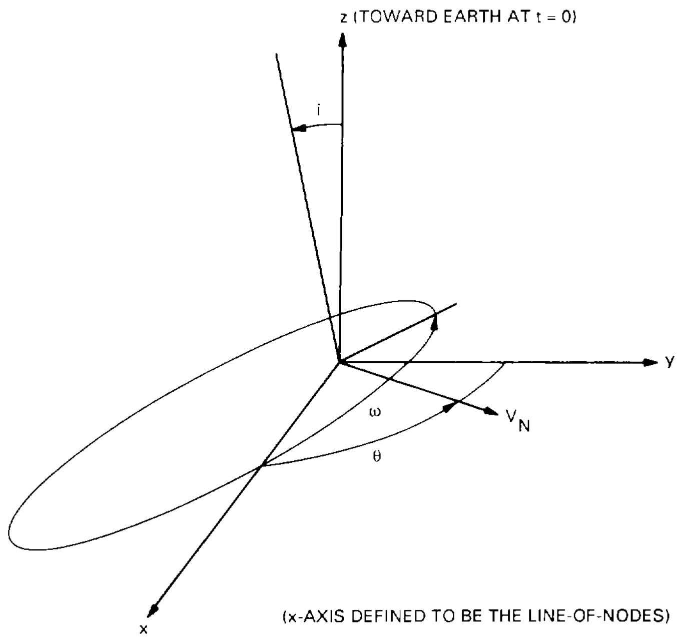

Fig. 1. Spacecraft orbital geometry in the plane of the sky.

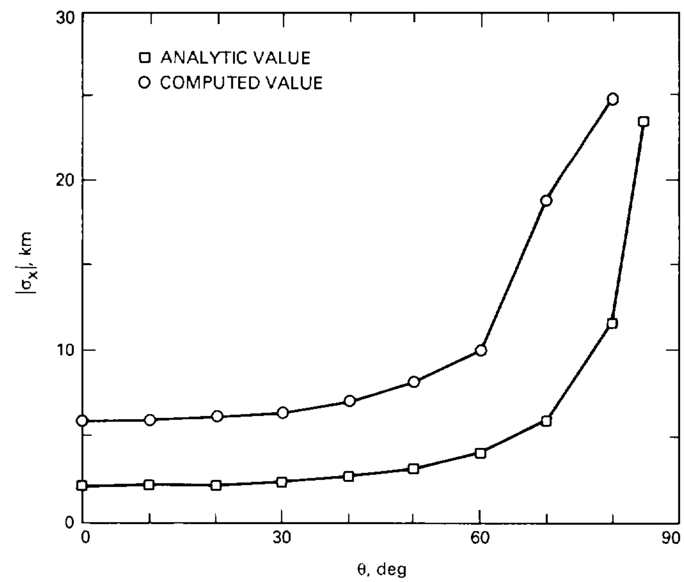

Fig. 2. Norm of position error versus θ (with inclination to the plane of the sky = 78.17 deg).

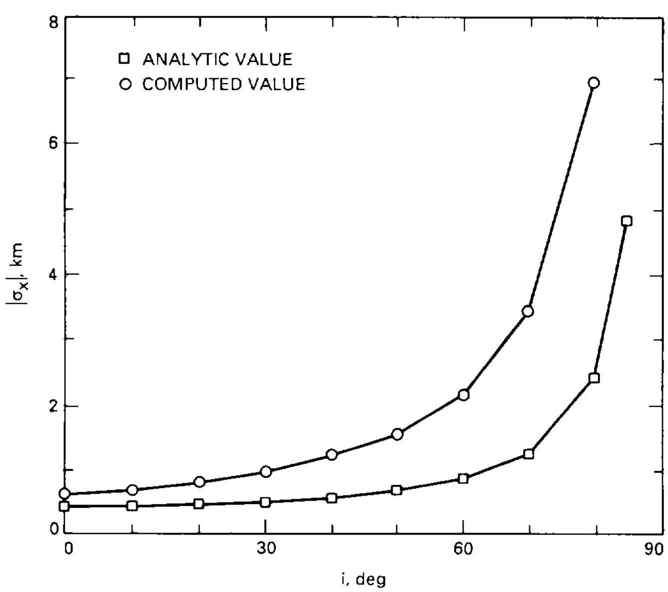

Fig. 3. Norm of position error versus inclination, i, to the plane of the sky (with θ = 0 deg).

Appendix A
Required Integrals I

To perform the accumulation of the information array, a large number of difficult integrals of mixed trigonometric and algebraic functions and their powers are required. Contained herein are the necessary results. Note that due to the limits of integration, all integrals of odd functions will vanish.

An important, general integral, from which some of the results contained herein are derived, is

$$I(a,b,n) = \int_{-\pi}^{\pi} \frac{dE}{(a + b\cos E)^{n+1}} = \frac{2\pi}{(a^2 - b^2)^{\frac{n+1}{2}}} P_n\left(\frac{a}{\sqrt{a^2 - b^2}}\right) \tag{A-1}$$

where P_n are the Legendre polynomials.

This integral is found in [1; p. 383, 3.661, 4].

For simplicity and convenience, we shall define:

$$\beta = \sqrt{1 - e^2} \tag{A-2}$$

$$\gamma = \left(\frac{1-\beta}{e}\right) \tag{A-3}$$

The integrals necessary to perform the accumulation are:

$$\int_{-\pi}^{\pi} (1 - e\cos E)dE = 2\pi \tag{A-4}$$

$$\int_{-\pi}^{\pi} (1 - e\cos E)\cos E dE = -\pi e \tag{A-5}$$

$$\int_{-\pi}^{\pi} (1 - e\cos E)\cos^2 E dE = \pi \tag{A-6}$$

$$\int_{-\pi}^{\pi} dE = 2\pi \tag{A-7}$$

$$\int_{-\pi}^{\pi} \cos E dE = 0 \tag{A-8}$$

$$\int_{-\pi}^{\pi} \cos^2 E dE = \pi \tag{A-9}$$

$$\int_{-\pi}^{\pi} \cos^3 E dE = 0 \tag{A-10}$$

$$\int_{-\pi}^{\pi} E\sin E dE = 2\pi \tag{A-11}$$

$$\int_{-\pi}^{\pi} E\sin E\cos E dE = -\left(\frac{\pi}{2}\right) \tag{A-12}$$

$$\int_{-\pi}^{\pi} \frac{dE}{(1 - e\cos E)} = \left(\frac{2\pi}{\beta}\right) \tag{A-13}$$

$$\int_{-\pi}^{\pi} \frac{\cos E dE}{(1 - e\cos E)} = \left(\frac{2\pi}{e\beta}\right)(1-\beta) = \left(\frac{2\pi}{\beta}\right)\gamma \tag{A-14}$$

$$\int_{-\pi}^{\pi} \frac{\cos^2 E dE}{(1 - e\cos E)} = \left(\frac{2\pi}{e^2\beta}\right)(1-\beta) = \left(\frac{2\pi}{e\beta}\right)\gamma \tag{A-15}$$

$$\int_{-\pi}^{\pi} \frac{\cos^3 E dE}{(1 - e\cos E)} = \left(\frac{\pi}{e^3\beta}\right)(1-\beta)^2(2+\beta) = \left(\frac{\pi}{e\beta}\right)\gamma^2(2+\beta) \tag{A-16}$$

$$\int_{-\pi}^{\pi} \frac{\cos^4 E dE}{(1 - e\cos E)} = \left(\frac{\pi}{e^4\beta}\right)(1-\beta)^2(2+\beta) = \left(\frac{\pi}{e^2\beta}\right)\gamma^2(2+\beta) \tag{A-17}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E dE}{(1 - e\cos E)} = -\left(\frac{2\pi}{e}\right)\ln\left\{\frac{1+\beta}{2(1+e)}\right\} \tag{A-18}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos E dE}{(1 - e\cos E)} = -\left(\frac{2\pi}{e}\right)\left[1 + \frac{1}{e}\ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right] \tag{A-19}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos^2 E dE}{(1 - e\cos E)} = \left(\frac{\pi}{2e^2}\right)\left[e - 4\left[1 + \frac{1}{e}\ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right]\right] \tag{A-20}$$

For a full development of the following integral, see Appendix B.

$$\int_{-\pi}^{\pi} \frac{E^2 dE}{(1 - e\cos E)} = \left(\frac{2\pi}{\beta}\right)S(\gamma) \tag{A-21}$$

where

$$S(\gamma) = \left[\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\gamma^n}{n^2}\right] \tag{A-22}$$

Continuing,

$$\int_{-\pi}^{\pi} \frac{E^2\cos^2 E dE}{(1 - e\cos E)} = \left(\frac{2\pi}{e^2\beta}\right)\left[S(\gamma) + \beta\left(2e - \frac{\pi^2}{3}\right)\right] \tag{A-23}$$

$$\int_{-\pi}^{\pi} \frac{dE}{(1 - e\cos E)^2} = \left(\frac{2\pi}{\beta^3}\right) \tag{A-24}$$

$$\int_{-\pi}^{\pi} \frac{\cos E dE}{(1 - e\cos E)^2} = \left(\frac{2\pi e}{\beta^3}\right) \tag{A-25}$$

$$\int_{-\pi}^{\pi} \frac{\cos^2 E dE}{(1 - e\cos)^2} = \left(\frac{2\pi}{e^2\beta^3}\right)(1-\beta)(1+\beta-\beta^2) = \left(\frac{2\pi}{e\beta^3}\right)\gamma(1+\beta-\beta^2) \tag{A-26}$$

$$\int_{-\pi}^{\pi} \frac{\cos^3 E dE}{(1 - e\cos E)^2} = \left(\frac{2\pi}{e^3\beta^3}\right)(1-\beta)^2(1+2\beta) = \left(\frac{2\pi}{e\beta^3}\right)\gamma^2(1+2\beta) \tag{A-27}$$

$$\int_{-\pi}^{\pi} \frac{\cos^4 E dE}{(1 - e\cos E)^2} = \left(\frac{\pi}{e^4\beta^3}\right)(1-\beta)^2(2+4\beta-2\beta^2-\beta^3) = \left(\frac{\pi}{e^2\beta^3}\right)\gamma^2(2+4\beta-2\beta^2-\beta^3) \tag{A-28}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E dE}{(1 - e\cos E)^2} = \left(\frac{2\pi}{e}\right)\left[\frac{1}{\beta} - \frac{1}{1+e}\right] = \left(\frac{2\pi}{\beta^2}\right)(1-\gamma) \tag{A-29}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos E dE}{(1 - e\cos E)^2} = \left(\frac{2\pi}{e^2}\right)\left[\frac{1}{\beta} - \frac{1}{1+e} + \ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right] \tag{A-30}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos^2 E dE}{(1 - e\cos E)^2} = \left(\frac{2\pi}{e^3}\right)\left[\frac{1}{\beta} - \frac{1}{1+e} + e + 2\ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right] \tag{A-31}$$

$$\int_{-\pi}^{\pi} \frac{E^2 dE}{(1 - e\cos E)^2} = \left(\frac{2\pi}{\beta^3}\right)\left[S(\gamma) - 4\beta\ln(1+\gamma)\right] \tag{A-32}$$

$$\int_{-\pi}^{\pi} \frac{dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{\beta^5}\right)(3-\beta^2) \tag{A-33}$$

$$\int_{-\pi}^{\pi} \frac{\cos E dE}{(1 - e\cos E)^3} = \left(\frac{3\pi e}{\beta^5}\right) \tag{A-34}$$

$$\int_{-\pi}^{\pi} \frac{\cos^2 E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{\beta^5}\right)(3-2\beta^2) \tag{A-35}$$

$$\int_{-\pi}^{\pi} \frac{\cos^3 E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{e^3\beta^5}\right)(1-\beta)^2(3+6\beta+2\beta^2-2\beta^3) = \left(\frac{\pi}{e\beta^5}\right)\gamma^2(3+6\beta+2\beta^2-2\beta^3) \tag{A-36}$$

$$\int_{-\pi}^{\pi} \frac{\cos^4 E dE}{(1 - e\cos E)^3} = \left(\frac{3\pi}{e^4\beta^5}\right)(1-\beta)^2(1+2\beta-2\beta^3) = \left(\frac{3\pi}{e^2\beta^5}\right)\gamma^2(1+2\beta-2\beta^3) \tag{A-37}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{e}\right)\left[\frac{1}{\beta^3} - \frac{1}{(1+e)^2}\right] \tag{A-38}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{e^2}\right)\left[\frac{1}{\beta^3}(1-2\beta^2) + \frac{(1+2e)}{(1+e)^2}\right] \tag{A-39}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos^2 E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{e^3}\right)\left[\frac{1}{\beta^3}(1-4\beta^2) + \frac{(3+4e)}{(1+e)^2} - 2\ln\left\{\frac{1+\beta}{2(1+e)}\right\}\right] \tag{A-40}$$

$$\int_{-\pi}^{\pi} \frac{E^2 dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{\beta^5}\right)\left[(3-\beta^2)S(\gamma) - 12\beta\ln(1+\gamma) - 2e\beta(1-\gamma)\right] \tag{A-41}$$

$$\int_{-\pi}^{\pi} \frac{E^2\cos E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{e\beta^5}\right)\left[3(1-\beta^2)S(\gamma) - 4\beta(3-2\beta^2)\ln(1+\gamma) - 2e\beta(1-\gamma)\right] \tag{A-42}$$

$$\int_{-\pi}^{\pi} \frac{E^2\cos^2 E dE}{(1 - e\cos E)^3} = \left(\frac{\pi}{e^2\beta^5}\right)\left[(3-5\beta^2+2\beta^4)S(\gamma) - 4\beta(3-4\beta^2)\ln(1+\gamma) - 2e\beta(1-\gamma)\right] \tag{A-43}$$

$$\int_{-\pi}^{\pi} \frac{dE}{(1 - e\cos E)^4} = \left(\frac{\pi}{\beta^7}\right)(5 - 3\beta^2) \tag{A-44}$$

$$\int_{-\pi}^{\pi} \frac{\cos E dE}{(1 - e\cos E)^4} = \left(\frac{\pi e}{\beta^7}\right)(5 - \beta^2) \tag{A-45}$$

$$\int_{-\pi}^{\pi} \frac{\cos^2 E dE}{(1 - e\cos E)^4} = \left(\frac{\pi}{\beta^7}\right)(5 - 4\beta^2) \tag{A-46}$$

$$\int_{-\pi}^{\pi} \frac{\cos^3 E dE}{(1 - e\cos E)^4} = \left(\frac{\pi e}{\beta^7}\right)(5 - 2\beta^2) \tag{A-47}$$

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\cos^4 E dE}{(1 - e\cos E)^4} &= \left(\frac{\pi}{e^4\beta^7}\right)(1 - \beta)^2(5 + 10\beta - 10\beta^3 - 4\beta^4 + 2\beta^5) \\ &= \left(\frac{\pi}{e^2\beta^7}\right)\gamma^2(5 + 10\beta - 10\beta^3 - 4\beta^4 + 2\beta^5)\end{aligned} \tag{A-48}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E dE}{(1 - e\cos E)^4} = \left(\frac{\pi}{3e}\right)\left[\frac{(3 - \beta^2)}{\beta^5} - \frac{2}{(1 + e)^3}\right] \tag{A-49}$$

$$\int_{\pi}^{\pi} \frac{E\sin E\cos E dE}{(1 - e\cos E)^4} = \left(\frac{\pi}{3e^2}\right)\left[\frac{1}{\beta^5}(3 - 4\beta^2) + \frac{(1 + 3e)}{(1 + e)^3}\right] \tag{A-50}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos^2 E dE}{(1 - e\cos E)^4} = \left(\frac{\pi}{3e^3}\right)\left[\frac{1}{\beta^5}(3 - 7\beta^2 + 6\beta^4) - \frac{2}{(1 + e)^3}(1 + 3e + 3e^2)\right] \tag{A-51}$$

$$\int_{-\pi}^{\pi} \frac{dE}{(1 - e\cos E)^5} = \left(\frac{\pi}{4\beta^9}\right)(35 - 30\beta^2 + 3\beta^4) \tag{A-52}$$

$$\int_{-\pi}^{\pi} \frac{\cos E dE}{(1 - e\cos E)^5} = \left(\frac{5\pi e}{4\beta^9}\right)(7 - 3\beta^2) \tag{A-53}$$

$$\int_{-\pi}^{\pi} \frac{\cos^2 E dE}{(1 - e\cos E)^5} = \left(\frac{\pi}{4\beta^9}\right)(35 - 35\beta^2 + 4\beta^4) \tag{A-54}$$

$$\int_{-\pi}^{\pi} \frac{\cos^3 E dE}{(1 - e\cos E)^5} = \left(\frac{5\pi e}{4\beta^9}\right)(7 - 4\beta^2) \tag{A-55}$$

$$\int_{-\pi}^{\pi} \frac{\cos^4 E dE}{(1 - e\cos E)^5} = \left(\frac{\pi}{4\beta^9}\right)(35 - 40\beta^2 + 8\beta^4) \tag{A-56}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E dE}{(1 - e\cos E)^5} = \left(\frac{\pi}{4e}\right)\left[\frac{(5 - 3\beta^2)}{\beta^7} - \frac{2}{(1 + e)^4}\right] \tag{A-57}$$

$$\int_{-\pi}^{\pi} \frac{E\sin E\cos E dE}{(1 - e\cos E)^5} = \left(\frac{\pi}{12e^2}\right)\left[\frac{1}{\beta^7}(15 - 21\beta^2 + 4\beta^4) + \frac{2}{(1 + e)^4}(1 + 4e)\right] \tag{A-58}$$

$$\int_{-\pi}^{\pi} \frac{E \sin E \cos^2 E dE}{(1 - e \cos E)^5} = \left(\frac{\pi}{12e^3}\right) \left[\frac{1}{\beta^7}(15 - 33\beta^2 + 20\beta^4) - \frac{2}{(1+e)^4}(1 + 4e + 6e^2)\right] \tag{A-59}$$

$$\int_{-\pi}^{\pi} \frac{E^2 dE}{(1 - e \cos E)^5} = \left(\frac{\pi}{12\beta^9}\right) \Big[3(35 - 30\beta^2 + 3\beta^4)S(\gamma) - 20\beta(21 - 11\beta^2)\ln(1+\gamma)$$
$$-2e\beta(1-\gamma)(57 + 12\beta - 10\beta^2) + 2e\beta^2(12 + \beta)\Big] \tag{A-60}$$

$$\int_{-\pi}^{\pi} \frac{E^2 \cos E dE}{(1 - e \cos E)^5} = \left(\frac{\pi}{12e\beta^9}\right) \Big[15(1-\beta^2)(7 - 3\beta^2)S(\gamma) - 4\beta(105 - 115\beta^2 + 16\beta^4)\ln(1+\gamma)$$
$$-2e\beta(1-\gamma)(57 + 12\beta - 38\beta^2 - 4\beta^3) + 2e\beta^2(12 + \beta - 4\beta^2)\Big] \tag{A-61}$$

$$\int_{-\pi}^{\pi} \frac{E^2 \cos^2 E dE}{(1 - e \cos E)^5} = \left(\frac{\pi}{12e^2\beta^9}\right) \Big[3(1-\beta^2)(35 - 35\beta^2 + 4\beta^4)S(\gamma) - 4\beta(105 - 175\beta^2 + 68\beta^4)\ln(1+\gamma)$$
$$-2e\beta(1-\gamma)(57 + 12\beta - 66\beta^2 - 8\beta^3 + 12\beta^4) + 2e\beta^2(12 + \beta - 8\beta^2)\Big] \tag{A-62}$$

Appendix B
Required Integrals II

Out of all these difficult integrals, only one class is not expressible in closed form. These are integrals of the form

$$I_{n,m} = \int_{-\pi}^{\pi} \frac{E^2 \cos^m E dE}{(1 - e\cos E)^n} \tag{B-1}$$

We shall confine our interest to the lowest order of the above, namely where $n = 1$ and $m = 0$. The other integrals can be simply obtained from this one, which is

$$I = \int_{-\pi}^{\pi} \frac{E^2 dE}{1 - e\cos E} \tag{B-2}$$

If the square function is expanded in a Fourier series over the interval $-\pi$ to π, one obtains

$$E^2 = \frac{\pi^2}{3} - 4\left[\cos E - \frac{1}{2^2}\cos 2E + \frac{1}{3^2}\cos 3E - \cdots\right], \quad -\pi \le E \le \pi \tag{B-3}$$

This now can be made to yield the integral we are seeking:

$$\int_{-\pi}^{\pi} \frac{E^2 dE}{1 - e\cos E} = \frac{\pi^2}{3}\int_{-\pi}^{\pi} \frac{dE}{1 - e\cos E} + 4\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\left(\frac{\cos nE}{1 - e\cos E}\right) dE \tag{B-4}$$

$$= \frac{2\pi^3}{3\beta} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\int_{-\pi}^{\pi} \frac{\cos nE dE}{1 - e\cos E} \tag{B-5}$$

Thus,

$$\int_{-\pi}^{\pi} \frac{\cos nE dE}{1 - e\cos E} = 2\int_{0}^{\pi} \frac{\cos nE dE}{1 - e\cos E} \tag{B-6}$$

This latter integral can be found in [1; p. 366, 3.613] as

$$\int_{0}^{\pi} \frac{\cos nE dE}{1 - e\cos E} = \frac{\pi}{\beta}\left(\frac{1 - \beta}{e}\right)^n \tag{B-7}$$

Thus, finally

$$\int_{-\pi}^{\pi} \frac{\cos nE dE}{1 - e\cos E} = \frac{2\pi}{\beta} S(\gamma) \tag{B-8}$$

where

$$S(\gamma) = \left[\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\gamma^n\right] \tag{B-9}$$

$$\beta = \sqrt{1 - e^2} \tag{B-10}$$

$$\gamma = \left(\frac{1-\beta}{e}\right) \tag{B-11}$$

Reference

[1] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Corrected and Enlarged Edition, New York: Academic Press, Inc., 1980.