Choice of Integrators for Use With a Variation-of-Parameters Formulation

T. D. Talbot and E. A. Rinderle
Flight Operations and DSN Programming Section

A comparison of three integrators is made in an effort to choose the one best suited for variations-of-parameters computations of the orbits of satellites.

I. Introduction

In searching for better methods of computing the orbit of a satellite over many revolutions, a special perturbations theory using a variation-of-parameters formulation has been developed (Ref. 1). As a prelude to obtaining valid comparisons with the Cowell technique currently in use, a study of various integrators was made in an effort to find the integrator best suited to a variation-of-parameters formulation. This study was initiated in response to suggestions (Ref. 2) that for a given set of differential equations, there exists an optimum integrator.

This study considered three integrators. One of these is a variable-order integrator, DVDQ[1] (Ref. 3). The second is another polynomial-type integrator which has a fixed order. The third integrator, termed "Fourier" by Sheffield (Ref. 2), integrates trigonometric functions exactly.

[1]Double-precision variable-order differential equation solver.

These integrators were compared by the execution of two sets of test cases. The integrators were first compared in the predict-only mode, then in the predict-correct mode. The numerical results of these studies and the conclusions reached are presented in the following section.

II. Numerical Results of Comparing the Integrators

A. Test Problem

The test problem for this study was a Mars orbiter with a semimajor axis of about 12,000 km, an eccentricity of about 0.6, and a period of about 12 h. The perturbing forces included were the gravitational attraction of the sun and the Mars zonal harmonics J_2, J_3, J_4, and J_5. Integration was carried out for five days (ten orbits).

A classical variation-of-parameters formulation was used. The orbital elements that were integrated consisted of the following:

a = semimajor axis

e = eccentricity

i = inclination

M_0 = mean anomaly at epoch

ω = argument of periapsis

Ω = longitude of the ascending node

The trajectory was started at about 2 h past periapsis. Print was obtained at both apoapsis and periapsis for each orbit.

To obtain an "exact" solution to this problem, the variable-order integrator DVDQ was used with very tight local integration tolerances (10^{-8} to 10^{-11}). The resulting step sizes chosen varied from 900 sec at apoapsis down to less than 60 sec at periapsis. All of the other runs made were compared to this run to determine their accuracies.

B. Comparison of Predict-Only Cases

Table 1 contains the apoapsis and periapsis errors on each orbit for several predict-only cases. The left column contains a brief description of each case. The errors in six of the ten orbits are tabulated. The right column contains the total number of integration steps.

The description of the DVDQ case contains the maximum, minimum, and average step sizes and integration orders. In the other variable-step cases, the maximum step size was 4/3 h_0, the minimum was 1/3 h_0, and the average was slightly less than h_0, where h_0 is the initial integration

Table 1. Comparison of predict-only cases

Case number	Integrator	Apsis	Error = max {$\lvert\delta x\rvert, \lvert\delta y\rvert, \lvert\delta z\rvert$}, m						Number of steps
			Orbit number						
			1	2	4	6	8	10	
1	DVDQ, maximum order 10, minimum order 5, maximum step 2400, minimum step 75, average order 8, average step 380	Apoapsis Periapsis	0.00124 0.0810	0.200 2.65	1.87 4.24	2.95 6.43	4.32 9.36	5.57 12.14	1130
2	Fourier, fourth order Variable step h_0 = 300	Apoapsis Periapsis	0.0540 0.324	0.951 2.08	3.59 7.57	7.08 16.2	11.3 26.5	16.4 39.9	1711
3	Polynomial, fourth order Variable step h_0 = 300	Apoapsis Periapsis	0.0447 0.304	0.824 1.54	2.81 4.55	5.10 8.91	7.54 13.3	10.4 19.2	1711
4	Fourier, sixth order Variable step h_0 = 300	Apoapsis Periapsis	0.0143 0.0793	0.474 1.88	3.25 11.9	8.32 29.7	15.7 54.5	25.4 88.0	1712
5	Polynomial, sixth order Variable step h_0 = 300	Apoapsis Periapsis	0.0123 0.0682	0.484 1.89	3.26 11.8	8.29 29.4	15.6 54.0	25.2 85.2	1712
6	Polynomial, fourth order Fixed step h = 300	Apoapsis Periapsis	0.0402 1.77	0.178 4.21	1.00 9.37	2.55 14.2	4.79 18.3	7.73 21.9	1417
7	DVDQ, maximum order 9, minimum order 4, maximum step 2400, minimum step 150, average order 7, average step 430 $E = E_0K$ (Footnote a)	Apoapsis Periapsis	0.00194 0.259	0.0149 2.40	1.52 2.88	2.16 3.52	2.78 4.15	3.13 4.15	982

[a] E = local error

E_0 = initial local error

$K = \frac{a(1+e)}{r}$

step size. In each case, the minimum step occurred at periapsis and the maximum occurred at apoapsis. The maximum orders for DVDQ occurred near periapsis and the minimum orders occurred near apoapsis. The errors tabulated are the maximums of the absolute errors in the components of position (the error in z in each instance).

Two main results are evident from these studies. First, it does not appear to be very advantageous to use higher orders or variable step sizes. Further investigation is necessary to explain these results. Second, there seems to be little difference between the Fourier and polynomial integrators. This result becomes more apparent when the integration coefficients are examined. Table 2 contains these coefficients for both fourth and sixth orders with a starting step size $h_0 = 300$ sec.

Table 2. Polynomial and Fourier Integration Coefficients

	Fourth order	
	Polynomial	Fourier
β_{40}	792.0833333333333	791.1839790650130
β_{41}	−1155.833333333333	−1153.177337110016
β_{42}	1190.000000000000	1087.425341026402
β_{43}	−530.833333333333	−530.0560158768241
β_{44}	104.5833333333333	104.6240328954251
	Sixth order	
	Polynomial	Fourier
β_{60}	985.7192460317460	983.3672331781959
β_{61}	−2218.690476190476	−2207.020820533816
β_{62}	3499.747023809523	3476.647150904269
β_{63}	−3413.968253968254	−3391.241259327214
β_{64}	2019.538690476190	2008.540513368117
β_{65}	−667.0238095238095	−665.0451712873998
β_{66}	94.67757936507936	94.75235369784858

Table 2 reveals that the Fourier and polynomial integration coefficients are very similar. These results prompted further theoretical investigation which led to the following equivalence theorem for Fourier and polynomial integrators:

Theorem. *Let N be any positive integer. Let n be any positive real number and let h be any fixed positive step size such that $Nh < 2\pi/n$. Let $t_k = -kh$ for $k = 0, 1, \cdots, N$. Define the Fourier integration coefficients $\beta_j(n)$ as follows:*

$$\beta_j(n) = \int_0^h T_j(n, s)\, ds, \qquad j = 0, 1, \cdots, N$$

where

$$T_j(n,t) = \frac{\displaystyle\prod_{\substack{k=0\\k\neq j}}^{N} \sin\frac{1}{2}n(t - t_k)}{\displaystyle\prod_{\substack{k=0\\k\neq j}}^{N} \sin\frac{1}{2}n(t_j - t_k)}$$

Note that the inequality constraint on h prevents the denominator from being zero. Define the polynomial integration coefficients γ_j as follows:

$$\gamma_j = \int_0^h P_j(s)\, ds, 0, 1, \cdots, N$$

where

$$P_j(t) = \frac{\displaystyle\prod_{\substack{k=0\\k\neq j}}^{N} (t - t_k)}{\displaystyle\prod_{\substack{k=0\\k\neq j}}^{N} (t_j - t_k)}$$

Then

$$\lim_{n\to 0} \beta_j(n)\, \gamma_j$$

A proof of this theorem, which was previously established by Gautschi (Ref. 4), is given in Ref. 5. The theorem states that the Fourier integrator approaches the polynomial integrator as the mean motion n approaches zero.

Several test runs were made with values of n that varied from 10^{-2} sec^{-1} to 10^{-4} sec^{-1}. This range includes the largest values of n possible in current celestial mechanics applications. In every case, the two sets of coefficients agreed to two or three digits. Therefore, one may conclude that the Fourier method offers no advantage over the polynomial method in the integration of the differential equations of celestial mechanics. The Fourier integration coefficients also have the disadvantage of depending on both n and the step size h.

C. Comparison of Predictor-Corrector Cases

Table 3 contains a comparison of several predictor-corrector cases. This table has the same format as Table 1. The remarks in *Subsection B* regarding the maximum, minimum, and average step sizes in Table 1 also apply to Table 3.

Table 3. Comparison of predictor-corrector cases

Case number	Integrator	Apsis	Error = max $\{\lvert\delta x\rvert, \lvert\delta y\rvert, \lvert\delta z\rvert\}$, m						Number of steps
			Orbit number						
			1	2	4	6	8	10	
1	DVDQ, maximum order 10, minimum order 5, maximum step 2400, minimum step 75, average order 8, average step 380	Apoapsis	0.00124	0.200	1.85	2.93	4.30	5.56	1135
		Periapsis	0.0809	2.60	4.17	6.36	9.31	12.08	
2	Polynomial, fourth order Variable step $h_0 = 300$	Apoapsis	0.00183	0.0594	0.311	0.731	1.31	2.06	1711
		Periapsis	0.0126	0.198	1.00	2.37	4.22	6.56	
3	Polynomial, sixth order Variable step $h_0 = 400$	Apoapsis	0.00330	0.134	0.870	2.21	4.16	6.72	1286
		Periapsis	0.0180	0.513	3.18	7.84	14.4	22.7	
4	Polynomial, tenth order Variable step $h_0 = 400$	Apoapsis	0.0000644	0.0961	0.845	2.22	4.35	7.14	1288
		Periapsis	0.00279	0.644	3.85	9.10	16.8	26.2	
5	Polynomial, fourth order Fixed step $h = 300$	Apoapsis	0.00162	0.112	0.819	2.24	4.15	6.77	1417
		Periapsis	0.0614	0.554	3.36	8.49	15.8	25.0	
6	Polynomial, sixth order Fixed step $h = 300$	Apoapsis	0.000286	0.00639	0.0672	0.190	0.373	0.613	1418
		Periapsis	0.146	0.279	0.234	0.253	1.15	2.36	
7	Polynomial, sixth order Fixed step $h = 400$	Apoapsis	0.00313	0.222	1.48	3.84	7.32	12.0	1064
		Periapsis	0.603	2.94	9.29	17.8	28.6	41.7	
8	Polynomial, tenth order Fixed step $h = 400$	Apoapsis	0.000235	0.168	1.16	3.07	5.97	9.91	1066
		Periapsis	0.728	0.428	5.86	19.0	38.3	61.5	
9	Polynomial, tenth order Variable step $h_0 = 300$	Apoapsis	0.152E-4	0.226E-2	0.976E-2	0.205E-1	0.346E-1	0.527E-1	1714
		Periapsis	0.107E-3	0.445E-2	0.219E-1	0.573E-1	0.900E-1	0.142	
10	DVDQ, $E = E_0K$ (Footnote a)	Apoapsis	0.00193	0.0140	1.51	2.11	3.20	3.27	987
		Periapsis	0.279	2.38	2.79	3.38	4.12	4.46	
11	DVDQ, $E = E_0K$	Apoapsis	0.00173	1.35	5.16	4.75	3.84	2.84	893
		Periapsis	0.184	7.61	7.20	3.91	0.476	8.25	
12	DVDQ, $E = E_0K$	Apoapsis	0.0107	0.566	2.54	1.56	0.624	0.00193	872
		Periapsis	0.248	1.57	7.43	13.2	17.3	23.3	

[a] E = local error

E_0 = initial local error

$$K = \frac{a(1+e)}{r}$$

Several conclusions may be made from these studies. For instance, cases 1 through 9 again indicate that taking fixed steps requires less work (fewer steps) to achieve a given accuracy than taking variable steps. Hence, it seems that more error can be introduced at periapsis than at apoapsis without increasing the total error in the computed solution. Cases 10 through 12 were run as a check on this hypothesis. Case 10, in particular, tends to substantiate the hypothesis.

These studies also indicate that higher order (tenth order) integration formulas are not necessarily more efficient for this type of problem. Also, a comparison of Tables 1 and 3 indicates that the predict-only mode is perhaps more efficient (that is, takes fewer derivative evaluations to achieve a given accuracy) than the predict-correct mode. But further study is needed in these areas.

Finally, it appears from these studies that DVDQ is approximately fifteen to 20% more efficient than the fixed-order polynomial type integrator. Therefore, since the Fourier method is essentially equivalent to the polynomial method, DVDQ seems to be the most efficient of the integrators studied for use in a variation-of-parameters formulation. These studies indicate that for maximum integrator efficiency DVDQ should be run in the predict-only mode, with the error introduced at each step proportional to $1/r$.

References

1. Dallas, S., and Rinderle, E., *A Special Perturbations Theory Using the Variation of Parameters Formulation,* April 1, 1970 (JPL internal document).
2. Sheffield, C., "Generalized Multi-Step Method With an Application to Orbit Computations," *Celest. Mech.,* Vol. I, pp. 46–58, 1969.
3. Krogh, F., *On Testing a Subroutine for the Numerical Integration of Ordinary Differential Equations,* May 9, 1969 (JPL internal document).
4. Gautschi, W., "Numerical Integration of Ordinary Differential Equations Based on Trigonometric Polynomials," *Numer. Math.,* Vol. 3, pp. 381–397, 1961.
5. Talbot, T., and Rinderle, E., *The Choice of Integrators for Use in the Variation of Parameters Formulation for DPTRAJ Research,* October 16, 1970 (JPL internal document).