Second Order Charged Particle Effects on Electromagnetic Waves in the Interplanetary Medium

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Possible influences on the measurements of the total electron content due to magnetic fields, spatial inhomogeneities, and pulse shape distortions are investigated and found to be negligibly small at Deep Space Network tracking frequencies.

I. Introduction

There are essentially two methods now in use to measure the effect of the interplanetary plasma on the propagation of radio signals used for tracking a spacecraft. One method is the differenced range versus integrated doppler (DRVID) calibration utilizing the difference in phase and group velocity of the electromagnetic waves (Ref. 1). The other method uses two different frequencies (dual frequency method) taking advantage of the frequency dependence of the phase and group velocity (Ref. 2). The effects of the interplanetary plasma on the electromagnetic wave propagation can be theoretically expressed by the fact that the (relative) dielectric constant ϵ is given by

$$\epsilon = 1 - \frac{\omega_P^2}{\omega^2} \tag{1}$$

where

$$\omega_P^2 = \frac{4\pi e^2}{m} N = \text{plasma frequency}$$

ω = angular frequency of the radio wave

and

N = number of electrons per cm^3

The refractive index is then given by

$$n = \sqrt{1 - \frac{\omega_P^2}{\omega^2}} \approx 1 - \frac{\omega_P^2}{2\omega^2} \tag{2}$$

since the plasma frequency is $\omega_P \approx 0.18\,\mathrm{MHz}$ assuming $N = 10\,\mathrm{cm}^{-3}$ and the carrier frequency ω is much larger than ω_P, the former being 400 MHz for the dual frequency experiments and 2000 MHz for the DRVID method. There ensues, then, a first order theory of the effects of the solar plasma on the electromagnetic wave propagation as put forward by Eshleman in comprehensive form (Ref. 3). This first order theory neglects magnetic field influences, influences due to spacial and temporal inhomogeneities in the electron density N, and, finally, pulse shape distortion due to dispersion. We shall address ourselves to these higher order effects and show that they are indeed negligible at the high carrier frequencies involved.

II. Development of the Theory

It can be shown (Ref. 4) that the equation for the electric field of a monochromatic electromagnetic wave in the presence of a constant magnetic field in convenient vector notation is given by

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} = \frac{\omega_P^2}{c^2}\left[A\mathbf{E} + B\hat{\mathbf{h}} \times \mathbf{E} + C\hat{\mathbf{h}}\hat{\mathbf{h}} \cdot \mathbf{E}\right] \tag{3}$$

where

$$A = \frac{i\omega(\nu - i\omega)}{(\nu - i\omega)^2 + \omega_c^2} \tag{4a}$$

$$B = -i\frac{\omega\,\omega_c}{(\nu - i\omega)^2 + \omega_c^2} \tag{4b}$$

$$C = \frac{\nu}{\nu - i\omega} - \frac{\nu(\nu - i\omega) + \omega_c^2}{(\nu - i\omega)^2 + \omega_c^2} \tag{4c}$$

and

ω_P = plasma frequency

ω = angular frequency of the wave

ν = effective collision frequency for the electrons

ω_c = cyclotron frequency

$\hat{\mathbf{h}}$ = unit vector in direction of the constant magnetic field

The magnitude of the interplanetary magnetic field is of the order of 10^{-9} tesla (10^{-5} gauss) resulting in a cyclotron frequency of $\omega_c \approx 176\ \text{sec}^{-1}$. The collision frequency is $\nu < 1\ \text{sec}^{-1}$. Therefore, for $\omega \approx 2 \times 10^9\ \text{sec}^{-1}$, the influence of the magnetic field and the influence of collisions are totally negligible as far as the wave propagation is concerned. This is even true for the ionosphere and solar corona where magnetic field strengths are of the order of 10^{-4} tesla (1 gauss) so that $\omega_c \approx 10^6\ \text{sec}^{-1}$. For these cases Eq. (3) reduces readily to

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} = \frac{\omega_P^2}{c^2}\left(i\frac{\omega_c}{\omega}\hat{\mathbf{h}} \times \mathbf{E} - \mathbf{E}\right) \tag{5}$$

where the ratio $\omega_c/\omega = 5 \times 10^{-4}$ indicates the smallness of the magnetic field influence. The first term on the right-hand side of Eq. (5) leads to an *observable* Faraday rotation effect, however (Ref. 5). The reason for this is that the rotation of the polarization is proportional to the *difference* of the two wave vectors:

$$k_\pm = \frac{\omega}{c}\left(1 - \frac{\omega_P^2}{2\omega^2} \pm \frac{1}{2}\frac{\omega_P^2}{\omega^2}\frac{\omega_c}{\omega}\right) \tag{6}$$

whereas the electron content measurements, as in the dual frequency and DRVID methods, essentially depend on the wave vectors k_+ and k_- separately, in which case the magnetic field influence is negligibly small.

The field equation (3) reduces, therefore, to

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2}\mathbf{E} = -\frac{\omega_P^2}{c^2}\mathbf{E} = -\frac{4\pi e^2}{c^2 m}N\mathbf{E} \tag{7}$$

Turning now to the spacial inhomogeneities of the electron density, it must be realized that the smallest scale of such fluctuations is at worst of the order of kilometers. Measurable changes rather take place over thousands of kilometers. Therefore, considering the small carrier wavelengths involved, a first order Wentzel–Kramers–Brillouin method is quite adequate for the solution of Eq. (7). Suppose that the electron density varies in the direction of propagation of the signal (the x direction). With the ansatz

$$E = A(x)\exp\left[i(kx - \omega t) + i\phi(x)\right] \tag{8}$$

where

$$k^2 = \frac{\omega^2}{c^2} - \frac{4\pi e^2}{c^2 m}N_0 \tag{9}$$

with N_0 a reference electron density chosen such as to make the difference $N(x) - N_0$ small, we obtain an expression for the correction in amplitude $A(x)$ and phase $\phi(x)$ from Eq. (7):

$$2i(k + \phi')A' + (i\phi'' - 2k\phi')A - \left[(\phi')^2 - \frac{4\pi e^2}{c^2 m}(N - N_0)\right]A = 0 \tag{10}$$

We neglected the second derivative of the amplitude A'' because of its smallness (see, for instance, Ref. 6). Setting

$$\phi' = \sqrt{\frac{4\pi e^2}{c^2 m}(N - N_0)} \tag{11}$$

we obtain from Eq. (10)

$$\frac{A'}{A} = \frac{i}{2}\,\frac{i\phi'' - 2k\phi'}{k + \phi'} \tag{12}$$

Noting that ϕ' is of the order of ϕ/L where L is the scale length of changes in the electron density which measures in kilometers and that on the other hand k^{-1} is of the order of centimeters, we see that Eq. (12) can be replaced by

$$\frac{A'}{A} = -i\phi' \tag{13}$$

the error being of the order of 10^{-5} ($\phi/kL \approx 10^{-5}$). But Eq. (13) yields as solution

$$A = e^{-i\phi} \tag{14}$$

disregarding an immaterial integration constant. From Eq. (8) we see immediately that there is no effect in first order.

In second order we obtain from Eq. (12)

$$\frac{A'}{A} = -i\phi' + \frac{i}{2k}\left[2(\phi')^2 + i\phi''\right] \tag{15}$$

which when integrated yields

$$A = e^{-i\phi} \exp\left\{k^{-1}\int^{x} d\tau\left[i(\phi')^2 - \frac{1}{2}\phi''\right]\right\} \tag{16}$$

The first term under the integral sign in Eq. (16) is always imaginary (see Eq. 11) and will, therefore, always lead to a phase shift.

If the electron density $N(x)$ is always larger than the reference electron density N_0 throughout the ray path, the signal will always be attenuated. In the opposite case the second term under the integral becomes imaginary and leads to an additional phase shift. However, the effect is extremely small, being of the order $\lambda/L \approx 10^{-5}$ at best.

Turning now to a situation in which the electron density inhomogeneities are transverse (z direction) to the radio beam direction, we put

$$E = A(z)\exp\left[i(kx - \omega t) + i\phi(z)\right] \tag{17}$$

and obtain in complete analogy to Eq. (11)

$$\phi' = \sqrt{\frac{4\pi e^2}{c^2 m}\left[N(z) - N_0\right]} \tag{18}$$

where of course the prime means differentiation with respect to z. The amplitude turns out to be

$$A(z) = A_0(\phi')^{-1/2} \tag{19}$$

We assume for convenience that $N(z) > N_0$. It is clear then that a surface of constant phase, the wave front, is given by Eqs. (17) and (18):

$$kx + \int_{z_0}^{z}\sqrt{\frac{4\pi e^2}{c^2 m}\left[N(\tau) - N_0\right]}\,d\tau = \text{constant} \tag{20}$$

so that

$$\frac{dx}{dz} = -\frac{1}{k}\sqrt{\frac{4\pi e^2}{c^2 m}\left[N(z) - N_0\right]} = \tan\alpha \approx \alpha \tag{21}$$

where α is the angle between the direction of propagation and the x coordinate (the undisturbed direction of propagation). To roughly estimate the angle α, let us put $N(z) = N_0(1 + z^2/D^2)$ where D, the scale length, is assumed to be 10^3 km. For a beamwidth B of 26 m, the diameter of the DSN antennas, we obtain from Eq. (21)

$$\alpha = \frac{\omega_P}{\omega}\,\frac{B}{D} = 1.8 \times 10^{-9}\ \text{radians} \tag{22}$$

assuming $N_0 = 10\ \text{cm}^{-3}$ and $\omega = 2 \times 10^9\ \text{sec}^{-1}$.

The apparent change in the range due to such small angles is very small indeed. To estimate its value, it suffices to consider that for a distance of 1 AU, the lateral deviation of the ray path is only about 0.2 km, assuming an angle as given by Eq. (22). The change in the range or length of the ray path is then only fractions of a centimeter.

Solar plasma clouds originating from solar flares are also of concern because of their vastly larger electron density. Typically, their dimensions are of the order of 10^6 km and their electron densities are 10 to 100 times that of the ambient plasma wind. Here we use a ray tracing technique to estimate their possible influence on the deflection of radio beams. A simplified geometry of a plasma cloud is given in Fig. 1.

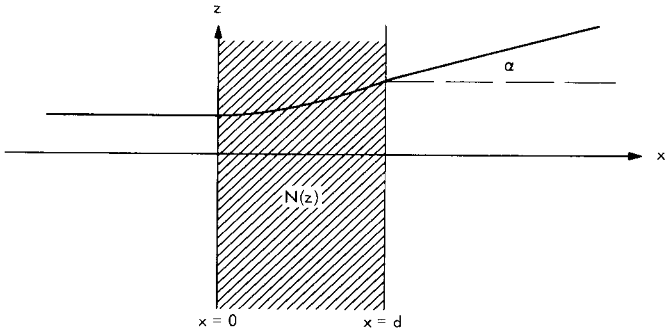

Fig. 1. Simplified geometry of a plasma cloud

A radio signal enters a plasma slab, confined between the planes $x = 0$, and $x = d$, normally to its surface. Due to a lateral electron density inhomogeneity $N(z)$, the beam will be deflected toward a region of *lower* electron concentration and will eventually emerge in a different direction characterized by the angle α in Fig. 1. The refractive index of the plasma is again given by

$$n = \sqrt{1 - \frac{4\pi e^2}{m\omega^2} N(z)} = 1 - \frac{2\pi e^2}{m\omega^2} N(z) \tag{23}$$

since the radar frequency ω is so much larger than the plasma frequency $4\pi e^2 N/m$. In order to obtain an expression for the angle α and to ultimately evaluate the effect of the bending on the ray path, it is quite sufficient to use Fermat's principle

$$\delta \int n\, ds = 0 \tag{24}$$

If we express the ray in Fig. 1 by $z = z(x)$, the Euler equation for it becomes very simple:

$$\frac{1}{n}\frac{dn}{dz} = \frac{z''}{1 + (z')^2} \tag{25}$$

where a prime means differentiation with respect to x. Integrating Eq. (25) once yields

$$[1 + (z')^2]^{1/2} = \frac{n}{n_0} \tag{26}$$

$n_0 = n$ at the position $x = 0$ (see Fig. 1), and $z' = dz/dx = 0$ at $x = 0$. It follows from Eq. (26) that

$$x = \int_{z_0}^{z} \frac{dz}{\sqrt{\left(\frac{n}{n_0}\right)^2 - 1}} \tag{27}$$

is the desired relationship between z and x. Since x is by definition real, n must be greater than n_0. The beam is indeed deflected into regions of a higher refractive index or a lower electron concentration.

According to P. F. MacDoran[1] the time rate of change of the electron content I of a typical solar plasma cloud can be as much as

$$\frac{dI}{dt} = 3 \times 10^{15}\, m^{-2}\, \sec^{-1} \tag{28}$$

Assuming a cloud velocity of 300 km-sec^{-1} and an extension of some 10^6 km (values which are generally accepted; see, for instance, Ref. 7), we obtain a characteristic value for the electron density gradient of about

$$\frac{dN}{dz} \approx 10\, m^{-4} \tag{29}$$

To estimate the angle α in Fig. 1, we now use Eq. (23) together with a linear relationship $N = N_0(1 - \gamma z)$. The integral (27) becomes

$$x = \int_{z_0}^{z} \frac{dz}{\sqrt{\left(1 + \gamma \frac{2\pi e^2}{m\omega^2} N_0 z\right)^2 - 1}} \tag{30}$$

Defining

$$\beta = \gamma \frac{2\pi e^2}{m\omega^2} N_0 \tag{31}$$

and putting $z_0 = 0$ for convenience, we obtain from Eq. (30)

$$\beta x = \ln\{1 + \sqrt{\beta^2 z^2 + 2\beta z} + \beta z\} \tag{32}$$

We also have (see Fig. 1):

$$\tan\alpha = \left(\frac{dz}{dx}\right)_{x=d} = \sqrt{(1 + \beta z_1)^2 - 1} \tag{33}$$

where z is the solution of Eq. (32) with $x = d$.

The solution of Eq. (32) in turn is given by

$$\beta z = \cosh(\beta x) - 1 \tag{34}$$

[1]Private communication.

and, finally,

$$\tan \alpha = \sinh (\beta d) \tag{35}$$

Just to see how small the angle α is, we take typical values for the plasma cloud as given by MacDoran. Let $N_0 = 300\,\mathrm{cm}^{-3}$ and $\omega = 4\pi\, 10^9$ Hz. From Eq. (29) the value of γ turns out to be

$$\gamma = \frac{1}{3} 10^{-7} m^{-1} \tag{36}$$

Let $d = 10^9$ m, the scale length of a typical plasma cloud, then

$$\tan \alpha = \sinh \beta d = \sinh (10^{-7}) \tag{37}$$

so that α is 10^{-7} rad signifying a lateral deflection of the beam at a distance of 1 AU of about 10 km, somewhat larger than before (Eq. 22) but still totally negligible.

Finally, we address ourselves to the question of pulse degradation by dispersion in the interplanetary plasma. It is clear that a pulse or pulse train may be represented by a Fourier integral. Accordingly the electric field of the radio signal is

$$E \sim I = \int F(\omega)\, d\omega \exp [i\phi(\omega)] \tag{38}$$

where

$$\phi(\omega) = \frac{x}{c}\left(\omega - \frac{\omega_P^2}{2\omega}\right) - \omega t \tag{39}$$

which is a solution of Eq. (7) using the approximation

$$\left(1 - \frac{\omega_P^2}{\omega^2}\right)^{1/2} = 1 - \frac{\omega_P^2}{2\omega^2}$$

The pulse spectrum $F(\omega)$ is sharply peaked at the carrier frequency ω_0 with an inverse bandwidth ω_1 small compared to the carrier. For the DSN transmitter $\omega_1/\omega_P = 2.5 \times 10^{-4}$, surely a small number. Because of these properties of $F(\omega)$, the phase $\phi(\omega)$ can be expanded in a Taylor series about ω_0 and we have

$$\phi(\omega) = \left(\frac{x}{c} - t\right)\omega_0 - \frac{\omega_P^2 x}{2c\omega_0} + \left(\frac{x}{c} - t + \frac{\omega_P^2 x}{2c\omega_0^2}\right)(\omega_0 - \omega) - \frac{\omega_P^2 x}{2c\omega_0^3}(\omega_0 - \omega)^2 + \cdots \tag{40}$$

With the substitution $\omega_0 - \omega = \omega'$, we obtain from Eqs. (38) and (40)

$$I = \exp\left\{i\left[\frac{x}{c}\left(\omega_0 - \frac{\omega_P^2}{2\omega_0}\right) - \omega_0 t\right]\right\}\int d\omega' F(\omega') \times \exp\left\{i\left[\frac{x}{c} - t + \frac{\omega_P^2 x}{2c\omega_0^2}\right]\omega' - i\frac{\omega_P^2 x}{2c\omega_0^3}(\omega')^2\right\} \tag{41}$$

where it is recognized that the first exponential determines the phase velocity and the first term in the second exponential determines the group velocity and the last term represents a distortion of phase and amplitude. For mathematical convenience we take a gaussian pulse

$$F(\omega') = \frac{1}{\sqrt{\pi}\,\omega_1}\exp\left[-\frac{(\omega')^2}{\omega_1^2}\right] \tag{42}$$

The integral (41) becomes with the pulse spectrum (42)

$$I = \left(1 + i\frac{\omega_P^2 \omega_1^2 x}{2c\omega_0^3}\right)^{-1/2} \exp\left\{-\frac{\left[\frac{x}{c} - t + \frac{\omega_P^2 x}{2c\omega_0^2}\right]^2}{4\left(\frac{1}{\omega_1^2} + i\frac{\omega_P^2 x}{2c\omega_0^3}\right)}\right\} \times \exp\left\{i\left[\frac{x}{c}\left(\omega_0 - \frac{\omega_P^2}{2\omega_0}\right) - \omega_0 t\right]\right\} \tag{43}$$

The term $\epsilon = \omega_P^2\omega_1^2 x/2c\omega_0^3$ responsible for the pulse degradation can quickly be seen to be ridiculously small even for distances x of the order of 1 AU. For $\omega_1 \approx \frac{1}{2} 10^6\,\mathrm{sec}^{-1}$, $\omega_0 \approx 2 \times 10^9\,\mathrm{sec}^{-1}$, $\omega_P \approx 1.8 \times 10^5\,\mathrm{sec}^{-1}$ (for $N_0 = 10\,\mathrm{cm}^{-3}$), we have

$$\epsilon = 1.6 \times 10^{-12} x$$

where x is measured in kilometers. The influence of dispersion on the pulse shape is negligible. The situation would change drastically if the carrier frequency would be lowered and the modulation frequency (inverse bandwidth) would be enlarged. For instance, a twice as high modulation frequency together with half the carrier frequency would result in a $4 \times 8 = 32$-fold enhancement of the pulse shape effect.

It must be realized that Eq. (43) is only valid if $\epsilon << 1$. In that case Eq. (43) may be written as:

$$I = \exp\left\{i\left[\frac{x}{c}\left(\omega_0 - \frac{\omega_P^2}{2\omega_0}\right) - \omega_0 t\right]\right\}$$
$$\times \exp\left\{-\frac{\omega_1^2}{4}\left[\frac{x}{c} - t + \frac{\omega_P^2 x}{2c\omega_0^2}\right]^2\right\}$$
$$\times \exp\left\{i\epsilon\left(\frac{\omega_1^2}{4}\left[\frac{x}{c} - t + \frac{\omega_P^2 x}{2c\omega_0^2}\right]^2 - \frac{1}{2}\right)\right\} \tag{44}$$

where of course

$$\epsilon = \frac{\omega_P^2\,\omega_1^2}{2\omega_0^3}\,\frac{x}{c} \tag{45}$$

In case of a slowly varying electron density along the ray path, we have

$$\omega_P^2\,x = \frac{4\pi e^2}{m} I \tag{46}$$

where

$$I = \int_0^x N(\tau)\,d\tau \tag{47}$$

is the electron content in cm^{-2} since we use cgs units throughout.

At the time

$$t' = \frac{x}{c} - t + \frac{\omega_P^2\,x}{2c\omega_0^2} \tag{48}$$

the pulse arrives at the receiver. In that case, x is the distance between the spacecraft and the earthbound receiver. If the pulse (44) is beat against a reference pulse of the same shape (Eq. 42), there results a signal which may be expressed as

$$v = \exp\left[-\frac{\omega_1^2}{4}(t')^2\right]\cos\left\{\epsilon\left(\frac{\omega_1^2}{4}(t')^2 - \frac{1}{2}\right)\right\}$$
$$- \exp\left[-\frac{\omega_1^2}{4}(t' + \delta)^2\right] \tag{49}$$

In Eq. (49) the high frequency component has been averaged out and δ is the delay time between the received pulse and the reference pulse. If the pulse maxima coincide ($\delta = 0$), there results a net signal

$$v = 2\exp\left[-\frac{\omega_1^2}{4}(t')^2\right]\sin^2\left\{\epsilon\left(\frac{\omega_1^2}{2}(t')^2 - \frac{1}{2}\right)\right\} \tag{50}$$

and, since ϵ is small, Expression (50) can be written as

$$v = -2\epsilon^2\left(\frac{\omega_1^2}{2}(t')^2 - \frac{1}{2}\right)^2 \exp\left[-\frac{\omega_1^2}{4}(t')^2\right] \tag{51}$$

The pulse Eq. (51) is symmetric about its maximum ($t' = 0$). If the delay time δ is not zero, the pulse shape is asymmetric. Therefore, by changing δ, it is possible to measure the group velocity accurately if the quantity ϵ is not too small. For the DSN, however, ϵ is far too small as we have seen previously and therefore a pulse shape analysis is out of the question.

III. Conclusion

In the foregoing, we have shown that higher order effects of the interplanetary plasma on radio signals as utilized in the DSN are negligible as far as a determination of the electron content is concerned. We have also seen that a lateral change in electron concentration engenders very small angular deviations of the ray path. They also lead to a negligible change in the apparent range. The first order theory is therefore completely adequate to deal with the charged particle content.

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