Sequential Ranging With the Viterbi Algorithm

U. Timor
Communications Systems Research Section

The performance of the sequential ranging system can be improved by using a maximum likelihood receiver; however, the complexity grows exponentially with the number of components N needed to determine the range unambiguously. A new truncated maximum-likelihood receiver, based on the Viterbi decoder for convolutional codes, is presented and is shown to achieve a maximum-likelihood performance while having a fixed complexity independent of N. The improvement in signal-to-noise ratio, compared to the present receiver, is 1.5 dB for $P_E < 10^{-2}$.

I. Introduction

Ranging systems for deep space applications achieve the required resolution by transmitting, either simultaneously or sequentially, a multi-component signal. Goldstein (Ref. 1) described a sequential ranging system, which transmits $N + 1$ squarewaves of increasing periods T, $2T, \cdots, 2^N T$. The highest frequency component yields the most accurate measurement, but with an added distance of $M \cdot R$, where R is proportional to T and M is an unknown integer. The other N components are used to estimate M and thus remove the ambiguity.

In the above system, the estimation is done sequentially, that is, at each step a binary number a_k is estimated from the signal component which is present at that step. The sequence $a_1, \cdots, a_N$, considered as a binary representation of M, yields an estimate of M.

The performance of the system—that is, the probability of estimating M correctly—can be improved by using a maximum-likelihood estimator, which estimates the whole sequence $a_1, \cdots, a_N$ simultaneously. However, the complexity of such a system is proportional to 2^N and is not practical for large N.

We will present a suboptimum estimation procedure, which outperforms the sequential receiver and approaches asymptotically, as the signal-to-noise ratio increases, the performance of the maximum-likelihood receiver. This method is based on the Viterbi algorithm for decoding convolutional codes of short constraint length (ν), and has a complexity of the order of 2^ν, no matter how large N is. The improvement in the signal-to-noise ratio required to achieve a given error probability P_E is 1.5 dB throughout the range of interest ($P_E < 10^{-2}$).

II. The Sequential Receiver

The time-of-flight (TOF) of the signal at t_0 can be represented by

$$\mathrm{TOF} = (M + \epsilon)\,T \tag{1}$$

where T is the period of the first (highest frequency) squarewave, M is a positive integer and $0 \leqq \epsilon < 1$.

Let $\{a_N\, a_{N-1}, \cdots, a_1\}$ be the binary representation of M, that is

$$M = \sum_{k=1}^{N} a_k\, 2^{(k-1)} \tag{2}$$

where a_k is 0 or 1.

To measure the TOF, and therefore the range, it is enough to measure ϵ and $\{a_1, \cdots, a_N\}$. The present receiver does this sequentially.

We start by transmitting the T-period squarewave to obtain an estimate $\hat{\epsilon}$ of ϵ. The receiver correlates the incoming signal with the receiver coder squarewave and its 90 deg-shift to obtain a pair of outputs x_0, y_0, from which ϵ is estimated. The correlator is then shifted by $\hat{\epsilon}$ to have a phase of $\hat{\epsilon} - \epsilon$. We will assume that the integration time is long enough to obtain $\epsilon - \hat{\epsilon} = 0$. To estimate $a_1, \cdots, a_N$, squarewaves of periods $2^k T$, $k = 1, \cdots, N$ are transmitted sequentially. The outputs of the in-phase and quadrature correlators at the kth step are

$$\begin{aligned} x_k &= s_k + n_k \\ y_k &= r_k + m_k \end{aligned} \tag{3}$$

respectively, where n_k and m_k are independent white gaussian noise samples of zero mean and variance σ^2,

$$s_k = \begin{cases} 1 - \alpha_k, & 0 \leqq \alpha_k < 2 \\ \alpha_k - 3, & 2 \leqq \alpha_k < 4 \end{cases}$$

$$r_k = \begin{cases} \alpha_k, & 0 \leqq \alpha_k < 1 \\ 2 - \alpha_k, & 1 \leqq \alpha_k < 3 \\ \alpha_k - 4, & 3 \leqq \alpha_k < 4 \end{cases} \tag{4}$$

and α_k depends on $a_1, \cdots, a_k$.

It was shown that the probability of error is minimized by shifting the correlator waveform by 90 deg at the kth step whenever $\hat{a}_{k-1} = 1$, where $\hat{a}_{k-1}$ is the estimate of the previous step. Thus

$$\alpha_k = 2a_k + \sum_{j=1}^{k-1} (a_j - \hat{a}_j)\, 2^{(j+1-k)} \tag{5}$$

and we estimate

$$\hat{a}_k = \begin{cases} 0 & \text{if } x_k \geqq 0 \\ 1 & \text{if } x_k < 0 \end{cases} \tag{6}$$

The procedure is terminated when $\hat{a}_N$ is obtained. The resulting

$$\hat{M} = \sum_{k=1}^{N} \hat{a}_k\, 2^{k-1} \tag{7}$$

and $\hat{\epsilon}$ yield the measured TOF.

III. Maximum-Likelihood Estimation

The "estimate-and-shift" sequential method has a strong error propagation property. Suppose the estimate $\hat{a}_1$ of a_1 is in error, e.g., $a_1 = 0$ but $\hat{a}_1 = 1$. The correlator waveform is shifted by 90 deg at the second step, therefore the no-noise outputs will be A' if $a_2 = 0$ and B' if $a_2 = 1$ instead of A and B, respectively (Fig. 1). Since the receiver estimates $a_2 = 0$ whenever $x_2 \geqq 0$, the probability of wrongly estimating a_2 is ½. It can be shown that the error in estimating a_1 will adversely affect the estimation

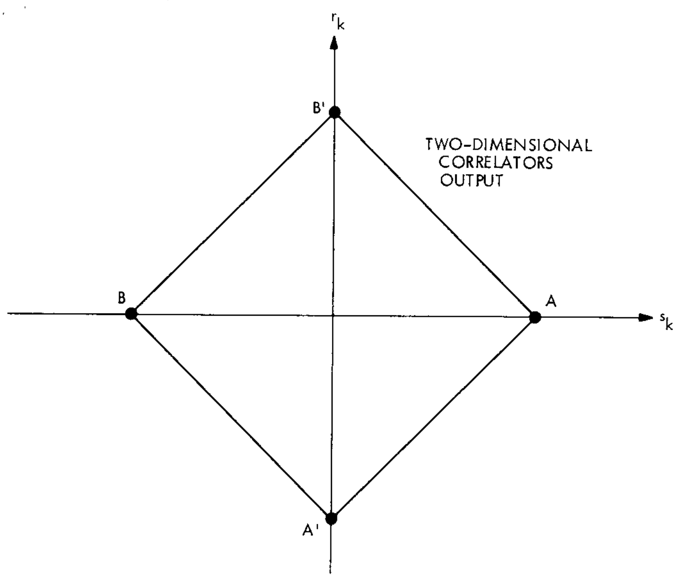

Fig. 1. Correlators output at second step: A or B if $\hat{a}_1 = a_1$ and A' or B' if $\hat{a}_1 \neq a_1$

of $a_3, a_4, \cdots$. However, the effect will diminish as k increases.

This error propagation does not affect the performance of the system, since one wrong estimation is enough to cause an error in the measurement of the TOF. However, since a_k affects all (x_n, y_n) for $n \geqq k$, an estimate of a_k based on $\{x_n, y_n; n \geqq k\}$ is superior to one which is performed at the kth step, and therefore depends on (x_k, y_k) only.

Assume, as before, that $\hat{\epsilon} \approx \epsilon$ and let

$$\mathbf{x} = (x_1, \cdots, x_N)$$

$$\mathbf{y} = (y_1, \cdots, y_N)$$

$$\mathbf{a} = (a_1, \cdots, a_N)$$

be the correlators output vectors and the TOF binary vector respectively. By Bayes rule we have

$$p(\mathbf{a} | \mathbf{x}, \mathbf{y}) = \frac{p(\mathbf{x}, \mathbf{y} | \mathbf{a})}{p(\mathbf{x}, \mathbf{y})} p(\mathbf{a}) \tag{8}$$

and the maximum-likelihood estimate is the binary vector $\mathbf{a}^*$ which maximizes

$$p(\mathbf{x}, \mathbf{y} | \mathbf{a}) = \frac{1}{(\sqrt{2\pi}\,\sigma)^{N/2}} \times \exp\left\{ -\frac{1}{2\sigma^2} \sum_{k=1}^{N} \left[(x_k - s_k)^2 + (y_k - r_k)^2 \right] \right\} \tag{9}$$

or equivalently minimizes

$$\ell(\mathbf{a}) = \sum_{k=1}^{N} \left[(x_k - s_k)^2 + (y_k - r_k)^2 \right] \tag{10}$$

where $s_k, r_k, k = 1, \cdots, N$ can be expressed in terms of $\mathbf{a}$ (Eqs. 4 and 5).

Note that the $\hat{a}_j, j = 1, \cdots, k-1$ that appear in α_k (and hence in s_k and r_k) are on-the-spot estimates which determine whether the correlator waveforms are shifted in the next step. They depend on the x_j's alone and not on $\mathbf{a}$, and are known to the receiver when $\ell(\mathbf{a})$ is evaluated.

Since we have to choose the most likely of 2^N possibilities, the complexity grows as 2^N, and the method is not practical for large N.

IV. Truncated Maximum-Likelihood Estimate and the Viterbi Algorithm

The contribution of a_k to (s_n, r_n) is halved at every successive step, since its coefficient in α_n is proportional to $2^{(k-n)}$ (Eq. 5). In other words, the value of a_k affects the correlator outputs for all $n \geqq k$; however, this effect diminishes exponentially as n increases. Thus, there exists some integer ν, which depends on the signal-to-noise ratio, such that the contribution of a_k to (s_n, r_n) for $n > k + \nu$ is negligible. We therefore can approximate Eq. (5) by

$$\alpha_k = 2a_k + \sum_{j=k-\nu}^{k-1} (a_j - \hat{a}_j)\, 2^{(i+1-k)} \tag{11}$$

for all $k > \nu + 1$.

Thus each (s_k, r_k) depends on a_k and the previous ν components and we have a finite state machine with 2^ν states, corresponding to all possible binary vectors $(a_{k-1}, \cdots, a_{k-\nu})$, and two outputs per state depending on a_k. The progress of this machine, during few successive steps can be depicted by its trellis diagram (Fig. 2).

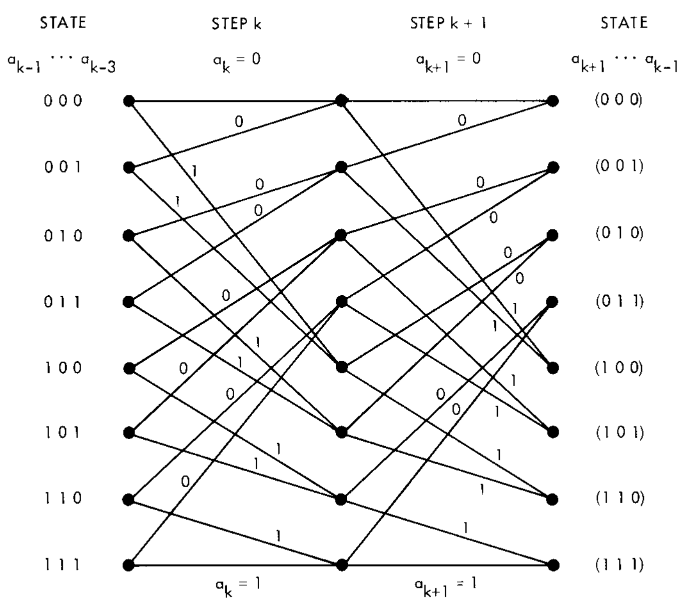

Fig. 2. Trellis diagram for $\nu = 3$ during steps k and $k + 1$

During step k, each state $(a_{k-1}, \cdots, a_{k-\nu})$ can advance to one of two states $(0, a_{k-1}, \cdots, a_{k-\nu+1})$ or $(1, a_{k-1}, \cdots, a_{k-\nu+1})$ depending whether a_k is 0 or 1. The corresponding correlation outputs (s_k, r_k) will depend on the starting state as well as on a_k.

The similarity to convolutional coding with a constraint length of ν is immediate. Many decoding procedures have been developed for convolutional codes, however, for short constraint length ($\nu < 10$) the Viterbi algorithm (Ref. 2) is the most efficient, and is actually a maximum-likelihood estimate of the truncated estimation problem (Ref. 3).

The Viterbi algorithm can be briefly described as follows: With every state we associate a metric (accumulated likelihood function up to this step) and a survivor (the most likely sequence leading to this state). At step k, each state $\hat{S}_k = (a_k, \cdots, a_{k-\nu+1})$ can be reached from two states $S^{\delta}_{k-1} = (a_{k-1}, \cdots, a_{k-\nu+1}, \delta)$, where δ is 0 or 1. After the received signal is correlated to yield (x_k, y_k), we compare the two possible ways to reach $\hat{S}_k$ and keep the most likely of them, by properly updating the metric and the survivor of $\hat{S}_k$. This is done for each one of the 2^ν states, and is repeated every step. The final decision is made after the Nth (last) step, by comparing the 2^ν metrics and selecting the survivor of the largest, to yield the most likely estimate of $\mathbf{a}$.

Thus a ranging receiver based on the Viterbi algorithm can yield a (truncated) maximum-likelihood performance with a complexity of 2^ν, which is independent of the number of components N to be estimated.

V. Error Probabilities

The signal is received in the presence of additive white gaussian noise of zero mean and spectral density N_0. If the signal power is S and the integration time of each component is $\tau_k = \tau$, $k = 1, 2, \cdots, N$ the probability that the sequential receiver will correctly estimate the whole sequence $\{a_1, \cdots, a_N\}$, is given by

$$P_c = 1 - \left\{1 - \mathrm{erfc}\left[\left(\frac{S\tau}{2N_0}\right)^{1/2}\right]\right\}^N$$

where

$$\mathrm{erfc}\,(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp\,(-x^2/2)\, dx \qquad (12)$$

If $\hat{\epsilon} - \epsilon \not\approx 0$ the performance is degraded; however, we can assume that the integration time of the highest frequency component is long enough to obtain $\hat{\epsilon} \approx \epsilon$. An analytic expression for the performance of the maximum-likelihood receiver or for the Viterbi algorithm cannot be obtained in closed form. Therefore, the error probabilities for various signal-to-noise ratios were obtained by computer simulations. The noise was generated by a multiplicative congruential generator (Ref. 4), and quantized in steps of $\sigma^2/32$, where σ^2 is the noise-to-signal ratio.

The results for the maximum-likelihood estimate of 10 components ($N = 10$), and a truncation to $\nu = 5$ are shown in Fig. 3 together with the performance of the sequential receiver. The improvement gained by maximum-likelihood estimation compared to sequential estimation is 1.5 dB for the measured range of error probabilities. This improvement is also achieved by the Viterbi algorithm (with $\nu = 5$ truncated memory and therefore a

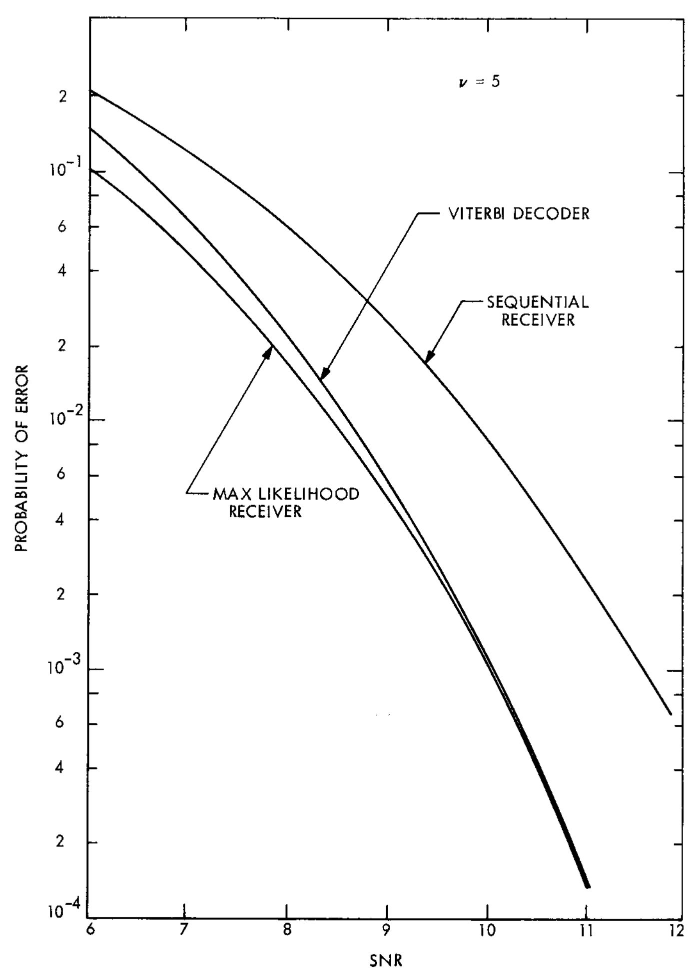

Fig. 3. Performance of various schemes for $N = 10$

smaller complexity), for signal-to-noise ratios which yield $P_E < 10^{-2}$.

V. Conclusions

A truncated maximum likelihood receiver for sequential ranging has been presented. The performance of the sequential ranging system can be improved by using maximum-likelihood techniques; however, the complexity grows exponentially with the number of components N needed to determine the range unambiguously. The suggested method, which is based on the Viterbi decoder for convolutional codes, performs like the maximum-likelihood receiver while having a finite complexity independent of N. The improvement in signal-to-noise ratio, compared to the present receiver, is 1.5 dB.

References

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