A First-Order Theory for Use in Investigating the Information Content Contained in a Few Days of Radio Tracking Data

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An approximation to the topocentric range rate of a spacecraft is developed which is first order in both the time past epoch and the ratio between the distance of an observing station from the geocenter and the geocentric range. This approximation is compared with a numerical integrated trajectory to obtain some idea of the duration over which it may be reliable. The development is extended to include an analytical determination of the errors in the spacecraft state produced by errors in the range rate data. It is also shown how range data may be incorporated into this cursory error analysis. The partial derivatives of the gravitational geocentric acceleration with respect to range, declination, and right ascension are obtained analytically and shown graphically.

I. Introduction

The determination of an orbit from range and range rate data and obtaining a measure of the accompanying errors is a complicated process. The physical understanding of this process is greatly enhanced if it is possible to develop a simple analytical theory, which although only an approximation, contains the pertinent features of the real problem. A major step in this direction was taken by Hamilton and Melbourne (Ref. 1) in their classical paper describing the information content of a single pass of doppler data when the geocentric range rate, declination, and right ascension may be assumed to be constant. Curkendall and McReynolds (Ref. 2) extended the theory to include first-order temporal variation in these quantities. The development which will be undertaken here is a refinement of the Curkendall and McReynolds approach and not conceptually a new method. Some new features which are presented are:

(1) Analytical derivation of the partial derivatives of the geocentric gravitational acceleration with respect to range, declination, and right ascension.

(2) Completely analytical description of errors in the full spacecraft state produced by station location errors.

An attempt is also made to obtain a better idea of how and when this first-order theory is no longer reliable.

II. Expansion of the Topocentric Range and Range Rate in terms of r_s/r and z_s/r

From the coordinate system shown in Fig. 1, it is easily seen that the magnitude of the topocentric range vector ρ may be written as

$$\rho = [(x - r_s \cos \varphi)^2 + (y - r_s \sin \varphi)^2 + (z - z_s)^2]^{1/2} \tag{1}$$

For a distant spacecraft the range equation may be put in a more convenient form by expanding the right-hand side of Eq. (1) in terms of the small quantities r_s/r and z_s/r as given below:

$$\begin{aligned}
\rho = r \Bigg\{ & 1 - \frac{r_s}{r}\left(\frac{x}{r}\cos\varphi + \frac{y}{r}\sin\varphi\right) - \frac{z_s}{r}\frac{z}{r} \\
& + \frac{1}{2}\frac{r_s^2}{r^2}\left[1 - \frac{1}{2}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right)\right. \\
& \left. - \frac{1}{2}\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\cos 2\varphi\right) - \frac{xy}{r^2}\sin 2\varphi\right] \\
& - \frac{r_s z_s z}{r\,r}\left(\frac{x}{r}\cos\varphi + \frac{y}{r}\sin\varphi\right) \\
& + \frac{1}{2}\frac{z_s^2}{r^2}\left(1 - \frac{z^2}{r^2}\right) + \cdots \Bigg\}
\end{aligned} \tag{2}$$

Deleting terms higher than the first order in r_s/r and z_s/r from the above equation and expressing the spacecraft coordinates in terms of a spherical system result in a very simple approximation for ρ, which is shown below. Starting with this equation, techniques may be developed that are extremely useful in interpreting various physical situations.

$$\rho \approx r - [r_s \cos\delta \cos(\varphi - \alpha) + z_s \sin\delta] \tag{3}$$

where

r = geocentric range

δ = declination

α = right ascension

The range rate to the same accuracy may be obtained by differentiating Eq. (3) with respect to time and is given below:

$$\begin{aligned}
\dot{\rho} \approx \dot{r} - z_s \dot{\delta} \cos\delta + r_s(\dot{\varphi} - \dot{\alpha})\cos\delta \sin(\varphi - \alpha) \\
+ r_s \dot{\delta} \sin\delta \cos(\varphi - \alpha)
\end{aligned} \tag{4}$$

III. Expansion of the Range Rate to the First Order in Time

After deriving an expression similar to Eq. (4), Hamilton and Melbourne (Ref. 1) proceed under the assumption that $\dot{r}$, δ, and α are constant. Curkendall and McReynolds (Ref. 2) have shown that the analysis is improved if the spacecraft variables in Eq. (4) are expressed as first-order expansions in time, as given below:

$$\begin{aligned}
\dot{r} &= \dot{r}_0 + \ddot{r}_0 t \\
\delta &= \delta_0 + \dot{\delta}_0 t \\
\dot{\delta} &= \dot{\delta}_0 + \ddot{\delta}_0 t \\
\alpha &= \alpha_0 + \dot{\alpha}_0 t \\
\dot{\alpha} &= \dot{\alpha}_0 + \ddot{\alpha}_0 t
\end{aligned}$$

where a_0 denotes that the quantity a is evaluated at $t = 0$. Substituting the above set of equations into Eq. (4), and assuming, for the present, that

$$\varphi = \varphi_0 + \dot{\varphi}_0 t$$

yields the following first-order expansion of the topocentric range rate in both r_s/r, z_s/r and time:

$$\begin{aligned}
\dot{\rho}(t) \approx \dot{r}_0 - z_s \dot{\delta}_0 \cos\delta_0 &+ r_s(\dot{\varphi}_0 - \dot{\alpha}_0)\cos\delta_0 \\
&\times \sin(\varphi_0 + \dot{\varphi}_0 t - \alpha) \\
&+ r_s \dot{\delta}_0 \sin\delta_0 \cos(\varphi_0 + \dot{\varphi}_0 t - \alpha_0) \\
&+ [\ddot{r}_0 + z_s \dot{\delta}_0^2 \sin\delta_0 - z_s \ddot{\delta}_0 \cos\delta_0]t \\
&+ r_s[-(\dot{\varphi}_0 - 2\dot{\alpha}_0)\dot{\delta}_0 \sin\delta_0 \\
&- \ddot{\alpha}_0 \cos\delta_0]\, t \sin(\varphi_0 + \dot{\varphi}_0 t - \alpha_0) \\
&+ r_s[-(\dot{\varphi}_0 - \dot{\alpha}_0)\dot{\alpha}_0 \cos\delta_0 \\
&+ \dot{\delta}_0^2 \cos\delta_0 + \ddot{\delta}_0 \sin\delta_0]\, t \cos(\dot{\varphi}_0 + \varphi_0 t - \alpha_0)
\end{aligned} \tag{5}$$

This equation is not in a suitable form for analysis because $\ddot{r}_0$, $\ddot{\delta}_0$, and $\ddot{\alpha}_0$ are not independent of r_0, α_0, δ_0, $\dot{r}_0$, $\dot{\delta}_0$, and $\dot{\alpha}_0$.

IV. Calculation of $\ddot{r}$, $\ddot{\delta}$, and $\ddot{\alpha}$

Before initiating the derivation for $\ddot{r}$, $\ddot{\delta}$, and $\ddot{\alpha}$, it is convenient to obtain some relations between the unit vectors, **i**, in the cartesian and spherical coordinate systems. These relations may be obtained by obtaining from Fig. 1 the following equations:

$$\left.\begin{aligned}
\mathbf{i}_r &= \cos\delta\cos\alpha\,\mathbf{i}_x + \cos\delta\sin\alpha\,\mathbf{i}_y + \sin\delta\,\mathbf{i}_z \\
\mathbf{i}_\alpha &= -\sin\alpha\,\mathbf{i}_x + \cos\alpha\,\mathbf{i}_y \\
\mathbf{i}_\delta &= -\sin\delta\cos\alpha\,\mathbf{i}_x - \sin\delta\sin\alpha\,\mathbf{i}_y + \cos\delta\,\mathbf{i}_z
\end{aligned}\right\} \quad (6)$$

Since the cartesian coordinate system is assumed to be an inertial frame of reference,

$$\dot{\mathbf{i}}_x = \dot{\mathbf{i}}_y = \dot{\mathbf{i}}_z = 0$$

and

$$\left.\begin{aligned}
\dot{\mathbf{i}}_r &= \dot{\delta}\,\mathbf{i}_\delta + \cos\delta\,\dot{\alpha}\,\mathbf{i}_\alpha \\
\dot{\mathbf{i}}_\delta &= -\dot{\delta}\,\mathbf{i}_r - \sin\delta\,\dot{\alpha}\,\mathbf{i}_\alpha \\
\dot{\mathbf{i}}_\alpha &= -\cos\delta\,\dot{\alpha}\,\mathbf{i}_r + \sin\delta\,\dot{\alpha}\,\mathbf{i}_\delta
\end{aligned}\right\} \quad (7)$$

The geocentric range vector **r** may be written as

$$\mathbf{r} = r\,\mathbf{i}_r$$

Taking the derivative with respect to time and using Eq. (7) yields

$$\dot{\mathbf{r}} = \dot{r}\mathbf{i}_r + r\,(\dot{\delta}\,\mathbf{i}_\delta + \cos\delta\,\dot{\alpha}\,\mathbf{i}_\alpha)$$

Similarly

$$\begin{aligned}
\ddot{\mathbf{r}} = {} & [\ddot{r} - r\,(\dot{\delta}^2 + \dot{\alpha}^2\cos^2\delta)]\,\mathbf{i}_r \\
& + (2\dot{r}\dot{\delta} + r\dot{\alpha}^2\sin\delta\cos\delta + r\,\ddot{\delta})\mathbf{i}_\delta \\
& + (2\dot{r}\dot{\alpha}\cos\delta - 2r\dot{\delta}\,\dot{\alpha}\sin\delta + r\cos\delta\,\ddot{\alpha})\mathbf{i}_\alpha
\end{aligned} \quad (8)$$

From Fig 2, **r** may also be written as

$$\mathbf{r} = \mathbf{r}_p - \mathbf{r}_e \quad (9)$$

If both the earth and spacecraft are assumed to move under the influence of the sun only, the second derivative of **r** may be written as

$$\ddot{\mathbf{r}} = -\mu\left(\frac{\mathbf{r}_p}{r_p^3} - \frac{\mathbf{r}_e}{r_e^3}\right)$$

where μ is the sun's gravitational constant. Substituting Eq. (9) into the above equation gives

$$\ddot{\mathbf{r}} = -\mu\left[\frac{\mathbf{r}}{r_p^3} - \mathbf{r}_s\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right)\right] \quad (10)$$

where $\mathbf{r}_s$ is the earth to sun vector, which in component form may be written as

$$\begin{aligned}
\mathbf{r}_s = {} & r_e\cos\delta_s\cos\alpha_s\mathbf{i}_x \\
& + r_e\cos\delta_s\sin\alpha_s\mathbf{i}_y + r_e\sin\delta_s\,\mathbf{i}_z
\end{aligned} \quad (11)$$

where

r_e = earth–sun distance

δ_s = declination of the sun

α_s = right ascension of the sun

Using the inverses of Eq. (6) allows Eq. (11) to be written as

$$\begin{aligned}
\mathbf{r}_s = {} & r_e\,[\cos\delta\cos\delta_s\cos(\alpha - \alpha_s) + \sin\delta\sin\delta_s]\mathbf{i}_r \\
& + r_e\,[-\sin\delta\cos\delta_s\cos(\alpha - \alpha_s) + \cos\delta\sin\delta_s]\,\mathbf{i}_\delta \\
& - r_e\cos\delta_s\sin(\alpha - \alpha_s)\mathbf{i}_\alpha
\end{aligned}$$

Substituting this equation into Eq. (10) results in

$$\begin{aligned}
\ddot{\mathbf{r}} = -\mu\Bigg\{ & \left[\frac{r}{r_p^3} - r_e\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right) < \cos\delta\cos\delta_s \right. \\
& \left. \times\cos(\alpha - \alpha_s) + \sin\delta\sin\delta_s > \right]\mathbf{i}_r \\
& - \left[r_e\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right) < -\sin\delta\cos\delta_s \right. \\
& \left. \times\cos(\alpha - \alpha_s) + \cos\delta\sin\delta_s > \right]\mathbf{i}_\delta \\
& + \left[r_e\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right) < \cos\delta_s \right. \\
& \left. \times\sin(\alpha - \alpha_s) > \right]\mathbf{i}_\alpha \Bigg\}
\end{aligned} \quad (12)$$

Comparing the right-hand side of this equation with the right-hand side of Eq. (8) gives the following expressions for $\ddot{r}$, $\ddot{\delta}$ and $\ddot{\alpha}$ in terms of r, δ, α, $\dot{r}$, $\dot{\delta}$, and $\dot{\alpha}$:

$$\left.\begin{aligned}\ddot{r} &= r\,(\dot{\delta}^2 + \dot{\alpha}^2\cos^2\delta) + \ddot{r}_g \\ \ddot{\delta} &= -\,\dot{\alpha}^2 \sin\delta\cos\delta - 2\,\frac{\dot{r}}{r}\,\dot{\delta} + \ddot{\delta}_g \\ \cos\delta\,\ddot{\alpha} &= 2\,\dot{\delta}\,\dot{\alpha}\sin\delta - 2\,\frac{\dot{r}}{r}\,\dot{\alpha}\cos\delta + \ddot{\alpha}_g\end{aligned}\right\} \tag{13}$$

where

$$\ddot{r}_g = -\,\mu\left[\frac{r}{r_p^3} - r_e\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right)< \cos\delta\cos\delta_s \right.$$
$$\left. \times \cos(\alpha - \alpha_s + \sin\delta\sin\delta_s > \right]$$

$$\ddot{\delta}_g = -\,\mu\,\frac{r_e}{r}\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right)< \sin\delta\cos\delta_s\cos(\alpha - \alpha_s)$$
$$-\cos\delta\sin\delta_s$$

$$\ddot{\alpha}_g = -\,\mu\,\frac{r_e}{r}\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right)\cos\delta_s\sin(\alpha - \alpha_s)$$

$$r_p = \{r^2 + r_e^2 - 2r\,r_e\,[\cos\delta\cos\delta_s\cos(\alpha - \alpha_s)$$
$$+ \sin\delta\sin\delta_s]\}^{1/2} \tag{14}$$

Finally substituting Eq. (13) into Eq. (5) gives the following approximation of the topocentric range rate:

$$\begin{aligned}\dot{\rho} \approx\; & a' + b'\sin(\varphi_0 - \alpha_0 + \dot{\varphi}_0\,t) \\ & + c'\cos(\varphi_0 - \alpha_0 + \dot{\varphi}_0\,t) + d't \\ & + e'\,t\sin(\varphi_0 - \alpha_0 + \dot{\varphi}_0\,t) \\ & + f'\,t\cos(\varphi_0 - \alpha_0 + \dot{\varphi}_0\,t)\end{aligned} \tag{15a}$$

where

$$a' = \dot{r}_0 - z_s\,\dot{\delta}_0\cos\delta_0$$

$$b' = r_s\,(\dot{\varphi}_0 - \dot{\alpha}_0)\cos\delta_0$$

$$c' = r_s\,\dot{\delta}_0\sin\delta_0$$

$$\begin{aligned}d' =\; & r_0\,(\dot{\delta}_0^2 + \dot{\alpha}_0^2\cos^2\delta_0) \\ & + \ddot{r}_{g0} + z_s\left(\dot{\delta}_0^2\sin\delta_0 + \dot{\alpha}_0^2\cos^2\delta_0\sin\delta_0 \right. \\ & \left. + 2\,\frac{\dot{r}_0}{r_0}\,\dot{\delta}_0\cos\delta_0 - \ddot{\delta}_{g0}\cos\delta_0\right)\end{aligned}$$

$$e' = r_s\left(-\,\dot{\varphi}_0\,\dot{\delta}_0\sin\delta_0 + 2\,\frac{\dot{r}_0}{r_0}\,\dot{\alpha}_0\cos\delta_0 - \ddot{\alpha}_{g0}\right)$$

$$\begin{aligned}f' = r_s\Big(& -\,\dot{\varphi}_0\,\dot{\alpha}_0\cos\delta_0 + \dot{\alpha}_0^2\cos^3\delta_0 \\ & + \dot{\delta}_0^2\cos\delta_0 - 2\,\frac{\dot{r}_0}{r_0}\,\dot{\delta}_0\sin\delta_0 + \sin\delta_0\,\ddot{\delta}_{g0}\Big)\end{aligned} \tag{15b}$$

V. Range Rate Partials and the Validity of the Approximations

From Eq. (15) it is very easy to obtain an approximation for the partial derivatives of the topocentric range rate with respect to the spacecraft coordinates at some epoch r_0, δ_0, α_0, $\dot{r}_0$, $\dot{\delta}_0$, and $\dot{\alpha}_0$, and also with respect to the station location coordinates r_s, z_s and the longitude λ, since as will be shown later $\partial/\partial\lambda = \partial/\partial\varphi$. Before any analysis is performed with partials obtained in this way, it would be desirable to have some idea of the validity of the approximation. To obtain a sample of such information, the procedure outlined in Table 1 was used.

The time behavior of some of the coefficients obtained by fitting $\Delta\dot{\rho}(t)$, generated by a station longitude error of 10^{-6} rad, is shown in Fig. 3 for a particular *Viking* type II trajectory. To easily see how well the coefficients generated in this way agree with the corresponding coefficients predicted from Eq. (15), Fig. 3 actually plots the ratio of these two sets of coefficients, or a quantity which is a function of this ratio.

If the expression given in Eq. (15) were an equality and not an approximation, the coefficients determined from the fit should be independent of the amount of data in the fit and should agree with the coefficients predicted from Eq. (15). An examination of Fig. 3 shows that the C and E coefficients determined by the fit are fairly constant over the 10-day interval and agree reasonably well with the coefficients predicted from Eq. (15). Unfortunately, the B and F coefficients determined by the fit, although initially agreeing fairly well with the coefficient predicted from Eq. (15), show a substantial time variation after a few days. The time variation of this parameter severely degrades any error analysis which involves a station longitude error and the B and F terms of Eq. (15), when more than a few days of data are under consideration.

The abnormal behavior of the first point on three of the curves shown in Fig. 3 probably receives contributions from both the fact that there are only a small number of points after the first day, and that the time is not large enough to give strength to the last three terms.

Computations similar to those used in generating Fig. 3 were made for perturbations in the spacecraft initial coordinates and the remaining station coordinates. Table 2 contains the coefficients obtained from fitting three and ten day's worth of $\Delta\dot{\rho}(t)$ data generated in this way. For comparison purposes the corresponding coefficients predicted by Eq. (15) are also included in Table 2.

An examination of Table 2 shows that for its particular type of trajectory, an error analysis based upon Eq. (15) may be very unreliable in several parameters if more than a few days worth of data is under consideration.

The various sets of $\Delta\dot{\rho}(t)$ used in Table 2 and generated by changes in spacecraft and station coordinates were also fitted by the following polynomial:

$$
\begin{aligned}
& A + B \sin(\varphi - \alpha + \dot{\varphi}\, t) + C \cos(\varphi - \alpha + \dot{\varphi}\, t) \\
& \quad + D\, t + E\, t \sin(\varphi - \alpha + \dot{\varphi}\, t) \\
& \quad + F t \cos(\varphi - \alpha + \dot{\varphi}\, t) \\
& \quad + G\, t^2 + H\, t^2 \sin(\varphi - \alpha + \dot{\varphi}\, t) \\
& \quad + I\, t^2 \cos(\varphi - \alpha + \dot{\varphi}\, t) \\
& \quad + J \sin[2(\varphi - \alpha + \dot{\varphi}\, t)] \\
& \quad + K \cos[2(\varphi - \alpha + \dot{\varphi}\, t)] \\
& \quad + L\, t \sin[2(\varphi - \alpha + \dot{\varphi} t)] \\
& \quad + M\, t \cos[2(\varphi - \alpha + \dot{\varphi}\, t)]
\end{aligned}
$$

where the $2(\varphi - \alpha + \dot{\varphi}\, t)$ terms were suggested by Eq. (2). The first six coefficients obtained from this fit had substantially less time behavior than the coefficients in Table 2 and the larger coefficients in each case agreed to at least 4 figures with the value predicted by Eq. (15), and almost all of the remaining non-zero coefficients agree to a few percent.

VI. Selection of the Dominant Terms in the Range-Rate Approximation

The purpose of finding analytical approximations to the range, range rate, and their associated partial derivatives with respect to the spacecraft and station coordinates is to gain a better understanding of the physical situation. If the range-rate approximation given by Eq. (15) is used, the understanding is clouded by the fact that almost all of the coefficients in Eq. (15) are functions of almost all of the spacecraft and station coordinates. The ease of understanding would be considerably improved if it were possible to isolate which terms in Eq. (15) contribute a negligible amount to an error analysis and may be deleted. To facilitate such a procedure, it is convenient to generate a quantity which compares the change in a particular coefficient of Eq. (15) due to a change in a particular spacecraft or station coordinate with the maximum change in this same coefficient due to a change in any spacecraft or station coordinate. For example, for the trajectory and changes in the nominal spacecraft and station coordinates used in Table 2, the change in e due to a change in r is $\Delta e(\Delta r) = 8 \times 10^{-14}$, and the maximum change in e, $\Delta e_{\max}$, due to the change in any of the spacecraft and station coordinates comes from a change in α, so that the quantity of interest is $\Delta e(\Delta r)/\Delta e_{\max}(\Delta\alpha) = 0.4$. Table 3 gives this ratio for all coefficients and changes in spacecraft or station coordinates for the nominals and changes used in Table 2.

We will make the rather arbitrary decision that if a particular element in the above table is less than 5% as big as the largest element in either its corresponding row or column, the partial derivative associated with that element may be ignored in an error analysis. This of course may change for different trajectories but probably not substantially. For example, since the (3,2) element in Table 3 is much less than the largest element in the $\Delta b'/\Delta b'_{\max}$ column or $\Delta\alpha$ row, the $\partial b/\partial\alpha$ partial may be ignored. Using this criterion allows the range-rate approximation given by Eq. (15) to be considerably simplified to the form given below:

$$
\begin{aligned}
\dot{\rho} \approx\ & a'' + b'' \sin(\varphi_0 - \alpha_0 + \dot{\varphi}_0\, t) \\
& + c'' \cos(\varphi_0 - \alpha_0 + \dot{\varphi}_0\, t) + d''\, t \\
& + e''\, t \sin(\varphi_0 - \alpha_0 + \dot{\varphi}_0\, t) \\
& + f''\, t \cos(\varphi_0 - \alpha_0 + \dot{\varphi}_0\, t) \qquad \text{(16a)}
\end{aligned}
$$

where

$$a'' = \dot{r}_0 - z_s\,\dot{\delta}_0 \cos\delta_0$$

$$b'' = r_s\,\dot{\varphi}_0 \cos\delta_0$$

$$c'' = 0$$

$$d'' = r_0\,(\dot{\delta}_0^2 + \dot{\alpha}_0^2 \cos^2\delta_0) + \ddot{r}_{g0}$$

$$e'' = -\,r_s\,\dot{\varphi}_0\,\dot{\delta}_0 \sin\delta_0$$

$$f'' = -\,r_s\,\dot{\varphi}_0\,\dot{\alpha}_0 \cos\delta_0$$

$$\ddot{r}_{g0} = -\,\mu\left[\frac{r_0}{r_{p0}^3} - r_e\left(\frac{1}{r_{p0}^3} - \frac{1}{r_{e0}^3}\right) < \cos\delta_0 \cos\delta_{s0} \times \cos(\alpha_0 - \alpha_{s0}) + \sin\delta_0 \sin\delta_{s0} > \right] \tag{16b}$$

and the second term in a'' is needed only in computing $\partial\dot{\rho}/\partial z_s$.

VII. Modification of the Trigonometric Arguments

The physical understanding which results from the use of Eq. (16) may be further increased by judiciously choosing the form of trigonometric arguments used in this equation. Recall from Eq. (3) that these quantities had their origin in the term $\varphi - \alpha$, where φ is the angle between the mean vernal equinox of date and the meridian of the tracking station as shown in Fig. 4.

The universal time t_u is related to the quantities in the above figure through the following equation:

$$\omega t_u = H + \omega \cdot 12\ \text{hrs} \tag{17}$$

where the rotation rate of the earth ω is

$$\omega = \frac{2\pi}{86400} = 0.72722 \times 10^{-4}\ \text{rad/sec} \tag{18}$$

and when the mean sun crosses the Greenwich meridian it is 12:00 universal time. Clearly the right ascension of the Greenwich meridian as a function of universal time may be written as

$$\theta = \omega t_u + (\alpha_\odot - 180^\circ) \tag{19}$$

The right ascension of the mean sun is given by (Ref. 3)

$$\alpha_\odot = 280^\circ 0755426 + 0^\circ\!.98564734\,d + 2^\circ 9015 \times 10^{-13}\,d^2 \tag{20}$$

where

$$d = \text{days past Jan. 1, 1950, 0 hrs}$$

The second term in Eq. (20) accounts for the annual motion of the earth about the sun; the third term is extremely small and may be neglected. Substituting Eq. (20) into Eq. (19) yields

$$\theta = (100^\circ\!.0755426 + 0^\circ\!.98564734\,d) + \omega t_u$$

From this equation it is easily seen that if the time is measured in units of universal time, the right ascension of Greenwich may be written as

$$\theta = \theta_0 + \dot{\theta}_0\,t$$

where

$$\theta_0 = \text{right ascension of Greenwich at some epoch } t = 0$$

$$\dot{\theta}_0 = (0.72722 + 0.00198)\ \text{rad/sec}$$

$$= 0.7\ \text{rad/sec} \tag{21}$$

Clearly, since

$$\varphi = \theta + \lambda$$

the argument of the trigonometric function in Eq. (16) may be written as

$$\varphi_0 - \alpha_0 + \dot{\varphi}_0\,t = \theta_0 + \lambda - \alpha_0 + \dot{\theta}_0\,t \tag{22}$$

Hamilton and Melbourne (Ref. 1) have shown that at this point in the development it is convenient to specify *a priori* information about α and λ, and to make the following definitions:

$$\left.\begin{aligned} \epsilon_\lambda &= \lambda - \lambda^* = (\text{true—}\textit{a priori})\ \text{station longitude} \\ \epsilon_a &= \alpha_0 - \alpha_0^* = (\text{true—}\textit{a priori})\ \text{spacecraft right ascension} \\ \epsilon &= \epsilon_\lambda - \epsilon_a \end{aligned}\right\} \tag{23}$$

Substituting these equations into Eq. (22) yields

$$\varphi_0 - \alpha_0 + \dot{\varphi}_0\,t = \theta_0 + \lambda^* - \alpha_0^* + \epsilon_\lambda - \epsilon_a + \dot{\theta}_0\,t \tag{24}$$

This equation may be considerably simplified if the time t is specified to be measured from a point when the spacecraft crosses the staiton's nominal meridian. Using this particular epoch the first three terms of Eq. (24) cancel, allowing Eq. (16) to be written as

$$\dot{\rho} \approx a'' + b'' \sin(\dot{\theta}_0 t + \epsilon) + c'' \cos(\dot{\theta}_0 t + \epsilon) + d'' t + e'' t \sin(\dot{\theta}_0 t + \epsilon) + f'' t \cos(\dot{\theta}_0 t + \epsilon)$$

where for this, and all future equations, $t = 0$ is understood to occur at a nominal meridian crossing.

Since ϵ is small enough so that ϵ^2 is negligible, the above equation may be rewritten as

$$\dot{\rho} \approx a'' + (b'' - \epsilon c'') \sin \dot{\theta} t + (c'' + \epsilon b'') \cos \dot{\theta} t + d'' t + (e'' - \epsilon f'') t \sin \dot{\theta} t + (f'' + \epsilon e'') t \cos \dot{\theta} t$$

or using Eq. (15b)

$$\dot{\rho} \approx a + b \sin \dot{\theta} t + c \cos \dot{\theta} t + d \dot{\theta} t + e \dot{\theta} t \sin \dot{\theta} t + f \dot{\theta} t \cos \dot{\theta} t \tag{25a}$$

where

$$a = \dot{r}_0 - z_s \dot{\delta}_0 \cos \delta_0$$

$$b = r_s \dot{\theta} \cos \delta_0$$

$$c = r_s \dot{\theta} \cos \delta_0 [(\lambda - \lambda^*) - (\alpha_0 - \alpha_0^*)]$$

$$d = [r_0 (\dot{\delta}_0^2 + \dot{\alpha}_0^2 \cos^2 \delta_0) + \ddot{r}_{y0}] \frac{1}{\dot{\theta}}$$

$$e = - r_s \{\dot{\delta}_0 \sin \delta_0 - [(\lambda - \lambda^*) - (\alpha_0 - \alpha_0^*)] \dot{\alpha}_0 \cos \delta_0\}$$

$$f = - r_s \{\dot{\alpha}_0 \cos \delta_0 + [(\lambda - \lambda^*) - (\alpha_0 - \alpha_0^*)] \dot{\delta}_0 \sin \delta_0\} \tag{25b}$$

where the z_s term of the first coefficient is nonnegligible only for the calculation of $\partial \dot{\rho} / \rho z_s$.

VIII. Error Analysis Using the Range-Rate Approximation

As pointed out by Curkendall and McReynolds (Ref. 2), any error analysis based upon Eq. (25) proceeds by treating the coefficients a, b, c, d, e, and f as data points which describe the range-rate observable. However, these "data" points are not independent, and in fact may be highly correlated. The correlations and appropriate weights associated with the coefficients may be found by first taking the partial of $\dot{\rho}$ with respect to $a \rightarrow f$ and forming the information matrix J_a in the usual manner. If data is taken often enough so that summations may be represented by integrals, J_a may be written as

$$J_a = \frac{N}{\sigma_{\dot{\rho}}^2} \frac{1}{\varphi} \begin{bmatrix} \int_p d\varphi & \int_p \sin\varphi \, d\varphi & \int_p \cos\varphi \, d\varphi & \int_p \varphi \, d\varphi & \int_p \varphi \sin \varphi \, d\varphi & \int_p \varphi \cos \varphi \, d\varphi \\ & \int_p \sin^2 \varphi \, d\varphi & \int_p \sin\varphi \cos\varphi \, d\varphi & \int_p \varphi \sin\varphi \, d\varphi & \int_p \varphi \sin^2 \varphi \, d\varphi & \int_p \varphi \sin\varphi \cos\varphi \, d\varphi \\ & & \int_p \cos^2 \varphi \, d\varphi & \int_p \varphi \cos\varphi \, d\varphi & \int_p \varphi \sin\varphi \cos\varphi \, d\varphi & \int_p \varphi \cos^2 \varphi \, d\varphi \\ & \text{sym.} & & \int_p \varphi^2 \, d\varphi & \int_p \varphi^2 \sin \varphi \, d\varphi & \int_p \varphi^2 \cos \varphi \, d\varphi \\ & & & & \int_p \varphi^2 \sin^2 \varphi \, d\varphi & \int_p \varphi^2 \sin\varphi \cos\varphi \, d\varphi \\ & & & & & \int_p \varphi^2 \cos^2 \varphi \, d\varphi \end{bmatrix} \tag{26}$$

where

$\varphi = \dot{\theta}_0 t$

$\sigma_{\dot{\rho}}$ = variance of the white noise associated with the range-rate measurements

N = number of range-rate data points

$\int_p$ indicates that the integral extends over the full tracking interval, but has a non-zero contribution only when data is being taken

In using the six coefficients $a \rightarrow f$ as data points, the orbit determination solution filter accepts changes in these coefficients which have been generated in some manner, and modifies six parameters from among the nine spacecraft and station coordinates so that the range-rate observable is changed as little as possible. For example, if the effect of the ionosphere on the range-rate observable could be represented by an error in the c parameter, a solution for the spacecraft state would make a compensating error in the right ascension of the spacecraft, so that the value of $\dot{\rho}(t)$ is best preserved. Using the classical least-squares technique this solution procedure may be represented by the following equation:

$$\Delta \mathbf{x} = [A^T W A]^{-1} A^T W \, \Delta \mathbf{a} \tag{27}$$

where

$\Delta \mathbf{x}$ = the solution vector for up to six parameters selected from among the spacecraft and station coordinates

$\Delta \mathbf{a}$ = a vector representing changes or residuals in the coefficients $a \rightarrow f$ which have been generated in some manner

$W = J_a$ = weighting matrix for the coefficients $a \rightarrow f$ which are being treated as data

$$A = \frac{\partial \mathbf{a}}{\partial \mathbf{x}}$$

The matrix of partials, A, is obtained by selecting columns from the matrix, A^*, whose elements are obtained by differentiating Eq. (25b) and which is given below:

$$A^* = \frac{\partial\,(a, b, c, d, e, f)}{\partial\,(r, \delta, \alpha, \dot{r}, \dot{\delta}, \dot{\alpha}, r_s, \lambda, z_s)}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\dot{\delta}_0 \sin \delta_0 \\ 0 & -r_s \dot{\theta} \sin \delta_0 & 0 & 0 & 0 & 0 & \dot{\theta} \cos \delta_0 & 0 & 0 \\ 0 & 0 & -r_s \dot{\theta} \cos \delta_0 & 0 & 0 & 0 & 0 & r_s \dot{\theta} \cos \delta_0 & 0 \\ \xi/\dot{\theta} & \eta/\dot{\theta} & \zeta/\dot{\theta} & 0 & 2r_0 \dot{\delta}_0/\dot{\theta} & 2r_0 \dot{\alpha}_0 \cos^2 \delta_0/\dot{\theta} & 0 & 0 & 0 \\ 0 & -r_s \dot{\delta}_0 \cos \delta_0 & -r_s \dot{\alpha}_0 \cos \delta_0 & 0 & -r_s \sin \delta_0 & 0 & -\dot{\delta}_0 \sin \delta_0 & r_s \dot{\alpha}_0 \cos \delta_0 & 0 \\ 0 & r_s \dot{\alpha}_0 \sin \delta_0 & r_s \dot{\delta}_0 \sin \delta_0 & 0 & 0 & -r_s \cos \delta_0 & -\dot{\alpha}_0 \cos \delta_0 & -r_s \dot{\delta}_0 \sin \delta_0 & 0 \end{bmatrix} \tag{28}$$

where

$$\xi = \partial \ddot{r}_{g0}/\partial r \,\big|_0 + (\dot{\delta}_0^2 + \dot{\alpha}_0^2 \cos^2 \delta_0)$$

$$\eta = \partial \ddot{r}_{g0}/\partial \delta \,\big|_0 - 2r_0 \dot{\alpha}_0^2 \cos \delta_0 \sin \delta_0$$

$$\zeta = \partial \ddot{r}_{g0}/\partial \alpha \,\big|_0$$

The partials of the gravitation acceleration with respect to the position of the spacecraft may be obtained by using the first and last of Eq. (14) and are

$$\frac{\partial \ddot{r}_g}{\partial r}\bigg|_0 = \frac{\mu}{r_{p0}^3}(2 - 3 \sin^2 \psi_0)$$

$$\frac{\partial \ddot{r}_g}{\partial \delta}\bigg|_0 = \mu \, r_e \cos \sigma \left[\left(\frac{1}{r_p^3} - \frac{1}{r_e^3}\right) - 3 \frac{r_0}{r_{p0}^5}(r_0 - r_e \cos \chi_0)\right]$$

$$\frac{\partial \ddot{r}_g}{\partial \alpha}\bigg|_0 = \mu \, r_e \cos \delta_0 \cos \delta_s \sin (\alpha - \alpha_s) \left[-\left(\frac{1}{r_{p0}^3} - \frac{1}{r_{e0}^3}\right) + 3 \frac{r_0}{r_{p0}^5}(r_0 - r_e \cos \chi_0)\right] \tag{29}$$

where

$$\cos \sigma = -\sin \delta \cos \delta_s \cos (\alpha - \alpha_s) + \cos \delta \sin \delta_s$$

$$\cos \chi = \cos \delta \cos \delta_s \cos (\alpha - \alpha_s) + \sin \delta \sin \delta_s$$

$$\sin \psi = r_e/r_p \sin \chi$$

χ = sun–earth–spacecraft angle

ψ = earth–spacecraft–sun angle

The curves of constant $\partial \ddot{r}_g/\partial r$ have been shown before by Curkendall and McReynolds (Ref. 2), but will be included here as Fig. 5 for completeness. The curves of constant $\partial \ddot{r}_g/\partial \delta$ and $\partial \ddot{r}_g/\partial \alpha$ are not as easily obtained because of their dependence on δ, δ_s, α, and α_s. However, some idea of the behavior of the constant $\partial \ddot{r}_g/\partial \delta$ and $\partial \ddot{r}_g/\partial \alpha$ curves may be obtained from Figs. 6 and 7 where the spacecraft declination has been specified to be zero. To obtain some feeling of how the gravitational field may influence the orbit determination solution for various missions, trajectories representative of *Mariner* Venus–Mercury 1973 and *Viking* 1975 have been included in Figs. 5, 6, and 7.

IX. Error Analysis for the Spacecraft State

Almost all orbit determination error analysis has as its goal an investigation of errors in the spacecraft state. Using the approximation techniques developed here and the classical least-squares method, the error analysis of the full spacecraft state, resulting from range-rate-only information, would proceed from the following equation:

$$\begin{bmatrix} \Delta r \\ \Delta\delta \\ \Delta\alpha \\ \Delta\dot{r} \\ \Delta\dot{\delta} \\ \Delta\dot{\alpha} \end{bmatrix} = \{A_s^T J_a A_s\}^{-1} A_s^T J_a \begin{bmatrix} \Delta a \\ \Delta b \\ \Delta c \\ \Delta d \\ \Delta e \\ \Delta f \end{bmatrix} \qquad (30)$$

where A_s is composed of the first six columns of the A^* matrix in Eq. (28). The state covariance is easily found by taking the inverse of the terms in braces in the above equation. As before $\Delta a \rightarrow \Delta f$ represented changes in the coefficients $a \rightarrow f$ produced by some physical phenomena, which degrades the range-rate observable. For example, if one wanted to investigate the errors in the spacecraft state which result from station location errors, the $\Delta a \rightarrow \Delta f$ which reflect this situation would be obtained by multiplying the partials of $a \rightarrow f$ with respect to r_s, λ, z_s (i.e., the last three columns of A^* in Eq. 28) by Δr_s, $\Delta\lambda$, and Δz_s, respectively.

Since A_s is a 6×6 matrix when the full spacecraft state is being included in the solution, the determination of $\Delta r \rightarrow \Delta\alpha$ is unique and Eq. (30) may be written as

$$\begin{bmatrix} \Delta r \\ \cdot \\ \cdot \\ \cdot \\ \Delta\dot{\alpha} \end{bmatrix} = A_s^{-1} \begin{bmatrix} \Delta a \\ \cdot \\ \cdot \\ \cdot \\ \Delta f \end{bmatrix}$$

Because of the many zero elements contained in A_s, it may be conveniently inverted. Hence, the changes in the full spacecraft state which result from changes in the six parameters describing the range-rate observable may easily be obtained from the following equation:

$$\begin{bmatrix} \Delta r \\ \Delta\delta \\ \Delta\alpha \\ \Delta\dot{r} \\ \Delta\dot{\delta} \\ \Delta\dot{\alpha} \end{bmatrix} = \frac{1}{\dot{\theta} r_s} \begin{bmatrix} 0 & \dfrac{2r\dot{\alpha}^2 \sin^2\delta\cos\delta + \dfrac{\partial d}{\partial\delta}\sin\delta - 2r\dot{\delta}^2\cos\delta}{\partial d/\partial r \sin^2\delta} & \dfrac{\dfrac{\partial d}{\partial\alpha}\sin\delta - 2r\dot{\alpha}\dot{\delta}\cos^3\delta}{\partial d/\partial r\cos\delta\sin\delta} & \dfrac{\dot{\theta}^2 r_s}{\partial d/\partial r} & \dfrac{2r\dot{\delta}\dot{\theta}}{\partial d/\partial r\sin\delta} & \dfrac{2r\dot{\theta}\cos\delta\dot{\alpha}}{\partial d/\partial r} \\ 0 & -1/\sin\delta & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\cos\delta & 0 & 0 & 0 \\ \dot{\theta} r_s & 0 & 0 & 0 & 0 & 0 \\ 0 & \dot{\delta}/\tan\delta\sin\delta & \dfrac{\dot{\alpha}}{\sin\delta} & 0 & -\dot{\theta}/\sin\delta & 0 \\ 0 & \dot{\alpha}/\cos\delta & -\dot{\delta}\tan\delta/\cos\delta & 0 & 0 & -\dot{\theta}/\cos\delta \end{bmatrix} \begin{bmatrix} \Delta a \\ \Delta b \\ \Delta c \\ \Delta d \\ \Delta e \\ \Delta f \end{bmatrix} \qquad (31)$$

When a range point is available in addition to the range-rate data, a cursory error analysis for the δ, α, $\dot{r}$, $\dot{\delta}$, $\dot{\alpha}$ portion of the spacecraft state may be performed by assuming the geocentric range to be known and deleting it from the solution. This technique is expressed in the following equation:

$$\begin{bmatrix} \Delta\delta \\ \Delta\alpha \\ \Delta\dot{r} \\ \Delta\dot{\delta} \\ \Delta\dot{\alpha} \end{bmatrix} = \{A_r^T J_a A_r\}^{-1} A_r J_a \begin{bmatrix} \Delta a \\ \Delta b \\ \Delta c \\ \Delta d \\ \Delta e \\ \Delta f \end{bmatrix} \qquad (32)$$

where A_r is a 6×5 matrix composed of the second through sixth columns of A^* contained in Eq. (28). This system is over-determined and the solution must be obtained by using least-squares techniques.

The classical $A^T W A$ form of the least-squares problem has been used in the last two sections because it was felt that it was probably very familiar to most potential readers. However, the inversion of $A^T W A$, generated by using the approximation discussed here may have numerical difficulties which would require recasting the problem in its square-root form.

X. Summary and Discussion

The preceding sections have been concerned with arriving at a first-order expansion of the topocentric range rate in terms of r_s/r, z_s/r, and time, which may be put in a form which is convenient to use for error analysis. Although at times the development was somewhat laborious and involved, the resulting error analysis procedure is quite easy to use. For example, over the range of validity of the approximations, a great deal of the state only–range rate only error analysis can be performed analytically. Although this technique may be used to obtain quantitative estimates regarding the inherent accuracy of particular orbit determination problems, it should always be borne in mind that the primary reason for undertaking the development was to provide a vehicle which can promote a better physical understanding of the orbit determination process.

The limiting feature of this approximation technique is the relatively short time periods over which it is reliably applicable. For example, for the *Viking* trajectory used here as an example, the method is severely degraded in many of the parameters after only a few days. This feature is particularly irritating when one wants to determine the information content contained within long arcs of data. It may be possible to develop techniques similar to the one achieved here, but applicable to long arc solutions, by using the closed form f and g expansions of celestial mechanics.

References

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2. Curkendall, D. W., and McReynolds, S. R., "A Simplified Approach for Determining the Information Content of Radio Tracking Data," *J. Spacecraft Rockets*, Vol. 6, No. 5, pp. 520–525, May 1969.

3. Holdridge, D. B., *Space Trajectories Program for the IBM 7090 Computer*, Technical Report 32-223, p. 56. Jet Propulsion Laboratory, Pasadena, Calif., Mar. 2, 1962.

Table 1. Procedure for verifying Eq. (15)

Step	Procedure
1	From a nominal trajectory, with only the mass of the sun $\neq 0$, obtain the topocentric range rate from one station every hour for 10 days.
2	Same as step 1, except perturb either the initial conditions of the spacecraft or the station location coordinates by some amount.
3	Difference the range rate values obtained in steps 1 and 2 to obtain $\Delta\dot{\rho}(t)$.
4	Starting with one day's data and then increasing the data arc a day at a time, make a least-squares fit of $\Delta\dot{\rho}(t)$ to the following polynomial: $A + B \sin(\varphi - \alpha + \dot{\varphi} t) + C \cos(\varphi - \alpha + \dot{\varphi} t) + D t + E t \sin(\varphi - \alpha + \dot{\varphi} t) + F t \cos(\varphi - \alpha + \dot{\varphi} t)$

Table 2. Changes in $a \to f$ resulting from changes in spacecraft and station coordinates

Coordinate change	$\Delta r = 100$ km			$\Delta \delta = 10^{-6}$ rad			$\Delta \alpha = 10^{-6}$ rad		
Coefficient source	Eq. (15)	3-day fit	10-day fit	Eq. (15)	3-day fit	10-day fit	Eq. (15)	3-day fit	10-day fit
$\Delta a'$ or A	—	-0.65×10^{-9}	-0.96×10^{-9}	0.352×10^{-10}	0.303×10^{-10}	0.238×10^{-10}	—	-0.20×10^{-8}	-0.24×10^{-7}
$\Delta b'$ or B	—	0.17×10^{-9}	0.91×10^{-9}	-0.1234×10^{-6}	-0.1235×10^{-6}	-0.1239×10^{-6}	0.417×10^{-10}	0.646×10^{-9}	0.244×10^{-8}
$\Delta c'$ or C	—	0.30×10^{-9}	0.47×10^{-9}	0.112×10^{-9}	0.381×10^{-11}	0.186×10^{-9}	-0.3320×10^{-6}	-0.3311×10^{-6}	-0.3314×10^{-6}
$\Delta d'$ or D	0.2042×10^{-11}	0.2058×10^{-11}	0.2108×10^{-11}	0.6512×10^{-12}	0.6464×10^{-12}	0.6338×10^{-12}	0.4892×10^{-11}	0.4942×10^{-11}	0.5068×10^{-11}
$\Delta e'$ or E	-0.244×10^{-15}	-0.724×10^{-15}	-0.126×10^{-14}	-0.821×10^{-14}	-0.739×10^{-14}	-0.5873×10^{-14}	-0.3176×10^{-13}	-0.3176×10^{-13}	-0.3279×10^{-13}
$\Delta f'$ or F	0.991×10^{-17}	-0.290×10^{-14}	-0.116×10^{-14}	0.109×10^{-13}	0.114×10^{-14}	0.9332×10^{-14}	0.270×10^{-14}	-0.417×10^{-14}	0.536×10^{-14}

Coordinate change	$\Delta \dot{r} = 10^{-6}$ km/s			$\Delta \dot{\delta} = 10^{-11}$ rad/s			$\Delta \dot{\alpha} = 10^{-11}$ rad/s		
Coefficient source	Eq. (15)	3-day fit	10-day fit	Eq. (15)	3-day fit	10-day fit	Eq. (15)	3-day fit	10-day fit
$\Delta a'$ or A	1.00000×10^{-6}	1.000049×10^{-6}	1.00075×10^{-6}	-0.3857×10^{-10}	-0.501×10^{-10}	-0.18×10^{-9}	—	-0.17×10^{-9}	-0.19×10^{-8}
$\Delta b'$ or B	—	-0.14×10^{-10}	-0.72×10^{-10}	—	0.24×10^{-11}	0.45×10^{-10}	-0.455×10^{-10}	0.75×10^{-11}	0.17×10^{-11}
$\Delta c'$ or C	—	-0.24×10^{-10}	-0.48×10^{-10}	0.169×10^{-10}	0.22×10^{-10}	0.213×10^{-10}	—	0.76×10^{-10}	0.11×10^{-12}
$\Delta d'$ or D	0.214×10^{-17}	0.119×10^{-14}	0.515×10^{-14}	0.43557×10^{-13}	0.4384×10^{-13}	0.4462×10^{-13}	0.13859×10^{-12}	0.142×10^{-12}	0.1532×10^{-12}
$\Delta e'$ or E	0.917×10^{-17}	0.58×10^{-16}	0.10×10^{-16}	-0.12354×10^{-15}	-0.1218×10^{-11}	-0.1148×10^{-15}	0.1578×10^{-16}	-0.213×10^{-15}	-0.405×10^{-15}
$\Delta f'$ or F	-0.941×10^{-18}	0.244×10^{-15}	0.18×10^{-16}	-0.3624×10^{-17}	-0.48×10^{-16}	0.529×10^{-17}	-0.3316×10^{-14}	-0.384×10^{-14}	-0.281×10^{-14}

Coordinate change	$\Delta r_s = 10$ m			$\Delta \lambda = 10^{-6}$ rad			$\Delta z_s = 10$ m		
Coefficient source	Eq. (15)	3-day fit	10-day fit	Eq. (15)	3-day fit	10-day fit	Eq. (15)	3-day fit	10-day fit
$\Delta a'$ or A	—	-0.16×10^{-11}	-0.88×10^{-11}	—	0.13×10^{-10}	-0.44×10^{-11}	-0.23048×10^{-9}	-0.23058×10^{-9}	-0.2308×10^{-9}
$\Delta b'$ or B	0.6827×10^{-6}	0.6827×10^{-6}	0.6829×10^{-6}	-0.4166×10^{-10}	-0.554×10^{-10}	-0.193×10^{-10}	—	0.13×10^{-10}	0.13×10^{-10}
$\Delta c'$ or C	0.856×10^{-10}	0.107×10^{-9}	0.359×10^{-10}	0.33196×10^{-7}	0.33200×10^{-7}	0.33210×10^{-7}	—	0.32×10^{-13}	0.23×10^{-13}
$\Delta d'$ or D	—	-0.49×10^{-16}	-0.90×10^{-17}	—	0.14×10^{-18}	0.16×10^{-18}	0.31731×10^{-16}	0.323×10^{-16}	0.339×10^{-16}
$\Delta e'$ or E	-0.558×10^{-14}	-0.683×10^{-14}	-0.847×10^{-14}	0.2957×10^{-13}	0.2989×10^{-13}	-0.3068×10^{-13}	—	-0.15×10^{-17}	-0.15×10^{-17}
$\Delta f'$ or F	-0.608×10^{-13}	-0.613×10^{-13}	-0.630×10^{-13}	-0.3078×10^{-14}	-0.3342×10^{-14}	-0.4121×10^{-14}	—	-0.12×10^{-17}	-0.10×10^{-17}

Table 3. The quantity used in selecting the dominate terms for error analysis

Coordinate	$\Delta a'/\Delta a'_{max}$	$\Delta b'/\Delta b'_{max}$	$\Delta c'/\Delta c'_{max}$	$\Delta d'/\Delta d'_{max}$	$\Delta e'/\Delta e'_{max}$	$\Delta f'/\Delta f'_{max}$
Δr	—	—	—	0.4	1×10^{-2}	2×10^{-3}
$\Delta \delta$	0.4×10^{-4}	0.4	0.3×10^{-1}	1	0.4	0.2
$\Delta \alpha$	—	0.6×10^{-4}	1	1	1	0.5×10^{-1}
$\Delta \dot{r}$	1	—	—	0.4×10^{-6}	0.3×10^{-3}	0.2×10^{-4}
$\Delta \dot{\delta}$	0.4×10^{-6}	—	0.7×10^{-4}	0.8×10^{-2}	0.3×10^{-1}	0.7×10^{-4}
$\Delta \dot{\alpha}$	—	0.6×10^{-4}	—	2×10^{-2}	0.3×10^{-3}	0.5×10^{-1}
Δr_s	—	1	0.3×10^{-3}	—	0.2×10^{-2}	1
$\Delta \lambda$	—	0.6×10^{-4}	1	—	1	0.5×10^{-1}
Δz_s	0.2×10^{-3}	—	—	0.6×10^{-5}	—	—

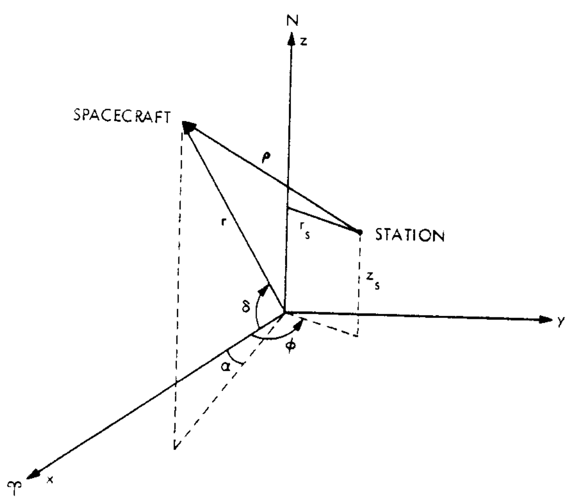

Fig. 1. Topocentric range in terms of spacecraft and station coordinates

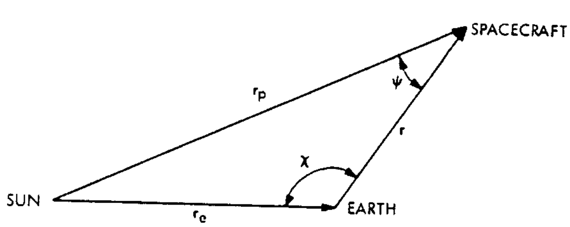

Fig. 2. Relative positions of the sun, earth, and spacecraft

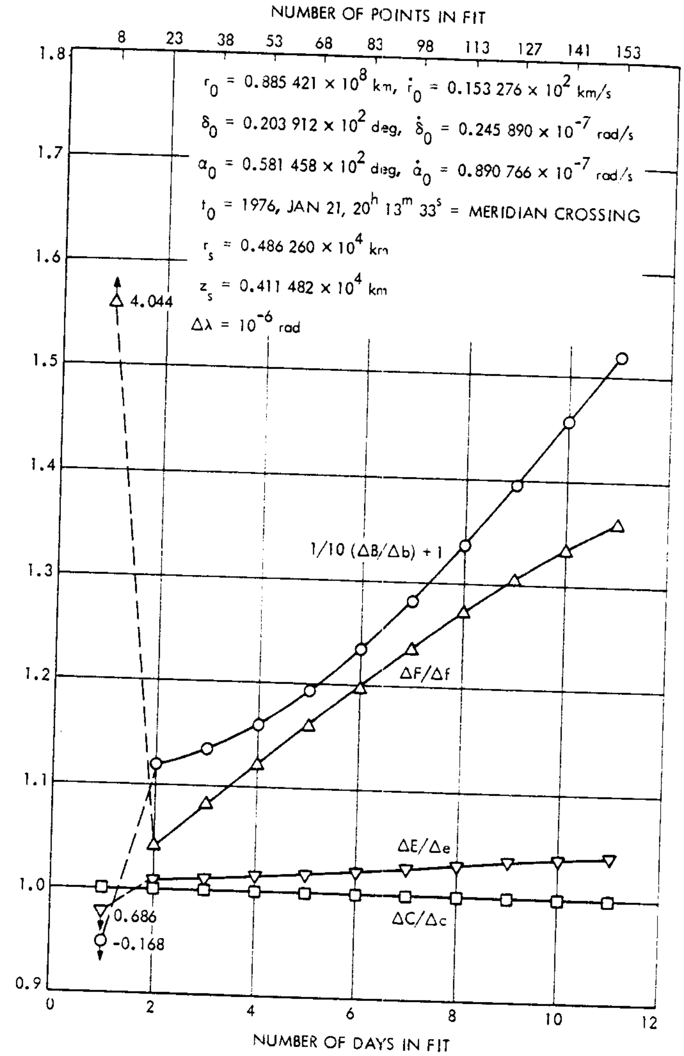

Fig. 3. Time behavior of six parameter fits for $\Delta\lambda = 10^{-6}$ rad

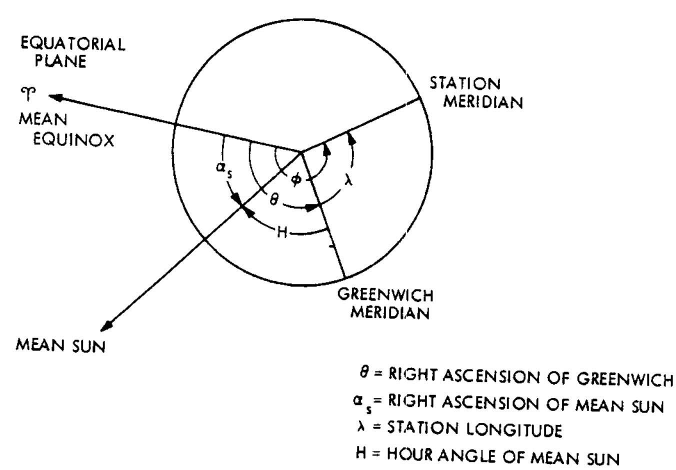

Fig. 4. Angles pertinent in determining behavior of φ

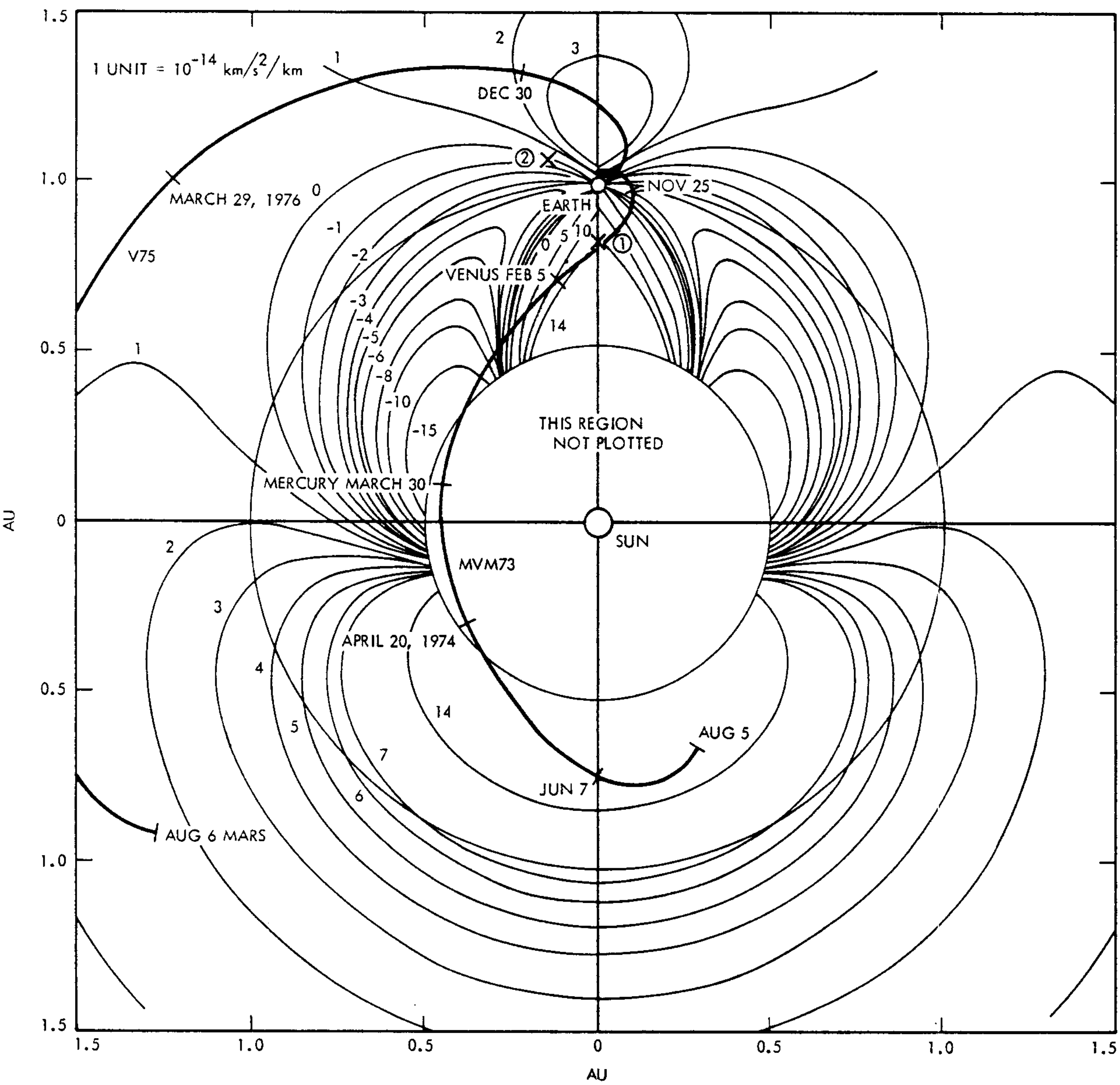

Fig. 5. Partial derivative of the gravitational acceleration of *r* with respect to *r*

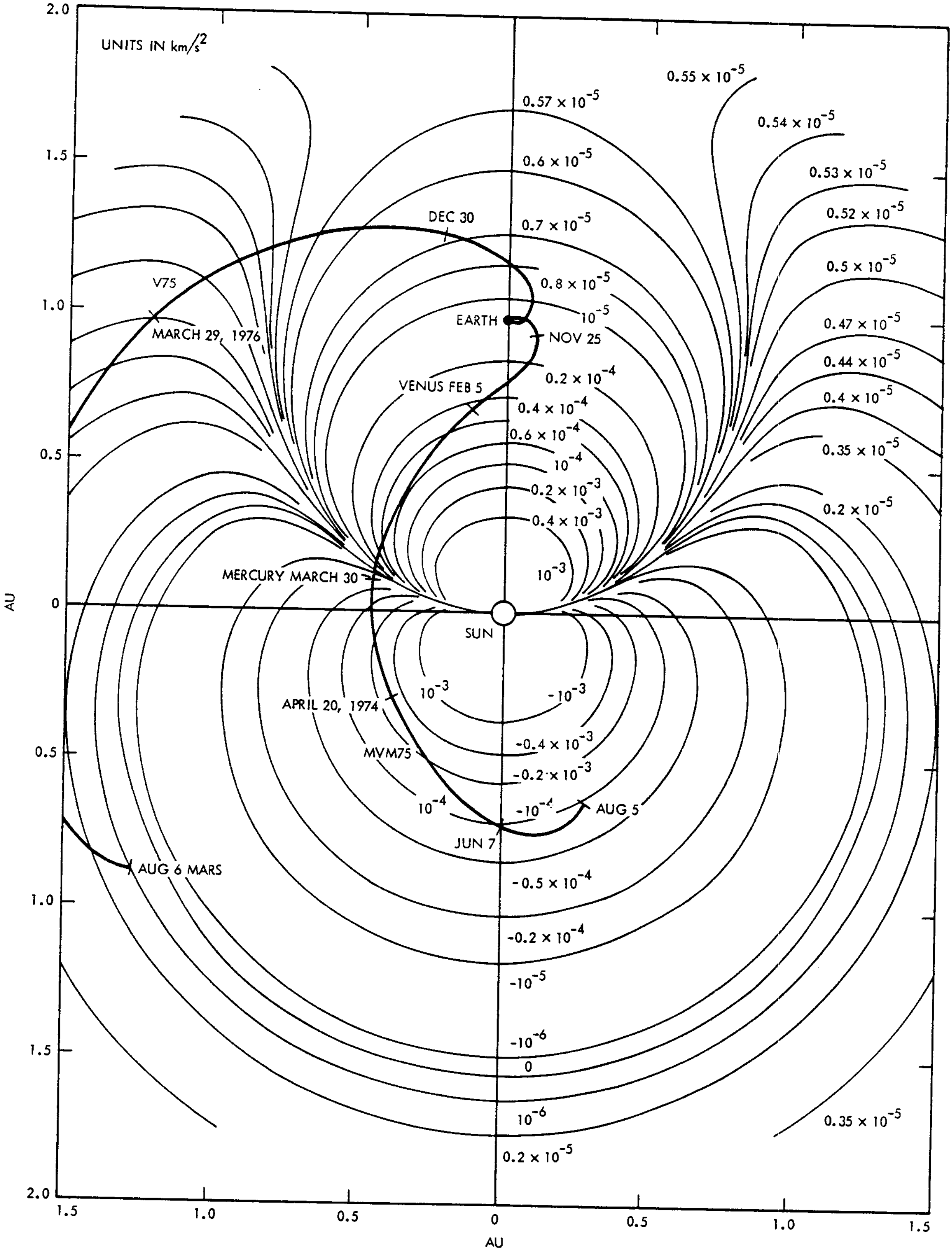

Fig. 6. Partial derivative of the gravitational acceleration of r with respect to δ divided by $\sin \delta_{*}$, when $\delta = 0$

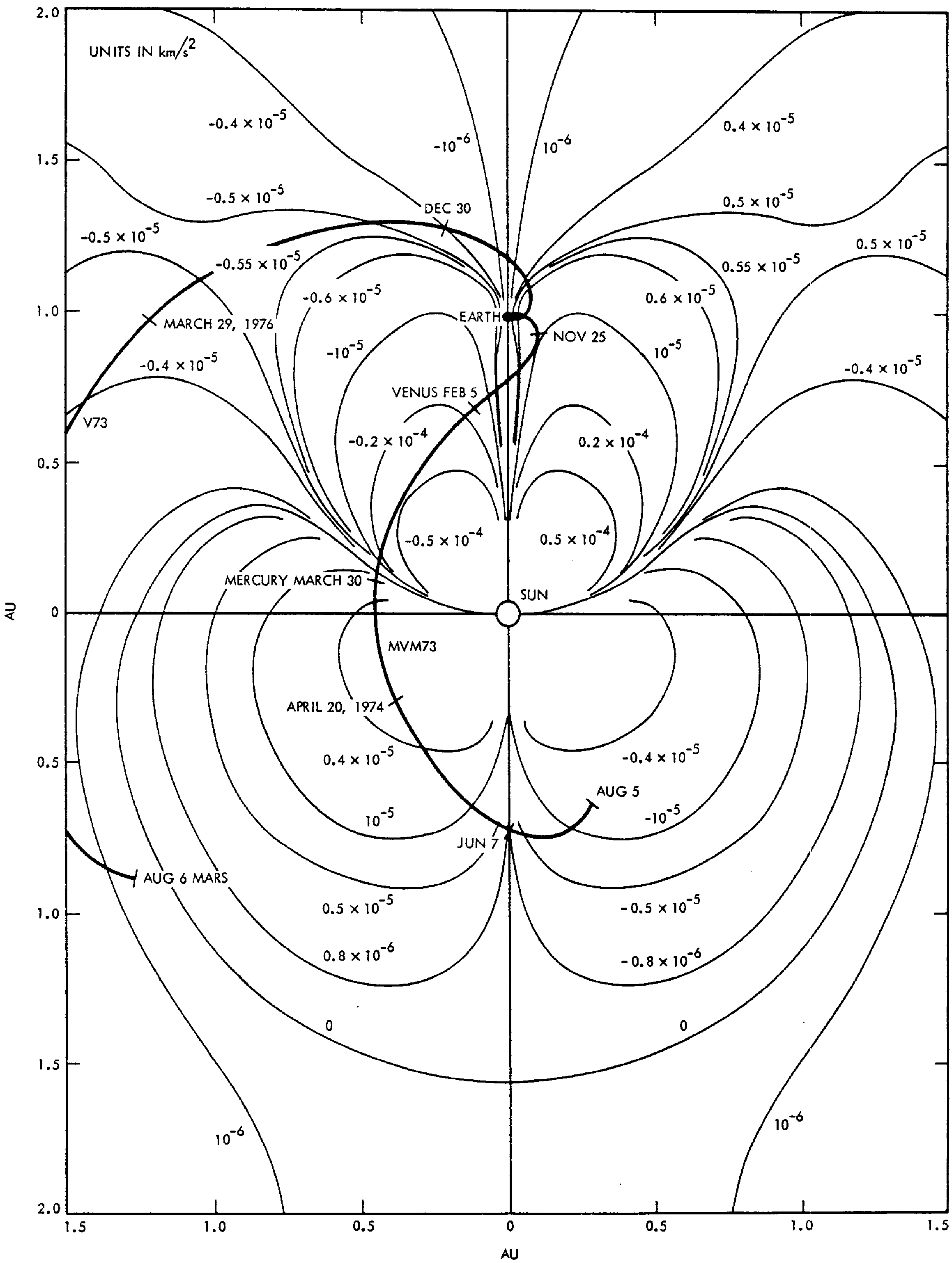

Fig. 7. Partial derivative of the gravitational acceleration of r with respect to α divided by cos δ_{x}, when $\delta = 0$