Numerical Evaluation of the Transient Response for a Third-Order Phase-Locked System

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A third-order phase-locked receiver is presently being investigated for possible use in tracking high doppler rates. This report presents additional data pertaining to the transient analysis of a model of a third-order phase-locked receiver.

Specifically, the instantaneous response of the system is calculated for an input phase function of the form

$$\theta(t) = \theta_0 + \Omega_0 t + \frac{1}{2}\Lambda_0 t^2$$

The results presented may be compared with those of the usual second-order loop. It is hoped that this report will contribute some insight into the nature of the operation of third-order loops at least in the in-lock region.

I. Introduction

A third-order phase-locked receiver is presently being investigated for possible use in tracking high doppler rates. This report presents additional data pertaining to the transient analysis of a model of a third-order phase locked receiver presented in Ref. 1.

Specifically, the instantaneous response of the system is calculated for an input phase function of the form

$$\theta(t) = \theta_0 + \Omega_0 t + \frac{1}{2}\Lambda_0 t^2$$

where

$$\theta_0 = \text{initial phase offset}$$
$$\Omega_0 = \text{initial frequency offset}$$
$$\Lambda_0 = \text{frequency rate, Hz/sec}$$

The results presented may be compared with those of the usual second-order loop. It is hoped that this report will contribute some insight into the nature of the operation of third-order loops at least in the in-lock region.

II. Mathematical Model

A. Linear Transfer Function

From Ref. 1, a "realizable" open-loop filter transfer function for third-order phase-locked systems is of the form

$$F(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s} + \frac{1}{(1 + \tau_1 s)(\delta + \tau_3 s)} \tag{1}$$

The definition of the parameters τ_1, τ_2, and τ_3 are given in Ref. 1. The resulting closed-loop transfer function $L(s)$

$$L(s) = \frac{rk(1+\delta) + r(1+\delta k)\tau_2 s + r(\tau_2 s)^2}{rk(1+\delta) + (r + r\delta k + \epsilon\delta k)\tau_2 s + (r + \epsilon + \delta k)(\tau_2 s)^2 + (\tau_2 s)^3} \tag{2}$$

where

$r = AK\tau_2^2/\tau_1$, AK is loop gain

$k = \tau_2/\tau_3$

$\epsilon = \tau_2/\tau_1$

In terms of the closed-loop transfer function $L(s)$ there is the following relation between the input phase $\theta(t)$ and the phase error $\phi(t)$

$$\phi(s) = [1 - L(s)]\,\theta(s) \tag{3}$$

where

$$\theta(s) = \frac{\theta_0}{s} + \frac{\Omega_0}{s^2} + \frac{\Lambda_0}{s^3}$$

In general there will be non-zero initial conditions which can be expressed in the form: (Ref. 1)

$$U(s) = -\frac{K'}{s}\left[\frac{U_1}{1 + \tau_1 s} + \frac{U_2}{(1 + \tau_1 s)(\delta + \tau_3 s)} + \frac{U_3}{\delta + \tau_3 s}\right] \tag{4}$$

Here, K' is the gain from the output of the open loop filter $F(s)$, and the values of U_1, U_2, U_3 depend on initial capacitor voltages.

Thus the total phase error satisfies the relation:

$$\phi(s) = [1 - L(s)]\,(\theta(s) + U(s)) \tag{5}$$

B. Calculation of Loop Parameters

The loop parameters k and r must be calculated in terms of the parameters δ, and ϵ.

$$k = \left(\frac{2+\delta}{\delta^2}\right)\left\{1 - \left[1 - \frac{\delta^2}{(2+\delta)^2}\right]^{1/2}\right\} \approx \frac{1}{2(2+\delta)} \tag{6}$$

$$r = \frac{1}{(1+\delta)k}\left(\frac{V}{3}\right)^3 [1 + (1 - 3W/V^2)^{1/2}]^2 \times [1 - 2(1 - 3W/V^2)^{1/2}] \tag{7}$$

$$\left.\begin{aligned} W &= r + \delta k(r + \epsilon) \\ V &= r + \epsilon + \delta k \end{aligned}\right\} \tag{8}$$

The relation (7) expresses the condition that $L(s)$ has a pair of critically damped roots.

C. Calculation of Transient Response

The calculation of the transient response is done by implementing the Heaviside expansion formulas:

$$\sum p(a_n)/q'(a_n)\exp(a_n t) \tag{9}$$

for the case $q(s)$ has no repeated roots and

$$\sum_{r=0}^{n} (\psi^{(n-r)}(a)/[(n-r)!\,r!])\,t^r \exp(at) + H(t) \tag{10}$$

for the case when $q(s)$ contains $n+1$ repeated linear factors.

Here

$$\psi(s) = (s-a)^{n+1}\frac{p(s)}{q(s)} \tag{11}$$

The inverse Laplace transform is obtained for

$$\begin{gathered}[1 - L(s)]\,\theta_0/s, \quad [1 - L(s)]\,\Omega_0/s^2, \\ [1 - L(s)]\,\Lambda_0/s^3, \quad [1 - L(s)]\left(\frac{-K'U_1}{s(1+\tau_1 s)}\right), \\ [1 - L(s)]\left(\frac{-K'U_2}{s(1+\tau_1 s)(\delta + \tau_3 s)}\right) \\ \text{and } [1 - L(s)]\left(-\frac{K'U_3}{s(\delta + \tau_3 s)}\right)\end{gathered} \tag{12}$$

Each of the above transforms is the quotient of polynomials $p(s)/q(s)$, where the degree of $q(s)$ is greater than the degree of $p(s)$. Also, it may be assumed that the leading coefficient of $q(s)$ is 1.

In case $q(s)$ has no repeated linear factors, the computation of the transient error, based on formula (9) is straightforward.

For the case when $q(s)$ has repeated linear factors,

$$q(s) = (s-a)^2 q_0(s) \tag{13}$$

where

$$q_0(a) \neq 0$$

Letting

$$\psi(s) = (s-a)^2 p(s)/q(s) \tag{14}$$

it is seen that

$$p(s)/q(s) = \frac{\psi'(a)}{s-a} + \frac{\psi(a)}{(s-a)^2} + h(s) \tag{15}$$

where

$$h(s) = \frac{p(s) - q_0(s)\,[\psi'(a)(s-a) + \psi(a)]}{(s-a)^2 q_0(s)} \tag{16}$$

is the sum of the partial fractions corresponding to the remaining factors of $q(s)$.

Since $|h(a)|$ is finite, one can write

$$p(s) - q_0(s)\,[\psi'(a)(s-a) + \psi(a)] = (s-a)^2 h_0(s) \tag{17}$$

Thus

$$h(s) = \frac{h_0(s)}{q_0(s)} \tag{18}$$

To find $h(s)$, one may equate the coefficients of like powers of s in Eq. (17). If $q_0(s)$ has repeated linear factors, the above procedure is applied to the rational function $h(s)$. This process may be continued until $p(s)/q(s)$ is decomposed into partial fractions. Once the partial fraction decomposition is completed, formulas (9) and (10) may be applied to obtain the inverse.

Thus the computational problem consists mainly of the numerical evaluation of the roots of $q(s)$, and in the determination of the numerical values of the polynomials $p(s_k)$ and $q_0'(s_k)$.

The JPL Library subroutine, POLZER, is used for the numerical evaluation of the roots s_k of $q(s)$. In every case considered, the roots were accurate within five decimal places. From a practical point of view, the errors made in evaluating the roots of $q(s)$ are insignificant, since the values of system parameters are seldom known to a high degree of accuracy.

The problem of implementing the general formula (10) on a digital computer appears to be quite difficult. Therefore, the evaluation of the transient response is limited to those cases where $q(s)$ has at most roots or order two.

III. Data Analysis

The data is presented in graphical form. The graphs display the response of the system to the inputs:

$$\frac{\theta(t)}{\Omega_0} = t \text{ (Figs. 1–5)}$$

and

$$\frac{\theta(t)}{\Lambda_0} = \frac{t^2}{2} \text{ (Figs. 6 and 7)}$$

for various values of the parameters ϵ and δ, and for zero initial conditions.

There are five curves per frame. These are numbered from 1 to 5 and corresponding parameters used to obtain the curve appear on the plot frame.

It is interesting to note that the maximum transient error is reasonably independent of the parameters ϵ and δ in the regions.

$$0 < \epsilon \leq 0.1, \qquad 0 \leq \delta \leq 0.1$$

As a practical example of the way these curves may be used, consider the case for the response to a frequency rate input when δ and ϵ are near zero, say $\delta = 0$ and $\epsilon = 0.001$. Then the peak response from Fig. 6 is

$$\frac{\phi_{ss}\omega_L^2}{\Lambda_0} = 1.22$$

Also, a reasonable assumption is that the maximum phase error for lock-on is 1 rad. If we also assume a DSIF receiver bandwidth of 10 Hz, then the maximum frequency rate is

$$\Lambda_{0_{\max}} = 13.05 \text{ Hz/sec}$$

It is interesting to note that this will not meet the maximum one-way doppler rate expected at Jupiter encounter, which is 30 Hz/sec. However, for a bandwidth of 20 Hz the maximum frequency rate is approximately 52.6 Hz/sec.

Figure 8 is a plot of the transient response for the second-order loop where the input phase function is $\theta(t)/\Lambda_0 = t^2/2$. The response in this case is independent of the parameter δ and approaches a stable value of about 1.56 as $\epsilon \to 0$. For non-zero values of ϵ the steady-state response is unbounded.

References

1. Tausworthe, R. C., and Crow, R. B., "Practical Design of Third-Order Tracking Loops," Interim Report 900-450, April 27, 1971 (JPL internal document).
2. Tausworthe, R. C. "Theory and Practical Design of Phase-Locked Receivers," Technical Report 32-819, Jet Propulsion Laboratory, Pasadena, Calif., Feb. 15, 1966.
3. Churchill, R. V., *Operational Mathematics*. McGraw Hill Book Co., Inc., 1958.

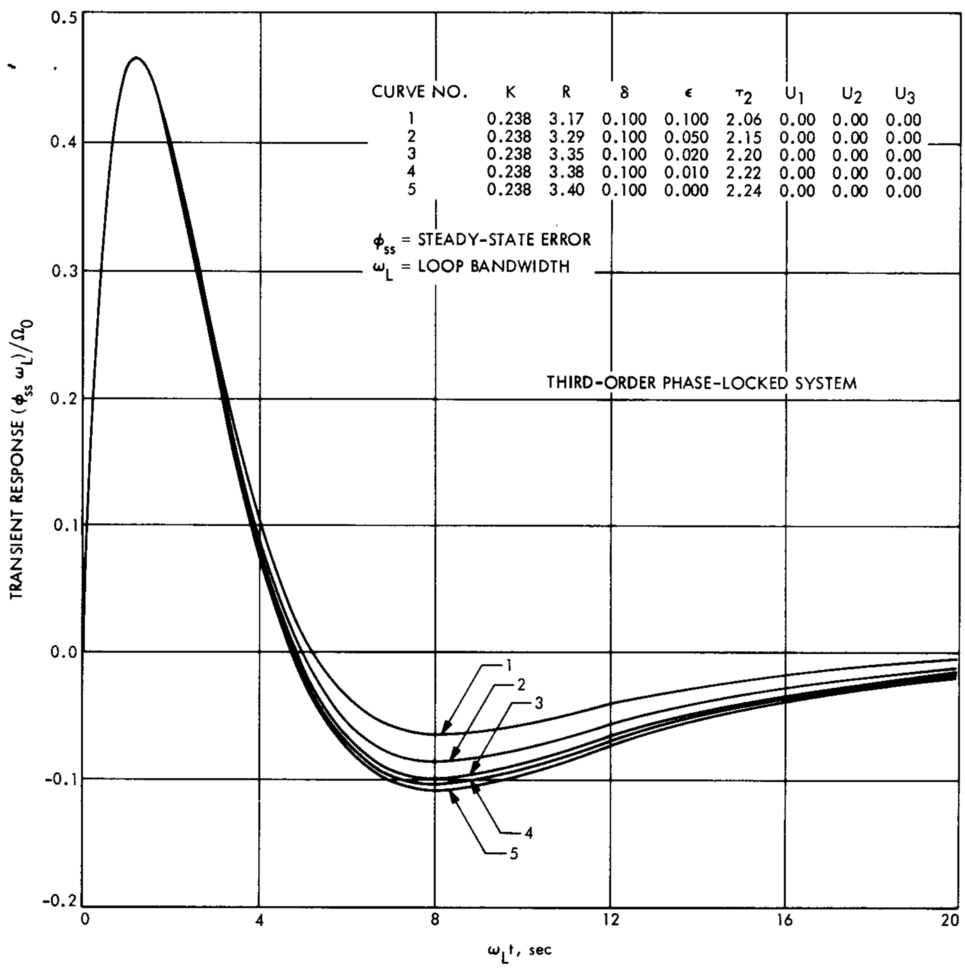


Fig. 1. Variable signal level, $\delta = 0.1$, $\epsilon = 0$ to 0.1 (third-order phase-locked system)

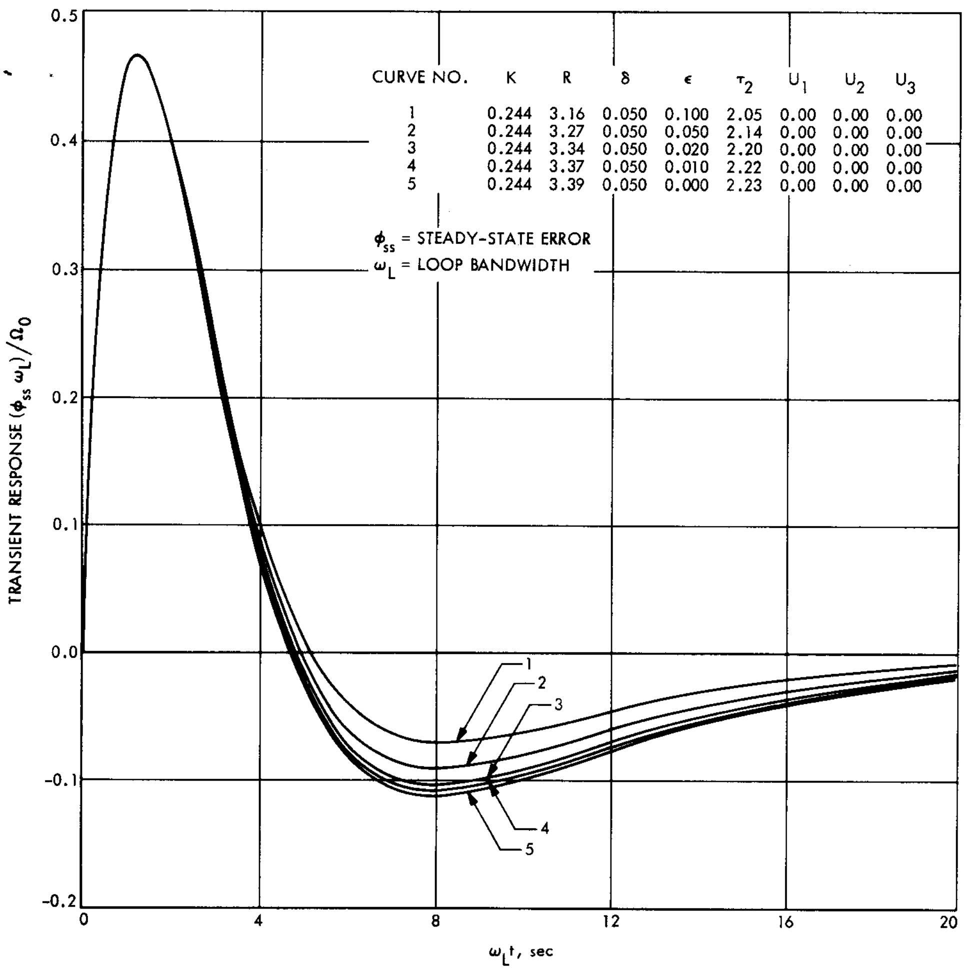


Fig. 2. Variable signal level $\delta = 0.05$, $\epsilon = 0$ to 0.1 (third-order phase-locked system)

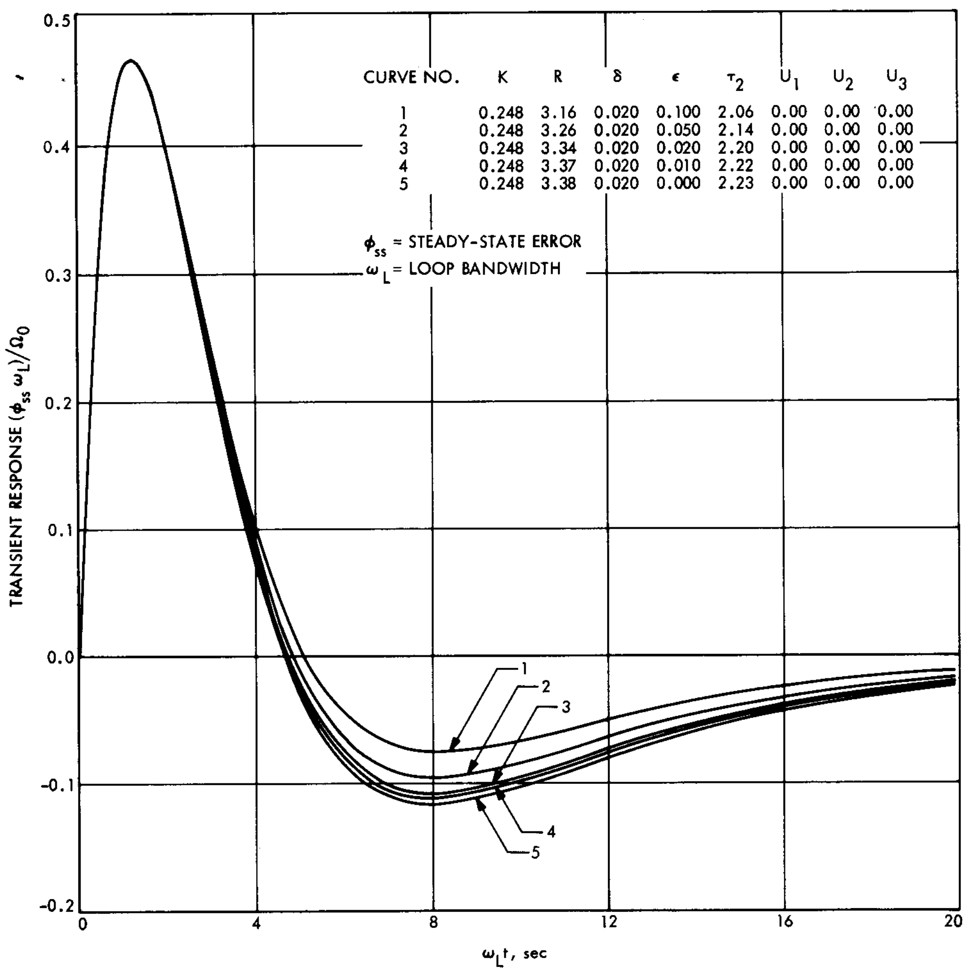


Fig. 3. Variable signal level $\delta = 0.02$, $\epsilon = 0$ to 0.1 (third-order phase-locked system)

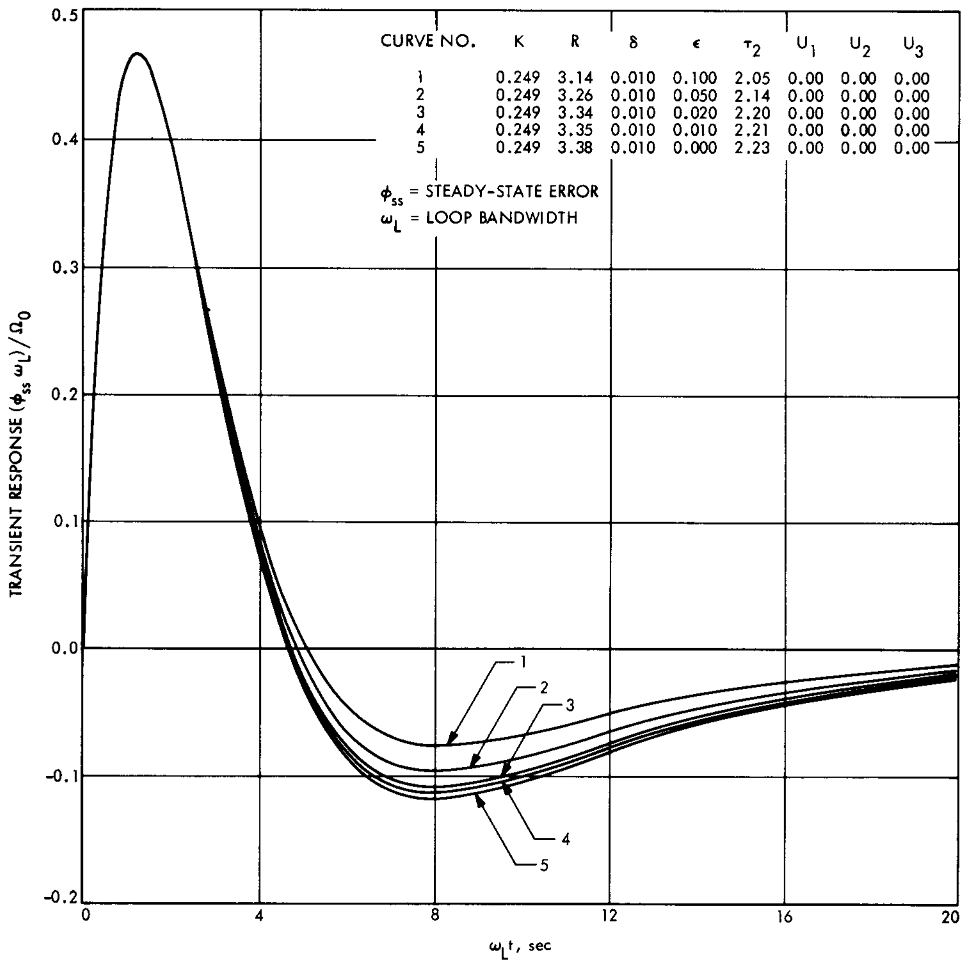


Fig. 4. Variable signal level, $\delta = 0.10$, $\epsilon = 0$ to 0.1 (third-order phase-locked system)

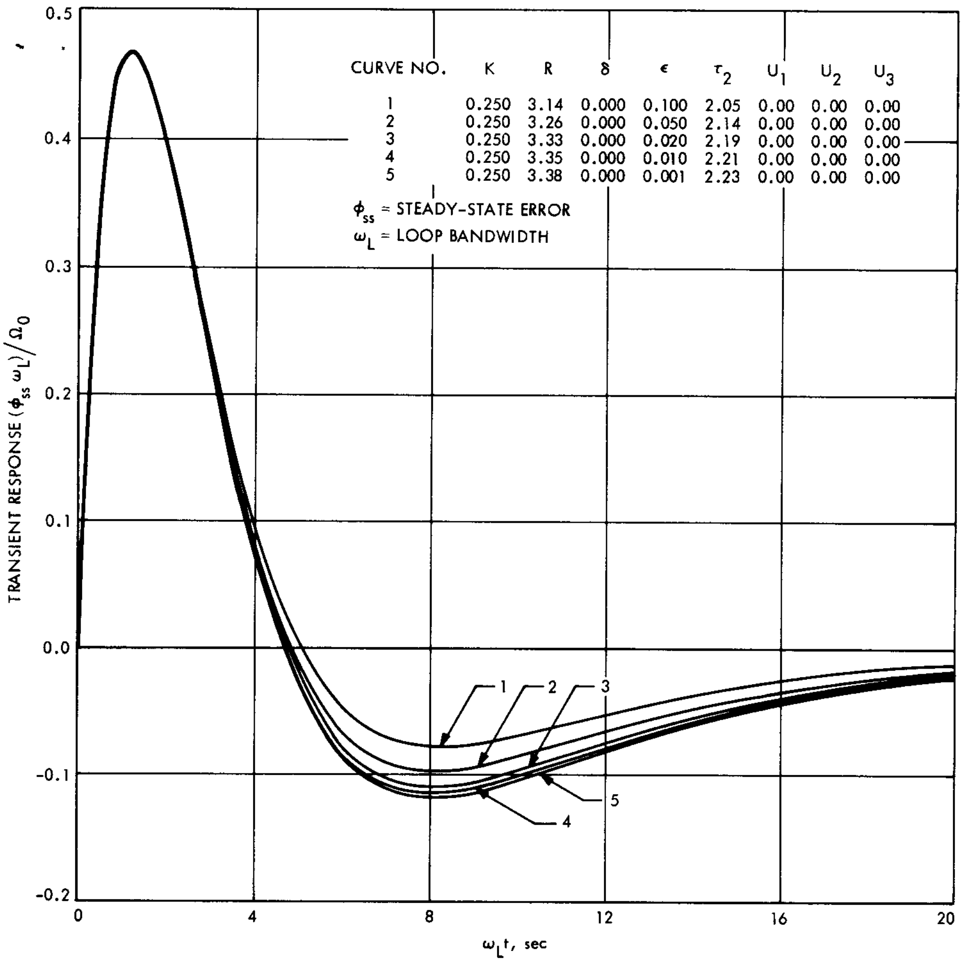
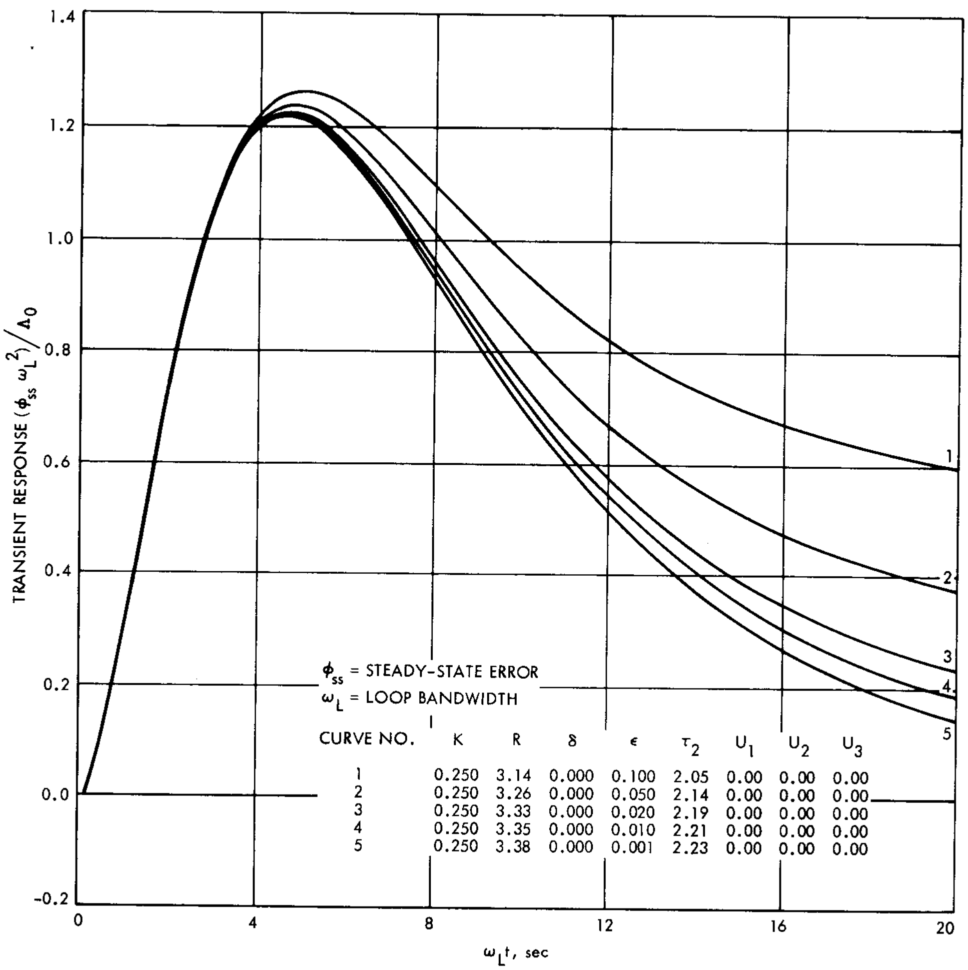


Fig. 5. Variable signal level, $\delta = 0$, $\epsilon = 0.001$ to 0.1, $\theta(t)/\Omega_0 = t$, $\theta(t)/\Lambda_0 = t^2/2$ (third-order phase-locked system)



CURVE NO.	K	R	δ	ϵ	τ_2	U_1	U_2	U_3
1	0.250	3.14	0.000	0.100	2.05	0.00	0.00	0.00
2	0.250	3.26	0.000	0.050	2.14	0.00	0.00	0.00
3	0.250	3.33	0.000	0.020	2.19	0.00	0.00	0.00
4	0.250	3.35	0.000	0.010	2.21	0.00	0.00	0.00
5	0.250	3.38	0.000	0.001	2.23	0.00	0.00	0.00

Fig. 6. Variable signal level, $\delta = 0$, $\epsilon = 0.001$ to 0.1 (third-order phase-locked system)

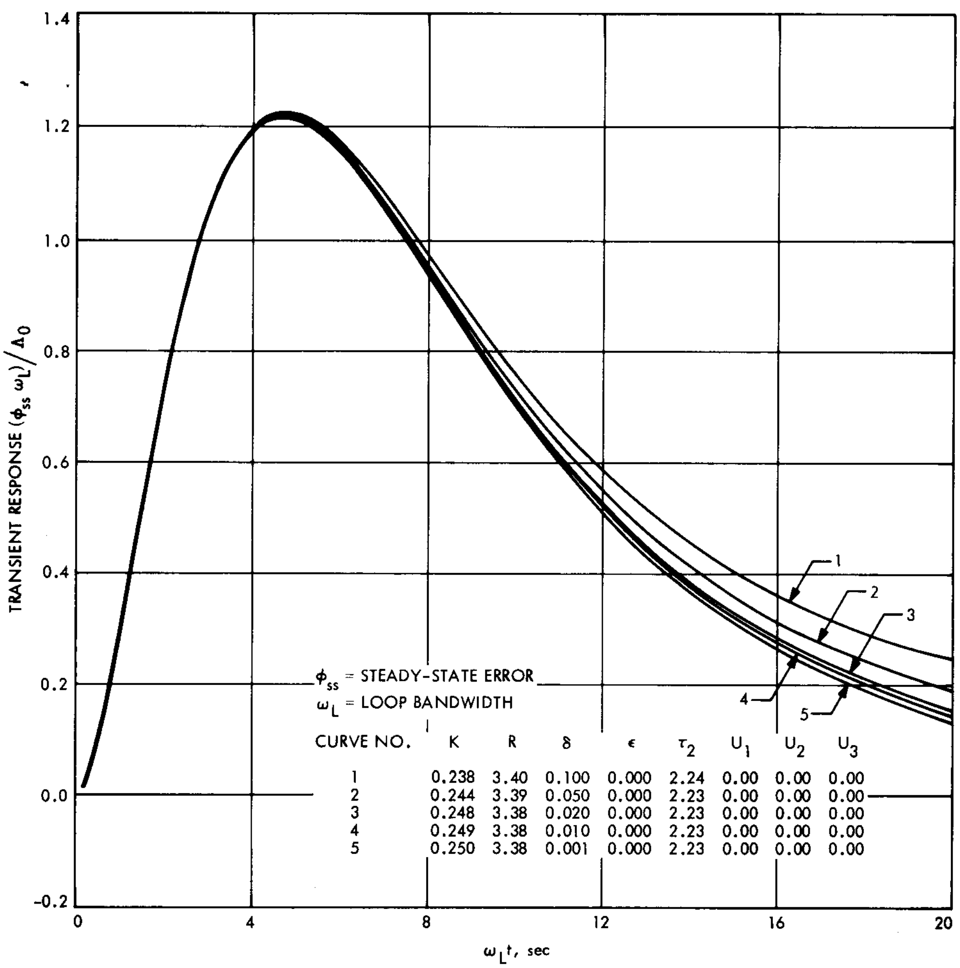
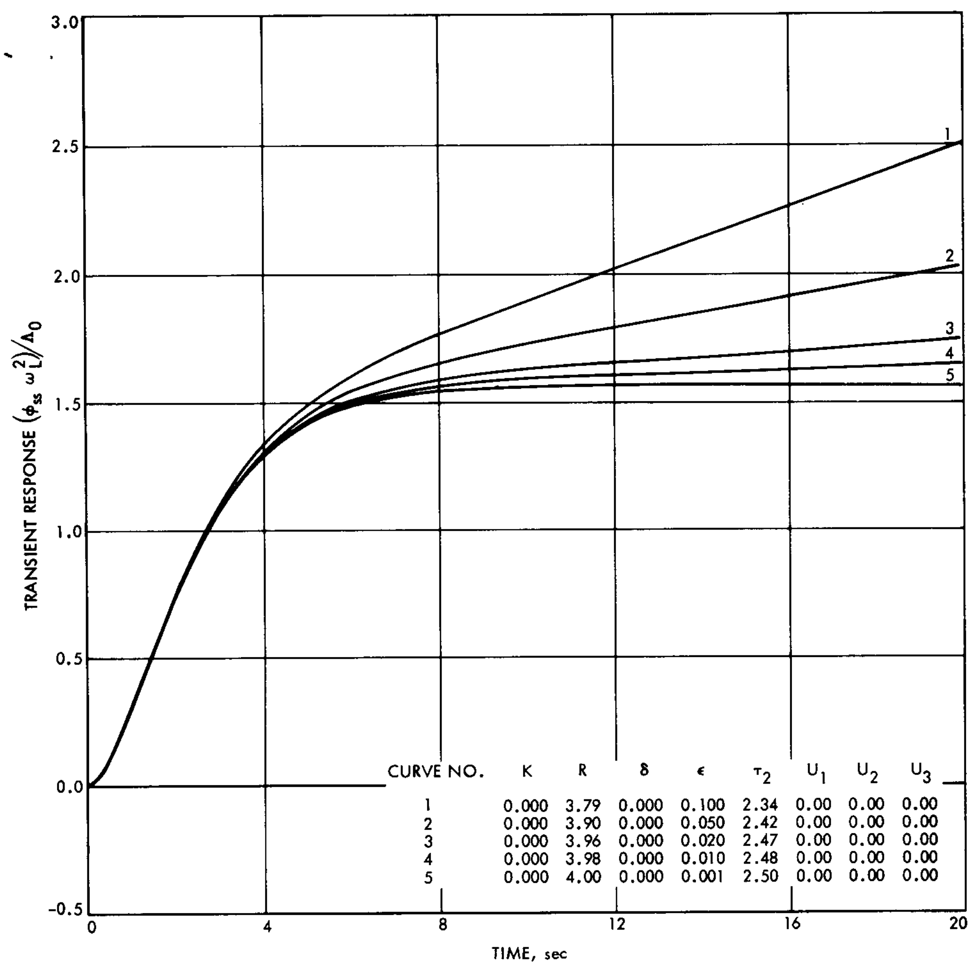


Fig. 7. Variable signal level, $\delta = 0.001$ to 0.1, $\epsilon = 0$ (third-order phase-locked system)



CURVE NO.	K	R	δ	ε	τ_2	U_1	U_2	U_3
1	0.000	3.79	0.000	0.100	2.34	0.00	0.00	0.00
2	0.000	3.90	0.000	0.050	2.42	0.00	0.00	0.00
3	0.000	3.96	0.000	0.020	2.47	0.00	0.00	0.00
4	0.000	3.98	0.000	0.010	2.48	0.00	0.00	0.00
5	0.000	4.00	0.000	0.001	2.50	0.00	0.00	0.00

Fig. 8. Variable signal level, $\delta = 0$, $\epsilon = 0.001$ to 0.1 (second-order phase-locked system)