A Comparison of Cowell's Method and a Variation-of-Parameters Method for the Computation of Precision Satellite Orbits

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A precision special perturbations program that uses either Cowell's method or a variation-of-parameters method to compute an elliptical orbit is analyzed to determine which mode is more efficient when computing satellite orbits. The results obtained indicate that the variation-of-parameters mode is significantly more efficient if the numerical integrator being used is optimized in that mode by varying the integration order and local error control and by using either a predictor or predictor-corrector algorithm.

The objective of this investigation is limited to determining if a variation-of-parameters method is superior to Cowell's method for computing satellite orbits in the environment of a complicated precision trajectory program with various interface complexities (e.g., planetary ephemeris file, sophisticated triggering options, comprehensive input and output options, etc.).

A realistic satellite orbit about the planet Mars (the *Mariner* Mars 1971 Mission A orbit) was chosen for the investigation. The initial state vector with respect to the Mars equator and equinox of date is

$$(a\ e\ i\ \omega\ \Omega\ T) = (12652.618\,\text{km}, 0.63300876, 80.0\,\text{deg}, 328.3937\,\text{deg}, 38.3701\,\text{deg}, 11/19/71\ 14^{h}\ 42^{m}\ \text{UTC})$$

The numerical data in this study was generated using the JPL Univac 1108 digital computer (the executive 8 operating system) and a modified research version of the JPL Double Precision Trajectory Program (DPTRAJ). The modification to DPTRAJ consisted of the inclusion of an option to compute the orbit using a variation-of-parameters method in addition to the standard Cowell method.

The variation-of-parameters method of orbit computation consists of the numerical integration of perturbative variations in terms of parameters such as the Keplerian elements a, e, i, ω, Ω, T. The parameters characterize an osculating orbit, which is a progressively changing reference orbit, that yields the actual position and velocity at any given instant of time. The particular parameters used for this study are as follows:

$$\mathbf{a} = \text{vector of length } e \text{ in direction of periapsis} = e\mathbf{P} \tag{1}$$

$$\mathbf{h} = \text{angular momentum vector} = |\mathbf{h}|\,\mathbf{W} \tag{2}$$

$$n = \text{mean motion} = \sqrt{\mu}/a^{3/2} \tag{3}$$

$$L_0 = \text{mean longitude} = M_0 + \Omega + \omega \tag{4}$$

where M_0 is the mean anomaly at $t = t_0$.

Note that in the case of retrograde motion ($i_0 > \pi/2$), $L_{r0} = M_0 - \Omega + \omega$ is used in place of L_0. Because of this particular choice of parameters, the variation-of-parameters option is just as flexible as Cowell's option in integrating satellite orbits. That is, any satellite orbit having an eccentricity in the range $0 \leq e < 1$ and an inclination in the range $0 \leq i \leq \pi$ can be integrated equally well by the two methods. This more general variation-of-parameters formulation must be used to obtain a valid comparison with Cowell's method even though its use requires more computation than a formulation using the Keplerian elements. In addition, note that eight first-order differential equations of motion are singly integrated to obtain an over-determined state vector of the satellite. These equations are as follows (see Ref. 1 for their derivations):

$$\frac{d\mathbf{a}}{dt} = \frac{1}{\mu}\left[2\,(\dot{s}\,\dot{s}^{\backprime})\,\mathbf{r} - (r\dot{r}^{\backprime})\,\dot{\mathbf{r}} - (r\dot{r})\,\dot{\mathbf{r}}^{\backprime}\right] \tag{5}$$

$$\frac{d\mathbf{h}}{dt} = \frac{1}{\sqrt{\mu}}\,(\mathbf{r} \times \dot{\mathbf{r}}^{\backprime}) \tag{6}$$

$$\frac{dn}{dt} = -\frac{3}{\sqrt{\mu a}}\,\dot{s}\dot{s}^{\backprime} \tag{7}$$

$$\begin{aligned}\frac{dL}{dt} = n &+ \frac{r\dot{b}^{\backprime}}{\sqrt{\mu p}}\,\frac{(r \sin u \sin i)}{1 + \cos i} - \frac{2\,(r\dot{r}^{\backprime})}{\sqrt{\mu a}} \\ &- \frac{1}{1 + \dfrac{\sqrt{\mu p}}{\sqrt{\mu a}}}\left[\frac{(r\dot{r}^{\backprime})}{\sqrt{\mu p}}\left(\frac{p}{r}\right)(e \cos v) \right. \\ &\left. - \frac{(r\dot{v}^{\backprime})}{\sqrt{\mu p}}\left(\frac{p}{r} + 1\right)(r\, e \sin v)\right]\end{aligned} \tag{8}$$

(or an equivalent expression for dL_r/dt) where

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}}, \qquad \sqrt{\mu p} = |\mathbf{r} \times \dot{\mathbf{r}}|, \qquad r\dot{r} = \mathbf{r} \cdot \dot{\mathbf{r}} \tag{9}$$

$$\mathbf{U} = \frac{\mathbf{r}}{r}, \qquad \mathbf{W} = \frac{1}{\sqrt{\mu p}}\,(\mathbf{r} \times \dot{\mathbf{r}}), \qquad \mathbf{V} = \mathbf{W} \times \mathbf{U} \tag{10}$$

$$r\dot{r}^{\backprime} = \mathbf{r} \cdot \dot{\mathbf{r}}^{\backprime}, \qquad r\dot{v}^{\backprime} = \mathbf{V} \cdot \dot{\mathbf{r}}^{\backprime}, \qquad r\dot{b}^{\backprime} = \mathbf{W} \cdot \dot{\mathbf{r}}^{\backprime}, \qquad \dot{s}\dot{s}^{\backprime} = \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}^{\backprime} \tag{11}$$

$$\frac{1}{\sqrt{\mu a}} = \frac{1}{\sqrt{\mu}}\left(\frac{2}{r} - \frac{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}{\mu}\right)^{1/2}, \qquad \frac{p}{r} = \frac{(\sqrt{\mu p})^2}{\mu r} \tag{12}$$

$$r \sin u \sin i = \mathbf{K} \cdot \mathbf{r}, \qquad \cos i = \mathbf{K} \cdot \mathbf{W}, \qquad n = \frac{\mu^2}{(\sqrt{\mu a})^3} \tag{13}$$

$$e \cos v = \frac{p}{r} - 1, \qquad r\, e \sin v = \frac{(r\dot{r})}{\mu}\sqrt{\mu p} \tag{14}$$

and where **I**, **J**, **K** is an inertial basis having the Earth mean equatorial plane of 1950.0 as the principal plane and the direction of the mean vernal equinox of 1950.0 as the principal direction. Note that the differential equations of motion are formulated directly in terms of the perturbative accelerations in the reference Cartesian coordinate system rather than in terms of the parameters being integrated. The quantity $\dot{\mathbf{r}}^{\backprime}$ represents the total perturbative acceleration acting upon the satellite. In this study, $\dot{\mathbf{r}}^{\backprime}$ consists of the perturbations due to the asphericity of the central body (J_2 only), N bodies other than the central body, and solar radiation pressure. In the course of this study, the best six of the eight parameters integrated will be determined. The intention is eventually to eliminate the integrations of the two less desirable parameters and to replace them with relationships derived from the following equations:

$$\mathbf{a} \cdot \mathbf{h} = 0 \tag{15}$$

$$\mathbf{a} \cdot \mathbf{a} + (n^2/\mu)^{1/3}\,\mathbf{h} \cdot \mathbf{h} - 1 = 0 \tag{16}$$

Note that in the case of L_0 (or L_{r0}), numerical integration is actually performed on L (or L_r). Then L_0 (or L_{r0}) is obtained by means of the equation

$$L_0 = L_{00} - n(t - t_0) + \int_{t_0}^{t} \frac{dL}{dt} d\tau \tag{17}$$

(or the equivalent expression for L_{r0}).

The complexity of the differential equations (5) through (8) and the necessity of solving a modified Kepler's equation by iteration at each integration step are clearly disadvantages in the variation-of-parameters method. Both these quantities adversely affect the amount of computer execution time required to advance the solution one step. The principal advantage of the variation-of-parameters method is that the derivatives are small and change slowly with time because of the absence of the central force term. Consequently, the integration tables converge more rapidly than in Cowell's method and therefore allow larger tabular intervals, particularly in the region near periapsis.

When using Cowell's method in the modified DPTRAJ, the numerical integration is performed using the tenth-order variable-step second-sum process that has proven so successful in the case of interplanetary trajectories (Ref. 2). When using the variation-of-parameters method in the modified DPTRAJ, the numerical integration is performed using the same process but with variations on the fixed integration order and local error control and with and without a corrector cycle. Talbot and Rinderle (Ref. 3) have compared the performance of this integrator with the performance of a variable-order polynomial-type integrator and a "Fourier"-type integrator in integrating a Mars orbiter using a classical variation-of-parameters method. They found that the performances of these integrators are similar. For about the same accuracy the runs varied by about 15 to 20% in the number of times the derivatives are evaluated. Note that a 15 to 20% variation in derivative evaluations does not translate into a 15 to 20% variation in cost because evaluating the derivatives is not the principal consumer of execution time in a complex program such as DPTRAJ.

The algorithms used to solve the two sets of equations are quite similar. In both cases, a multi-step process in summed form closely related to an Adams-type process is used. The algorithms consist basically of a starting procedure to compute the solution values at m points backward in time (where m is the number of backward differences used in the integration formulas and is denoted as the order of the process), and a stepping procedure to advance the solution one step in time using information at the previous m points. In Cowell's method, only tenth order was used, and both the predict-correct and predict-partial-correct (Ref. 4) modes were used. In the variation-of-parameters method, orders 4, 6, 8, and 10 were tried, and the predict only and predict-correct modes were used. In most cases, the automatic step size control based upon an estimate of the local truncation error (Ref. 2) was employed.

The standard of comparison for this study was obtained by entering the initial conditions for the *Mariner* orbit given previously into DPTRAJ and numerically integrating the differential equations of motion of the satellite with very tight error control ($ERMX = 10^{-13}$ and $ERMN = 10^{-18}$). Although the solution obtained in this manner is not perfectly accurate, the accuracy is much higher than the accuracy of the test cases, and, therefore, a valid standard of comparison was obtained. A measure of the accuracy of the test cases relative to the standard of comparison was obtained by computing the magnitudes of the position ($|\Delta \mathbf{r}|$) and velocity ($|\Delta \dot{\mathbf{r}}|$) vector errors. A measure of the cost of the test cases was chosen to be the Central Processing Unit (CPU) time corrected for any nonstandard production practices and known improvements in the variation-of-parameters method.[1]

A total of 34 cases was run in this study including the standard of comparison. Each of the cases was run with the same initial conditions (the *Mariner* orbit) using either Cowell's method or the variation-of-parameters method. The majority of the variation-of-parameters cases was predict only, sixth order, and variable step, and differed primarily in the local error control.

A preliminary comparison of the variation-of-parameters cases run to date shows that the best case is the predict-only, sixth-order, variable-step case with the local truncation error bounded by an $ERMX$ and $ERMN$ proportional to r_a/r (where r_a is the apoapsis distance). Table 1 shows a comparison of this case with a case determined by Cowell's method having the same accuracy after 20 revolutions. This comparison shows that the variation-of-parameters method requires approximately 59 seconds less CPU time (about 18%) than does Cowell's method. The test cases indicate that this savings in CPU time results in a significant savings in the total cost. These

[1]The two major corrections made were for the additional costs of special output and an inefficient iterative procedure for solving Kepler's equation.

results are preliminary since this study has not been completed. Consequently, the improvement of 18% in the CPU time should be regarded only as an indication of the improvement to be expected from using the variation-of-parameters method in place of Cowell's method when computing precision satellite orbits.

In conclusion, the significantly more efficient variation-of-parameters method for computing precision satellite orbits offers the potential for reducing the cost of satellite orbit computations and also for reducing the computer execution time during real-time mission operations for future orbiter projects. The final results of this study will include a measure of these reductions in cost and computer execution time and a recommendation as to whether the variation-of-parameters method should be included in the standard production and mission operations versions of DPTRAJ.

References

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Table 1. Comparison of Cowell's method and a variation-of-parameters method

Method	Order	Local error control		Revolution number		Accuracy[a]		Number of steps	Number of derivative evaluations	CPU time, s	
		ERMX	ERMN			$\lvert\Delta r\rvert$, m	$\lvert\Delta \dot{r}\rvert$, m/s			Output	Corrected[b]
(1) Cowell (predict-correct, variable step)	10	2×10^{-10}	2×10^{-15}	5	apoapsis	4.62	0.0003	1024	2288	—	—
					periapsis	8.46	0.0049				
				20	apoapsis	521.26	0.0577	3608	8126	395	335
					periapsis	2261.93	1.1008				
(2) Variation of parameters (predict only, variable step)	6	$10^{-8}(r_a/r)$	$10^{-12}(r_a/r)$	5	apoapsis	25.94	0.0027	745	871	—	—
					periapsis	122.99	0.0631				
				20	apoapsis	537.61	0.0606	2844	3330	405	276
					periapsis	2398.12	1.1662				

[a]These errors occur approximately at apoapsis ($t - t_0 = 54$ and 234 h) and periapsis ($t - t_0 = 60$ and 240 h) of revolutions five and twenty.
[b]Corrected for nonstandard production practices and known improvements.