Spectral Estimate Variance Reduction by Averaging Fast-Fourier Transform Spectra of Overlapped Time Series Data

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An analysis is made of the variance of the spectral estimates calculated in the DSN by two methods, namely the correlation method and the Fast Fourier Transform (FFT) method. It is shown that the FFT method using consecutive sequences of data samples produces the same variance as the correlation method. However, a reduction of over 20% in variance can be obtained by using the FFT method with overlapped sequences of data. A relationship is derived giving the variance reduction as a function of the amount of data sequence overlap.

I. Introduction

The ability to distinguish useful signal characteristics in a measured power spectrum of a signal with high noise content is limited by the variance of the individual spectrum point estimates resulting from the noise component of the signal. Reduction of this variance is accomplished by using large amounts of data in either of the two present methods of obtaining power spectra in the DSN. In one method, the correlation method, the autocorrelation function of the signal is accumulated over a long period of observation. After the observation time, the accumulated autocorrelation function is transformed into an estimate of the power spectrum. In the other method, the fast Fourier transform (FFT) method, consecutive portions of data are each individually transformed by the FFT and the squares of the magnitude of the transformed points are taken to represent an estimate of the power spectrum. These local spectral estimates are averaged over the entire observation time to obtain the final useful spectrum.

It will be shown that under equivalent conditions both methods provide the same variance which is inversely proportional to the observation time. The prime concern of the following is to determine the spectral estimate variance resulting from using the FFT method on overlapped sequences of data points rather than consecutive sequences of data points. Since there are more terms to be averaged using overlapped sequences of data, it might be expected that the spectrum variance would be proportionately smaller. Inhibiting this expectation is the fact that the individual terms of the final averaged spectrum are no longer statistically independent, and the inverse proportionality rule for the variance of the averaging statistic of independent terms no longer holds. Nevertheless, a limited amount of variance reduction can definitely be obtained using overlapped sets of data. The penalty is, of course, the need to do more computing.

II. The FFT Method

To find the variance of the average of spectra from overlapped data sequences, it is first necessary to find the covariance of corresponding spectral points in overlapped sequences. This covariance function is then used in the formula for the variance of the averaging statistic of nonindependent terms. The time series data are assumed to be stationary zero mean gaussian noise with samples uncorrelated to each other and with variance equal to σ^2. Using the FFT, the power spectrum, P_n, of a series of data points is

$$P_n = \frac{2}{M^2} \left| \sum_{k=0}^{M-1} X_k \, e^{-i\frac{2\pi nk}{M}} \right|^2 \tag{1}$$

where X_k is the kth data point, M is the number of data points in the sequence, and n is the number of the spectral point in the spectrum. M is considered to be a power of two, and the range of n is

$$0 \leq n \leq \frac{M}{2} \tag{2}$$

If Δ_f represents the frequency difference between adjacent power spectral points, and Δ_t is the time between data samples, then

$$\Delta_f \, \Delta_t = \frac{1}{M} \tag{3}$$

For sinusoidal data, the main lobe width is $2\Delta_f$ and the folding frequency is $1/(2\Delta_t)$. The proportionality term in Eq. (1) permits the total power, P_{total}, in the sampled signal to be found from

$$P_{\text{total}} = \tfrac{1}{2}\left(P_0 + P_{\frac{M}{2}}\right) + \sum_{n=1}^{\frac{M}{2}-1} P_n \tag{4}$$

Similar to Eq. (1), the power spectrum, $P_{n,\theta}$, of an overlapped data sequence is

$$P_{n,\theta} = \frac{2}{M^2} \left| \sum_{k=\theta}^{M-1+\theta} X_k \, e^{-i\frac{2\pi nk}{M}} \right|^2 \tag{5}$$

where θ is the number of data points between the starts of adjacent overlapped sequences. For θ equal to zero, the sequences are identical, and for θ equal to M, the sequences are consecutive. When θ is less than M, the sequences are overlapped, and for θ greater than M, there are data points between the sequences which are not included in either sequence.

Equations (1) and (5) may be manipulated to be

$$P_n = \frac{2}{M^2} \sum_{k=0}^{M-1} \sum_{\tau=-k}^{M-1-k} X_k \, X_{k+\tau} \cos \frac{2\pi n\tau}{M} \tag{6}$$

$$P_{n,\theta} = \frac{2}{M^2} \sum_{L=\theta}^{\theta+M-1} \sum_{\psi=\theta-L}^{\theta+M-1-L} X_L \, X_{L+\psi} \cos \frac{2\pi n\psi}{M} \tag{7}$$

Using braces to indicate the ensemble mean or expected value, the product of the means of Eqs. (6) and (7) is

$$\langle P_n \rangle \langle P_{n,\theta} \rangle = \frac{4}{M^4} \sum_{k=0}^{M-1} \sum_{\tau=-k}^{M-1-k} \sum_{L=\theta}^{\theta+M-1} \sum_{\psi=\theta-L}^{\theta+M-1-L} \beta(\tau)\,\beta(\psi) \cos \frac{2\pi n\tau}{M} \cos \frac{2\pi n\psi}{M} \tag{8}$$

where $\beta(\tau)$, the mean of the product of X_k and $X_{k+\tau}$, is the correlation function of the noise variable X_k. Similarly

$$\beta(\psi) = \langle X_L \, X_{L+\psi} \rangle \tag{9}$$

Because X_k has been specified as stationary, the argument of the correlation function is formed by differencing the subscripts of the two terms in the braces. The mean of the product of Eqs. (6) and (7) is

$$\langle P_n P_{n,\theta} \rangle = \frac{4}{M^4} \sum_{k=0}^{M-1} \sum_{\tau=-k}^{M-1-k} \sum_{L=\theta}^{\theta+M-1} \sum_{\psi=\theta-L}^{\theta+m-1-L} \langle X_k X_{k+\tau} X_L X_{L+\psi} \rangle \cos \frac{2\pi n\tau}{M} \cos \frac{2\pi n\psi}{M} \tag{10}$$

The mean of the product of four terms of a zero mean gaussian random variable (Ref. 1) is

$$\langle X_1X_2X_3X_4\rangle = \langle X_1X_2\rangle\langle X_3X_4\rangle + \langle X_1X_3\rangle\langle X_2X_4\rangle + \langle X_1X_4\rangle\langle X_2X_3\rangle \tag{11}$$

The covariance function, $K_{n,\theta}$, of the power spectral points as a function of θ is found by subtracting Eq. (8) from Eq. (10). Using Eq. (11) and the notation of Eq. (9) this becomes

$$K_{n,\theta} = \frac{4}{M^4}\sum_{k=0}^{M-1}\sum_{\tau=-k}^{M-1-k}\sum_{L=\theta}^{\theta+m+1}\sum_{\psi=\theta-L}^{\theta+M-1-L}\Big\{\beta(k-L)\,\beta(k-L+\tau-\psi) + \beta(k-L-\psi)\beta(k-L+\tau)\Big\}\cos\frac{2\pi n\tau}{M}\cos\frac{2\pi n\psi}{M} \tag{12}$$

Since uncorrelated noise samples were specified, the correlation function is zero for all arguments unequal to zero. For zero argument, the correlation function is

$$\beta(0) = \sigma^2 \tag{13}$$

where σ^2 is the variance of the noise signal. Using Eq. (13), Eq. (12) is laboriously simplified to

$$\begin{aligned} K_{n,\theta} &= \frac{8\sigma^4}{M^4}\left\{M-\theta+2\sum_{\tau=1}^{M-1-\theta}(M-\theta-\tau)\cos^2\frac{2\pi n\tau}{M}\right\} \qquad 0\le\theta\le M-2 \\ &= \frac{8\sigma^4}{M^4} \qquad \theta = M-1 \\ &= 0 \qquad \theta\ge M \\ &= K_{n,-\theta} \end{aligned} \tag{14}$$

With the help of the identity

$$\sum_{k=1}^{M-1}\cos kY = \frac{\sin\left\{(2M-1)\dfrac{Y}{2}\right\}}{2\sin\dfrac{Y}{2}} - ½ \tag{15}$$

the final form of Eq. (14) is found to be

$$\begin{aligned} K_{n,\theta} &= \frac{4\sigma^4}{M^4}\left\{(M-\theta)^2 + \frac{\sin^2\dfrac{2\pi n\theta}{M}}{\sin^2\dfrac{2\pi n}{M}}\right\} \qquad 0\le\theta\le M-1 \\ &= 0 \qquad \theta\ge M \\ &= K_{n,-\theta} \end{aligned} \tag{16}$$

The final spectrum, $\overline{P_{n,\theta}}$, is found by averaging the individual overlapped spectra.

$$\overline{P_{n,\theta}} = \frac{1}{N}\sum_{L=0}^{N-1} P_{n,L\theta} \tag{17}$$

where N is the number of overlapped spectra. If the total observation time T is

$$T = KM\Delta_t \tag{18}$$

where K is some large positive integer, and the time between successive spectra is $\theta\Delta_t$, the N is approximately

$$N = \frac{KM}{\theta} \tag{19}$$

The variance of $\overline{P_{n,\theta}}$, $D_{n,\theta}$, is given by

$$D_{n,\theta} = \frac{K_{n,0}}{N} + \frac{2}{N^2}\sum_{L=1}^{N-1}(N-L)\,K_{n,L\theta} \tag{20}$$

Since $K_{n,L\theta}$ is nonzero for only the lower values of L, Eq. (20) can be approximated by

$$D_{n,\theta} = \frac{K_{n,0}}{N} + \frac{2}{N}\sum_{L=1}^{\hat{L}} K_{n,L\theta} \tag{21}$$

where $\hat{L}$ is

$$\hat{L} = \left[\frac{M-1}{\theta}\right] \tag{22}$$

The brackets in Eq. (22) denote integer value. Substituting Eq. (16) and Eq. (19) in Eq. (21)

$$D_{n,\theta} = \frac{4\sigma^4\theta}{KM^3}\left\{1 + 2\sum_{L=1}^{\hat{L}}\left(\left(1-L\frac{\theta}{M}\right)^2 + \frac{\sin^2\dfrac{2\pi nL\theta}{M}}{M^2\sin^2\dfrac{2\pi n}{M}}\right)\right\} \tag{23}$$

If the sinusoidal term in Eq. (23) is omitted, the error in $D_{n,\theta}$ is significant only at the extreme ends of the power spectrum. For n equal to one, the maximum error is 4.5% which occurs at θ/M equal to 0.76. For n equal to two, the maximum error is 1.2% at θ/M equal to 0.87. Further reduced is the maximum error at n equal to three, which is 0.55% at θ/M equal to 0.915. Thus for n unequal to zero or $M/2$, a good engineering approximation to Eq. (23) is

$$
\begin{aligned}
D_{n,\theta} &= \frac{4\sigma^4}{KM^2}\frac{\theta}{M}\left\{1 + 2\sum_{L=1}^{\hat{L}}\left(1 - L\frac{\theta}{M}\right)^2\right\} \qquad \frac{1}{\hat{L}+1} \leq \frac{\theta}{M} < \frac{1}{\hat{L}} \\
&= \frac{4\sigma^4}{KM^2}\frac{\theta}{M} \qquad \frac{\theta}{M} \geq 1
\end{aligned}
\tag{24}
$$

Table 1 gives the value of $D_{n,\theta}$ for selected values of θ. The reduction in variance given by Eq. (24) can be equated to a corresponding dB increase in a signal-to-noise power ratio. If the variance of the power spectrum estimate was changed due to a change in noise power, the signal-to-noise power ratio would vary inversely to the square root of the variance change. Figure 1 is a plot of Eq. (24) as an equivalent signal-to-noise power ratio gain versus θ/M. Zero dB is defined for θ/M equal to one. When θ/M is equal to one half, $D_{n,\theta}$ is three fourths of its value at θ/M equal to one. The equivalent signal-to-noise power ratio gain is the square root of four thirds or a gain of 0.625 dB at a cost of doubling the number of transforms to be calculated. Values for θ/M equal to one and θ/M equal to one half have been verified using Monte Carlo simulation tests.

III. Correlation Method

Using the same total observation time as Eq. (18), the number of data samples is KM. The observed correlation function, R_k, is

$$
R_k = \frac{1}{KM}\sum_{L=0}^{LM-1} X_L X_{L+k} \tag{25}
$$

In order to form a consistent basis of comparison between the correlation method and the FFT method, it will be convenient to use a member of a discrete Fourier transform pair to convert R_k to a power spectrum P_n. Such a pair is derived from the general discrete Fourier transform pair

$$
\begin{aligned}
P_n &= \sum_{k=C}^{C+A-1} R_k \exp\left(-i\frac{2\pi nk}{A}\right) \\
R_k &= \frac{1}{A}\sum_{n=C}^{C+A-1} P_n \exp\left(i\frac{2\pi nk}{A}\right)
\end{aligned}
\tag{26}
$$

where C is any integer constant. Letting C equal $-B$, A equal $2B+1$, and noting that R_k and P_n are even real functions, the pair in Eq. (26) can be written

$$
\left.
\begin{aligned}
P_n &= R_0 + 2\sum_{k=1}^{B} R_k \cos\frac{2\pi nk}{2B+1} \\
R_k &= \frac{1}{2B+1}\left\{P_0 + 2\sum_{n=1}^{B} P_n \cos\frac{2\pi nk}{2B+1}\right\}
\end{aligned}
\right\}
\tag{27}
$$

Similar to Eq. (3)

$$
\Delta_f \Delta_t = \frac{1}{2B+1} \tag{28}
$$

An approximate comparison between Eq. (28) and Eq. (3) is established if B is set equal to $M/2$. The desired transform pair then becomes

$$
P_n = \frac{4}{M+1}\left\{\frac{R_0}{2} + \sum_{k=1}^{\frac{M}{2}} R_k \cos\frac{2\pi nk}{M+1}\right\} \tag{29}
$$

$$
R_k = \frac{P_0}{2} + \sum_{n=1}^{\frac{M}{2}} P_n \cos\frac{2\pi nk}{M+1} \tag{30}
$$

The proportionality term in Eq. (29) is chosen so that R_0 from Eq. (30) is

$$
R_0 = \frac{P_0}{2} + \sum_{n=1}^{\frac{M}{2}} P_n \tag{31}
$$

Since R_0 is the total power in the signal, a close similarity exists between Eq. (31) and Eq. (4). As in Eq. (3), Δ_f represents the frequency difference between adjacent power spectral points. The main lobe width for sinusoidal inputs is $2\Delta_f$ and the folding frequency is $1/(2\Delta_t)$. Corresponding to Eq. (3)

$$
\Delta_f \Delta_t = \frac{1}{M+1} \tag{32}
$$

As seen from Eq. (30), the range of n is as shown in Eq. (2). Thus a consistent basis of comparison has been established between the correlation method and the FFT method.

Substituting Eq. (25) into Eq. (29) gives

$$P_n = \frac{4}{KM(M+1)} \left\{ \tfrac{1}{2} \sum_{L=0}^{KM-1} X_L^2 + \sum_{k=1}^{\frac{M}{2}} \sum_{L=0}^{KM-1} X_L X_{L+k} \cos \frac{2\pi nk}{M+1} \right\} \tag{33}$$

The mean of P_n is

$$\langle P_n \rangle = \frac{4}{KM(M+1)} \left\{ \tfrac{1}{2} \sum_{L=0}^{KM-1} \beta(0) + \sum_{k=1}^{\frac{M}{2}} \sum_{L=0}^{KM-1} \beta(k) \cos \frac{2\pi nk}{M+1} \right\} \tag{34}$$

Using Eq. (13), Eq. (34) becomes

$$\langle P_n \rangle = \frac{2\sigma^2}{M+1} \tag{35}$$

and the square of the mean is

$$\langle P_n \rangle^2 = \frac{4\sigma^4}{(M+1)^2} \tag{36}$$

The mean of the square of Eq. (33) is

$$\begin{aligned} \langle P_n^2 \rangle = \frac{16}{K^2M^2(M+1)^2} \Bigg\{ & \tfrac{1}{4} \sum_{L=0}^{KM-1} \sum_{k=0}^{KM-1} \langle X_L^2 X_k^2 \rangle \\ & + \sum_{Q=0}^{KM-1} \sum_{k=1}^{\frac{M}{2}} \sum_{L=0}^{KM-1} \langle X_Q^2 X_L X_{L+k} \rangle \cos \frac{2\pi nk}{M+1} \\ & + \sum_{k=1}^{\frac{M}{2}} \sum_{s=1}^{\frac{M}{2}} \sum_{L=0}^{KM-1} \sum_{Q=0}^{KM-1} \langle X_L X_{L+k} X_Q X_{Q+s} \rangle \\ & \times \cos \frac{2\pi nk}{M+1} \cos \frac{2\pi ns}{M+1} \Bigg\} \end{aligned} \tag{37}$$

With the help of Eqs. (11), (13), and (15), Eq. (37) is found to be

$$\langle P_n^2 \rangle = \frac{4\sigma^4}{(M+1)^2} + \frac{4\sigma^4}{KM(M+1)} \tag{38}$$

The variance of the power spectrum, D_n, is found by subtracting Eq. (36) from Eq. (38).

$$D_n = \frac{4\sigma^4}{KM(M+1)} \tag{39}$$

From Table 1 it is seen that this is very close to the FFT method where θ equals M.

IV. Conclusion

It has been shown that the variance of spectral estimates is equivalent for either the correlation method of calculation, or the FFT method using consecutive sequences of data points. With the FFT method, however, it is possible to reduce the variance by a maximum amount of 33% by using overlapping sequences of data. An overlap of 50% will provide an improvement in signal-to-noise power ratio of 0.625 dB at a cost of doubling the required number of FFT calculations.

Although the above analysis has not included the effect of a correlation window, the use of such a window can be considered to be a post-measurement convolutional calculation. As such, the absolute variance of the spectral points in both calculation methods will be reduced, but the relative improvement shown in Fig. 1 will still be in effect.

Reference

1. Davenport, W. B., and Root, W. L., *An Introduction to the Theory of Random Signals and Noise*, p. 168. McGraw-Hill Book Co., Inc., New York, 1958.

Table 1. FFT variance values

θ	$\hat{L}$	$D_{n',\theta}$
M	0	$\frac{4\sigma^4}{KM^2}$
$\frac{M}{2}$	1	$\frac{3}{4}\,\frac{4\sigma^4}{KM^2}$
$\frac{M}{4}$	3	$\frac{11}{16}\,\frac{4\sigma^4}{KM^2}$
1	$M-1$	$\frac{2}{3}\,\frac{4\sigma^4}{KM^2}$

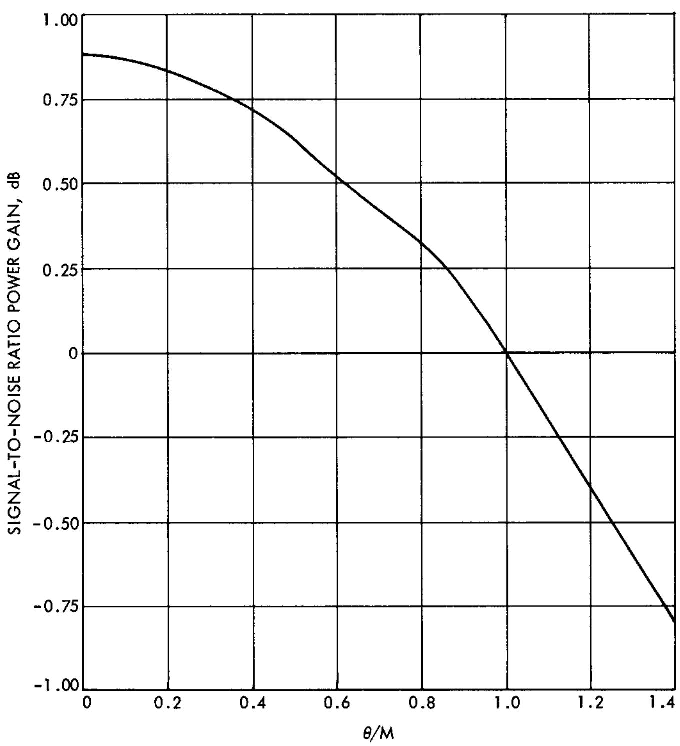

Fig. 1. Equivalent S/N dB gain for overlapped data sequences