An Analysis of Long Baseline Radio Interferometry, Part II

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This report continues the analysis of the cross-correlation procedure used in long baseline radio interferometry begun in Technical Report 32-1526, Vol. VII, pp. 37–50. It is assumed that the radio signal is generated by a very distant, completely incoherent, extended source. For both digital and analog recording systems, the normalized cross-correlation function is derived in terms of noise temperature, fringe visibility, and bandpass overlap. For very strong point sources and accurate model delays, it is shown that the digital cross-correlation function becomes a sawtooth time function whose extrema and zero crossings agree with the sinusoidal cross-correlation function produced by an analog system. For weak sources, such as those common to most very long baseline interferometry measurements, the digital cross-correlation function is identical to the normalized analog cross-correlation function, except for a loss of $2/\pi$ in amplitude.

General signal/noise (S/N) expressions are derived for both the digital and the analog cross-correlation functions. For a very strong point source, the S/N ratio in a digital system can be infinitely better than the S/N ratio in an analog system at time points of maximum correlation. However, at points of weak correlation, the digital S/N ratio is $2/\pi$ smaller than the analog value. In the case of small correlated amplitude, the digital system produces a S/N ratio that is uniformly $2/\pi$ worse than the analog system ratio.

I. Introduction

In very long baseline interferometer (VLBI) measurements, the radio signal produced by a distant source is recorded simultaneously at two widely separated antennas. These recorded signals are then cross-correlated to determine correlated amplitude, as well as delay and delay rate due to path differences. An earlier report (Ref. 1) presented an analysis of long baseline interferometry that included the following topics—time delay theory, source statistics, electronic factors, and a derivation of the analog cross-correlation functions for point sources and completely incoherent, extended sources. In this report, the analog cross-correlation function is simplified by normalization and expressed in terms of noise temperature, fringe visibility, and bandpass overlap. In addition, this report considers digital recording and signal-to-noise ratios for the following reasons.

In the digital recording systems found in many VLBI systems, the voltage signal is infinitely clipped before it is recorded. That is, each time the signal is sampled, only the sign of the voltage is recorded. Since this clipping process only preserves the zero crossings of the original analog signals, the digital cross-correlation function will deviate from the analog cross-correlation function derived in earlier work (Ref. 1). In order to assess the importance of this deviation, the digital cross-correlation procedure is investigated in the case of a completely incoherent, extended source. The statistical approach used in this analysis is based on a technique employed by Van Vleck and Middleton (Ref. 2) to investigate the *autocorrelation* of clipped Gaussian noise. Their technique is applied to the *cross-correlation* of infinitely clipped VLBI signals in order to obtain the digital cross-correlation function.

A signal/noise (S/N) analysis has been performed for two reasons. First, source brightness measurements are often based on the S/N values observed in VLBI experiments. In the case of infinite clipping, brightness measurements must depend to some extent on S/N expressions since all absolute amplitude information is lost in the clipping process. Second, estimates of the measurement precision for time delay and delay rate can be calculated on the basis of the S/N ratios for the cross-correlation function. For these reasons S/N expressions have been derived for both the analog and digital cross-correlation functions.

II. Normalized Cross-Correlation Function

In this section, the analog cross-correlation function for an extended source is simplified by a normalization process that introduces noise temperature and fringe visibility. We first derive expressions for the mean-square noise and signal voltages. These expressions are then used to normalize the cross-correlation function in terms of noise temperatures.

As indicated in previous work (Ref. 1), the voltage due to an extended source recorded at antenna j may be represented as the sum of a signal term and a noise term as follows:

$$V_j(t) = \int_{\hat{\mathbf{k}}} \int_0^\infty G_j(y_j)\,\kappa_j\,A(\hat{\mathbf{k}},\omega)\,e^{i\alpha_j}\,d\omega\,d\Omega + \text{c.c.} + \int_0^\infty G_j(y_j)\,H_j(\omega)\,e^{i\theta_j}\,d\omega + \text{c.c.} \tag{1}$$

where

$$y_j = \omega(1 - \hat{\mathbf{k}}\cdot\mathbf{x}_j/c)$$

$$\alpha_j = (\omega - \omega_j)\,t - \omega\hat{\mathbf{k}}\cdot\dot{\mathbf{x}}_j/c - \omega\tau_j + \phi_j$$

$$\theta_j = (\omega - \omega_j)\,t - \omega\tau_j + \phi_j$$

The quantity $A(\hat{\mathbf{k}},\omega)$ is the Fourier amplitude of the wave received from direction $\hat{\mathbf{k}}$ at frequency ω. Note that the signal term consists of a superposition of waves received from all parts of the source ($\hat{\mathbf{k}}$ integral) over all allowed frequencies (ω integral). The quantity $d\Omega$ represents a differential solid angle centered at $\hat{\mathbf{k}}$. The function G_j is the effective bandpass filter with an argument y_j that accounts for doppler shifting. The quantity ω_j is the effective mixing frequency, ϕ_j is the instrumental phase shift, and τ_j is the instrumental delay. The vector $\mathbf{x}_j$ is the position of station j, and c is the speed of light. We have assumed the antenna pattern is large compared to the source size and may therefore be neglected. More discussion of Eq. (1) is given in Ref. 1. This expression differs from the reference in two ways. First, the noise term has been represented in terms of its frequency components $H_j(\omega)$. In this representation, it has been assumed, without loss of generality, that all instrumental noise is effectively added at the first stage of amplification. This noise term will also include all background radio noise. Second, a factor κ_j has been included in order to separately account for antenna factors such as aperture and efficiency in the conversion from electrical field to voltage.

In order to calculate the mean-square voltages, we must know the average values of random products such as $A(k,\omega)\,A^*(k',\omega')$. If we assume that the source is completely incoherent, an ensemble average of the signal components is given by the expressions (Ref. 1)

$$\left.\begin{aligned} \langle A(\hat{\mathbf{k}},\omega)\,A^*(\hat{\mathbf{k}}',\omega')\rangle &= S_D(\hat{\mathbf{k}},\omega)\,\delta(\omega-\omega')\,\delta(\hat{\mathbf{k}}-\hat{\mathbf{k}}') \\ \langle A(\hat{\mathbf{k}},\omega)\,A(\hat{\mathbf{k}}',\omega')\rangle &= 0 \qquad \text{for } \omega \text{ and } \omega' > 0 \end{aligned}\right\} \tag{2}$$

where $S_D(\hat{\mathbf{k}},\omega)$ is the spectral power from direction $\hat{\mathbf{k}}$ and $\delta(z)$ is the Dirac delta function. That is, radio waves emitted by different areas of the source are uncorrelated. Furthermore, noise waves emitted by a given area of the source are stationary and therefore possess uncorrelated frequency components (Ref. 1). We will also assume that the system noise is stationary, so that

$$\left.\begin{aligned} \langle H_j(\omega)\,H_j^*(\omega')\rangle &= N_j(\omega)\,\delta(\omega-\omega') \\ \langle H_j(\omega)\,H_j(\omega')\rangle &= 0 \qquad \text{for } \omega' \text{ and } \omega > 0 \end{aligned}\right\} \tag{3}$$

where $N_j(\omega)$ is the power spectrum of the system noise at station j.

Under these assumptions, it is readily shown that an ensemble average of the square of the voltage is given by the expression

$$\langle V_j^2\rangle = \langle V_{S_j}^2\rangle + \langle V_{N_j}^2\rangle \tag{4}$$

where

$$\langle V_{S_j}^2\rangle = 2\int_0^\infty \int_{\hat{\mathbf{k}}} |G_j(y_j)|^2 \kappa_j^2 S_D(\hat{\mathbf{k}},\omega)\, d\Omega\, d\omega \tag{5}$$

and

$$\langle V_{N_j}^2\rangle = 2\int_0^\infty |G_j(\omega)|^2 N_j(\omega)\, d\omega \tag{6}$$

In analogy with the derivation of the cross-correlation function in Ref. 1, the cross terms between uncorrelated components have disappeared. As one would expect, the average signal power is given by the power received in the passband (ω integral) from all parts of the source ($\hat{\mathbf{k}}$ integral).

We may define the total spectral power of the source as the integral

$$S_p(\omega) = \int_{\hat{\mathbf{k}}} S_D(\hat{\mathbf{k}},\omega)\, d\Omega \tag{7}$$

so that the mean-square signal becomes

$$\langle V_{S_j}^2\rangle = 2\int_0^\infty |G_j(\widetilde{y}_j)|^2 \kappa_j^2 S_p(\omega)\, d\omega \tag{8}$$

We have neglected the insignificant doppler shift variations (<0.01 Hz) in y_j across the source and have chosen the source center to evaluate the doppler term $\hat{\mathbf{k}}\cdot\dot{\mathbf{x}}_j$ in the argument of the bandpass filter.

If both stations possess a square effective bandpass of width W and height $|G_j|$, and if the power spectra of the signal S_p and noise N_j are flat in the region of the bandpass, we obtain

$$\langle V_{S_j}^2\rangle = 4\pi |G_j|^2 \kappa_j^2 S_p W \tag{9}$$

$$\langle V_{N_j}^2\rangle = 4\pi |G_j|^2 N_j W \tag{10}$$

The analog cross-correlation function for a completely incoherent, extended source is given by the expression (Ref. 1)

$$\langle V_1(t)\, V_2(t+\tau_m)\rangle = \exp\{i[(\omega_2-\omega_1)t + \omega_2\tau_m + \phi]\}$$
$$\times \int_0^\infty R(u,v,\omega)\, G_1(\widetilde{y}_1)\, G_2^*(\widetilde{y}_2) \exp(i\omega\Delta\tau)\, d\omega + \text{c.c.} \tag{11}$$

where

$$\widetilde{y}_1 = \omega(1 - \hat{\mathbf{k}}_a\cdot\dot{\mathbf{x}}_1/c)$$

$$\widetilde{y}_2 = \omega(1 - \hat{\mathbf{k}}_a\cdot\dot{\mathbf{x}}_2/c)$$

$$\Delta\tau = \tau_g + \tau_e - \tau_m$$

In addition, the brightness transform is defined by the expression

$$R(u,v,\omega) \equiv \int_{-\infty}^{-\infty}\int_{-\infty}^{\infty} S_D(\beta,\gamma,\omega) \exp\{2\pi i[u(\beta-\beta_a) + v(\gamma-\gamma_a)]\}\, d\beta\, d\gamma \tag{12}$$

where

$$u \equiv \left.\frac{\partial\hat{\mathbf{k}}}{\partial\beta}\right|_a \cdot \mathbf{B}_r/\lambda$$

$$v \equiv \left.\frac{\partial\hat{\mathbf{k}}}{\partial\gamma}\right|_a \cdot \mathbf{B}_r/\lambda$$

$$\lambda = 2\pi c/\omega$$

The brightness transform determines the self-interference of the extended source by summing the point-source interferometer response over the total area of the source, using the source center as a zero-phase reference. The variables β and γ are two direction parameters chosen to describe the direction vector $\hat{\mathbf{k}}$. (In VLBI work, γ is usually declination δ, while β is right ascension multiplied times the cosine of the declination of the source center or $\alpha\cos\delta_a$.) The subscript a on γ, β, and $\hat{\mathbf{k}}$ refers to the source center where $\hat{\mathbf{k}}_a = \hat{\mathbf{k}}(\gamma_a,\beta_a)$. The quantity τ_g is the geometric delay, τ_e is the instrumental delay, and τ_m is the model delay. In these expressions, the following factors are evaluated at the center of the source $\hat{\mathbf{k}}_a$—the geometric time delay τ_g; the $\hat{\mathbf{k}}$ partials in u, v; and the doppler shifts in $\widetilde{y}_j$. The vector $\mathbf{B}_r$ is the retarded baseline, and λ is the RF wavelength. (For more detail concerning definitions and the derivation of this result, see Ref. 1.)

Note that the brightness transform for a "zero-length" baseline gives

$$R(0,0,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_D(\beta,\gamma,\omega)\, d\Omega$$
$$= S_p(\omega) \tag{13}$$

where S_p is the total spectral power of the source. The fringe visibility, which is defined by the expression

$$\gamma_v(u, v, \omega) = \frac{|R(u, v, \omega)|}{R(0, 0, \omega)} = \frac{|R(u, v, \omega)|}{S_p(\omega)} \tag{14}$$

measures the "correlated flux" in terms of the total flux. That is, the fringe visibility is the fractional amplitude that remains after self-interference. Note that γ_v equals one for a point source.

For unsymmetrical brightness distributions $S_D(\beta, \gamma, \omega)$, one must be careful when assigning an effective source center. For example, consider a source that consists of a broad diffuse disk-shaped background with a point-source at one edge. For a short baseline with weak resolution, the effective center will be approximately equal to the centroid of the total brightness distribution. For long baselines that totally resolve the diffuse component, the effective source center becomes the point-source location. In present VLBI measurements, this effect is generally not observed since instrumental and transmission media uncertainties in time delay measurements lead to source location errors that are several times greater than the interferometer resolution. For example, typical source location errors are of the order of 0.01 arc sec, while resolution is of the order of 0.001 arc sec for intercontinental baselines.

In general, both the magnitude and phase of the brightness transform depend on u, v, and ω. In the following work, we will assume that all parts of the source emit a flat power spectrum in the region of the passband. This means that the explicit frequency variation in $R(u, v, \omega)$ may be neglected in the ω integration in Eq. (11) if we make the explicit frequency ω equal to the bandpass center. In addition to this explicit frequency dependence, the variables u and v depend implicitly on ω. In the following derivation, we will assume this implicit frequency dependence is negligible over the passband for both the amplitude and phase of the brightness transform. That is, we will assume that R may be removed from the integral if we evaluate ω, u, and v at ω_0, the bandpass center frequency. The approximation is justified by the fact that the frequency typically changes about one part in 10^3 or 10^4 in integrating over the bandpass. For most brightness distributions, this approximation is very accurate. For example, in the case of diffuse, yet compact distributions, it can be shown that the fractional amplitude change will be of the order of 10^{-3} and the phase change of the order of 0.0001 cycle if the fractional frequency change is 10^{-3}. Both of these changes may presently be neglected in the ω integral.

If we express the brightness transform in terms of its real and imaginary parts,

$$R(u, v, \omega) = |R(u, v, \omega)| \exp[i\phi_R(u, v, \omega)] \tag{15}$$

the cross-correlation function becomes

$$\langle V_1(t) V_2(t + \tau_m)\rangle = \exp\{i[(\omega_2 - \omega_1)t + \omega_2\tau_m + \phi + \phi_R]\}$$

$$\times |R(u_0, v_0, \omega_0)| \int_0^\infty G_1(\widetilde{y}_1) G_2(\widetilde{y}_2) \exp(i\omega\Delta\tau)\, d\omega \tag{16}$$

where we have evaluated R and ϕ_R at the bandpass center and removed them from the integral. If we now assume that both stations possess a square bandpass filter of height $|G_j|$ and width W, we can perform the frequency integration in Eq. (16) to obtain (Ref. 1)

$$\langle V_1(t) V_2(t + \tau_m)\rangle =$$

$$4\pi\kappa_1\kappa_2 |G_1| |G_2| \gamma_v S_p W_D \frac{\sin \pi W_D \Delta\tau}{\pi W_D \Delta\tau} \cos\phi_f \tag{17}$$

where

$$\phi_f \equiv (\omega_2 - \omega_1)t + \omega_2\tau_m + \omega_0\Delta\tau + \phi + \phi_R$$

In this expression, W_D is the bandpass overlap after doppler shifting. The bandpass center ω_0 is the centroid of the bandpass product. In addition, $|R|$, the magnitude of the brightness transform, has been replaced by the fringe visibility γ_v and the total power spectrum S_p as indicated in Eq. (14). Both γ_v and S_p are evaluated at the bandpass center (ω_0, u_0, v_0).

The normalized cross-correlation function will be defined by the expression

$$r(t, \tau_m) = \frac{\langle V_1(t) V_2(t + \tau_m)\rangle}{\sqrt{\langle V_1^2\rangle\langle V_2^2\rangle}} \tag{18}$$

With this definition and Eqs. (9), (10), and (17), the normalized cross-correlation function becomes

$$r(t, \tau_m) = \gamma_v \sqrt{\frac{\langle V_{S_1}^2\rangle\langle V_{S_2}^2\rangle}{\langle V_1^2\rangle\langle V_2^2\rangle}} \frac{W_D}{W} \frac{\sin \pi W_D \Delta\tau}{\pi W_D \Delta\tau} \cos\phi_f \tag{19}$$

The last expression may be simplified by the concept of noise temperature. The system temperature (Ref. 3) for antenna j will satisfy the relation

$$T_j = \frac{p_j}{W}\langle V_j^2\rangle \tag{20}$$

while the signal temperature will be given by

$$T_{S_j} = \frac{p_j}{W} \langle V_{S_j}^2 \rangle \tag{21}$$

The quantity p_j is a proportionality constant whose magnitude will depend on the amplifier gains at station j. The actual value for p_j will not be of importance, since we are only interested in ratios of noise power and signal power. With the aid of Eqs. (20) and (21), the normalized cross-correlation function becomes

$$r(t, \tau_m) = \gamma_v \sqrt{\frac{T_{S_1} T_{S_2}}{T_1 T_2}} \frac{W_D}{W} \frac{\sin \pi W_D \Delta\tau}{\pi W_D \Delta\tau} \cos \phi_f \tag{22}$$

where

$$\phi_f = (\omega_2 - \omega_1) t + \omega_2 \tau_m + \omega_0 \Delta\tau + \phi + \phi_R$$

Thus, the normalized interferometer response to an extended source consists of the following factors. The geometric mean of the noise temperature ratios accounts for signal-to-noise factors. The bandpass overlap factor, W_D/W, accounts for power lost due to imperfect passband alignment. As discussed in Ref. 1, the $(\sin x)/x$ delay function indicates the accuracy with which the two signals have been aligned and peaks for zero delay error ($\Delta\tau = 0$). The fast fringes, $\cos \phi_f$, express the average overall phase behavior of the cross-correlated signals. The fringe visibility γ_v accounts for power lost due to self-interference of the extended sources.

Techniques for measuring system temperature are routinely employed at DSN stations. Note that the system temperatures, T_1 and T_2, in Eq. (20) are a sum of the instrumental noise temperature, the background radio noise temperature, and the total radio signal T_{s_j}. DSN instrumental noise temperatures are typically between 18 and 40°K and are due mainly to receiver noise. Background radio noise depends on sky temperature and elevation angle, but generally is less than 30°K for DSN antennas. Recent experimental results indicate that approximate values for signal temperature may be calculated by means of the expression

$$T_{s_j} \approx 0.0002\, S A_j \tag{23}$$

where T_{s_j} is the signal temperature in °K, S is the total source strength in flux units,[1] and A_j is the area of antenna j in meters. However, exact values for signal temperature may deviate from this expression because of differences in antenna and receiver efficiencies. The signal temperature for a 1-flux-unit source at DSS 14 (64-m-antenna) is approximately 0.65°K at S-band.

[1] 1 flux unit = 10^{-26} W/m²Hz.

III. Digital Cross-Correlation Function

In this section, an expression is derived for the cross-correlation function produced by infinitely clipped signals. The derivation is an application of a technique devised by Van Vleck and Middleton (Ref. 2) to analyze the *autocorrelation* of clipped noise. This section shows their derivation can be applied to the *cross-correlation* problem found in VLBI work.

Define the normalized voltage $X(t)$ for station 1 by the relation

$$X(t) = \frac{V_1(t)}{\sqrt{\langle V_1^2 \rangle}} \tag{24}$$

and the normalized voltage $Y(t)$ for station 2 by the relation

$$Y(t) = \frac{V_2(t)}{\sqrt{\langle V_2^2 \rangle}} \tag{25}$$

The normalized cross-correlation function for analog signals is then given by the expression

$$\begin{aligned} r(t, \tau_m) &\equiv \frac{\langle V_1(t) V_2(t + \tau_m) \rangle}{\sqrt{\langle V_1^2 \rangle \langle V_2^2 \rangle}} \\ &= \langle X(t) Y(t + \tau_m) \rangle \end{aligned} \tag{26}$$

As indicated in Eq. (1), the voltage at each station may be represented as the sum of integrals of random noise and signal components. Suppose that the components $A(\hat{k}, \omega)$ and $H_j(\omega)$ are normally distributed. Since the voltages are then a linear combination of normally distributed random variables, their joint probability distribution (Ref. 2) will be given by the following normalized bivariate Gaussian distribution:

$$P(X, Y) = \frac{1}{2\pi (1 - r^2)^{1/2}} \exp\left[-\frac{(X^2 + Y^2 - 2rXY)}{2(1 - r^2)} \right] \tag{27}$$

where, for conciseness, we have used the abbreviated notation

$$X = X(t), \qquad Y = Y(t + \tau_m)$$

and

$$r = r(t, \tau_m) = \langle XY \rangle$$

Suppose that, for a particular recording system, the voltage signal is subjected to amplitude distortion so that the recorded amplitude is $f(X)$ instead of X. If this recording system is deployed at both antennas, the cross-correlation function will be given by the expression

$$R(t, \tau_m) = \langle f(X) f(Y) \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(X) f(Y) P(X, Y)\, dX dY \tag{28}$$

In the case of extreme clipping, we have the distortion function

$$\begin{aligned} f(X) = f_c(X) &= +1, \qquad X > 0 \\ &= -1, \qquad X < 0 \end{aligned} \tag{29}$$

For this distortion, Eq. (28) may be integrated (Ref. 2) to give the digital cross-correlation function r_c.

$$\begin{aligned} r_c(t, \tau_m) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_c(X) f_c(Y) P(X, Y)\, dX dY \\ &= \frac{2}{\pi} \sin^{-1}[r(t, \tau_m)] \end{aligned} \tag{30}$$

Thus, after defining the content, statistics, and normalization of the voltage signals, the derivation of the digital cross-correlation function duplicates the technique employed by Van Vleck and Middleton (Ref. 2). However, in the present work, the correlation functions depend on time in addition to the model delay. The appearance of a time dependence is due to the fact that the VLBI procedure is a cross-correlation process, rather than the autocorrelation process in Ref. 2. Furthermore, the voltage signals and the normalization process involve both the radio signal and the additive noise. Note that this result, Eq. (30), is quite general, since it only assumes that the signals, X and Y, are sums of Gaussian random variables. In particular, the result is valid for both extended sources and point sources.

In particular cases, the digital cross-correlation function assumes a simpler form. First, suppose that the radio signals are very weak ($T_{S_j} << T_j$) or that the correlated flux is very small ($\gamma_v << 1$). (In typical VLBI work, the amplitude of the normalized cross-correlation function falls in the range 0.1 to 0.001.) In either case, the normalized cross-correlation function r is small compared to one, so that Eqs. (22) and (30) give

$$\begin{aligned} r_c(t, \tau_m) &\approx \frac{2}{\pi} r(t, \tau_m) \\ &= \frac{2}{\pi} \gamma_v \sqrt{\frac{T_{S_1} T_{S_2}}{T_1 T_2}}\, \frac{W_D}{W}\, \frac{\sin \pi W_D \Delta\tau}{\pi W_D \Delta\tau} \cos \phi_f \end{aligned} \tag{31}$$

Thus, in the case of small correlated amplitude, the normalized digital cross-correlation function is identical to the sinusoidal analog cross-correlation function, except for a loss of $2/\pi$ in amplitude. Schematic examples of the normalized analog and digital cross-correlation functions are shown in Fig. 1a. In this and following examples, we utilize the fact that the fringe frequency $\dot{\phi}_f$ is almost constant over small time intervals.

In the limit of a very strong point source, the system temperature is equal to the signal temperature ($T_j = T_{s_j}$). If, in addition, the bandpass alignment is perfect ($W_D = W$) and the model delay is very accurate ($\Delta\tau << 1/W$), the analog cross-correlation function, Eq. (22), becomes

$$r(t, \tau_m) = \sin \lambda_p \tag{32}$$

where

$$\lambda_p = \frac{\pi}{2} - (\omega_2 - \omega_1) t - \omega_2 \tau_m - \omega_0 \Delta\tau - \phi - \phi_R$$

The normalized digital cross-correlation function for strong signals then becomes

$$\begin{aligned} r_c(t, \tau_m) &= \frac{2}{\pi} \sin^{-1}[r(t, \tau_m)] \\ &= \frac{2}{\pi} \tilde{\lambda}_p \end{aligned} \tag{33}$$

where

$$\tilde{\lambda}_p = \sin^{-1}[\sin \lambda_p] \tag{34}$$

In the expression for $\tilde{\lambda}_p$, the sine inversion ambiguity is uniquely resolved as follows. Since the digital cross-correlation function r_c must be less than one in magnitude, $\tilde{\lambda}_p$ must lie between $-\pi/2$ and $+\pi/2$. Thus, given a value for λ_p, only one value for $\tilde{\lambda}_p$ is less than $\pi/2$ in magnitude and satisfies the equation $\sin \lambda_p = \sin \tilde{\lambda}_p$. A schematic plot of the analog cross-correlation function for a very strong point source is shown in Fig. 2a, while the corresponding digital cross-correlation function is shown in Fig. 3a. Note that the digital cross-correlation function consists of a sawtooth curve rather than the sinusoidal function associated with analog recording. However, the extrema and zero crossings of the digital system occur at the same time points found with the analog system. For this reason, the frequency characteristics of the digital cross-correlation function are the same as those found in the analog case if we neglect harmonics higher than the first.

IV. Signal-to-Noise Analysis

In this section, expressions are derived for the S/N ratios associated with the cross-correlation functions produced by both analog and digital recording systems. This S/N analysis is valid for both point sources and extended sources.

In the case of analog recording, the rms noise on the cross-correlation function is given by the expression

$$\sigma_r^2 = \langle (XY - r)^2 \rangle = \iint (XY - r)^2 P(X, Y)\, dXdY$$

$$= \iint X^2Y^2 P(X, Y)\, dXdY - r^2 \qquad (35)$$

where $P(X, Y)$ is given by Eq. (27). The last integration is assisted by the transformation

$$X' = X$$

$$Y' = \frac{(Y - rX)}{(1 - r^2)^{1/2}} \qquad (36)$$

which gives

$$\sigma_r = \sqrt{1 + r^2} \qquad (37)$$

In the case of digital recording, the rms noise on the cross-correlation function is given by

$$\sigma_{r_c}^2 = \langle [f_c(X) f_c(Y) - r_c]^2 \rangle$$

$$= \iint [f_c(X) f_c(Y) - r_c]^2 P(X, Y)\, dXdY \qquad (38)$$

which is easily integrated to give

$$\sigma_{r_c} = \sqrt{1 - r_c^2} \qquad (39)$$

The S/N ratio for the analog case is therefore given by the expression

$$\left.\frac{S}{N}\right|_a = \frac{r}{\sigma_r} = \frac{r}{\sqrt{1 + r^2}} \qquad (40)$$

where r is the normalized analog cross-correlation function. For digital recording, the S/N ratio is given by

$$\left.\frac{S}{N}\right|_c = \frac{r_c}{\sigma_{r_c}} = \frac{r_c}{\sqrt{1 - r_c^2}} \qquad (41)$$

where

$$r_c = \frac{2}{\pi} \sin^{-1} r$$

For small correlated amplitude, the S/N ratios assume simple forms. In this limit, we have $r << 1$ and $r_c << 1$, so that

$$\sigma_{r_c} = \sigma_r \approx 1 \qquad (42)$$

and

$$r_c = \frac{2}{\pi} r \qquad (43)$$

The S/N ratios in Eqs. (40) and (41) become

$$\left.\frac{S}{N}\right|_a = r(t, \tau_m) \qquad (44)$$

for analog recording and

$$\left.\frac{S}{N}\right|_c = \frac{2}{\pi} r(t, \tau_m) \qquad (45)$$

for digital recording. Thus, for small correlated amplitude, one would uniformly lose a factor of $2/\pi$ in signal-to-noise by using digital recording instead of analog recording. This fact has been mentioned in various VLBI papers. Schematic examples of the digital and analog cross-correlation functions and their rms noise are shown in Figs. 1a and 1b for the case of small correlated amplitude.

For the case of very strong point radio sources, plots of the cross-correlation function and its rms noise are shown in Fig. 2 for analog recording and in Fig. 3 for digital recording. Note that, for analog recording, the maximum S/N ratio is $1/\sqrt{2}$ due to self-noise. For digital recording, the S/N ratio goes to infinity at time points of maximum correlation. The zero noise at these points may be explained by considering the autocorrelation of infinitely clipped noise. When the autocorrelation signals are perfectly aligned ($\tau = 0$), the product of the clipped voltages is always $+1$ $[(+1)(+1)$ or $(-1)(-1)]$ with no noise. At points of weak correlation, the digital S/N is $2/\pi$ smaller than the analog S/N.

V. Summary

The analog cross-correlation function is normalized and expressed in terms of noise temperature, fringe visibility, and bandpass overlap for rectangular filters.

By defining the content, statistics, and normalization of the VLBI signals, the Van Vleck derivation is used to determine the digital cross-correlation function in terms of the normalized analog cross-correlation function. For very strong point sources, a digital recording system generates a sawtooth cross-correlation function in place of the sinusoidal function produced by an analog system. For small correlated amplitude, the digital cross-correlation function is identical to the normalized analog cross-correlation function, except for a loss of $2/\pi$ in amplitude.

General expressions for signal-to-noise ratios are derived for the analog and digital cross-correlation functions. When a very strong point source is recorded with a digital system, the S/N ratio is infinite at time points of maximum correlation, while, at points of weak correlation, the S/N ratio is $2/\pi$ smaller than that for the analog case. In addition, the S/N ratio for an analog system is, at most, only $1/\sqrt{2}$ due to self-noise. For small correlated amplitude, the S/N ratio for a digital system is uniformly $2/\pi$ smaller than the analog S/N ratio.

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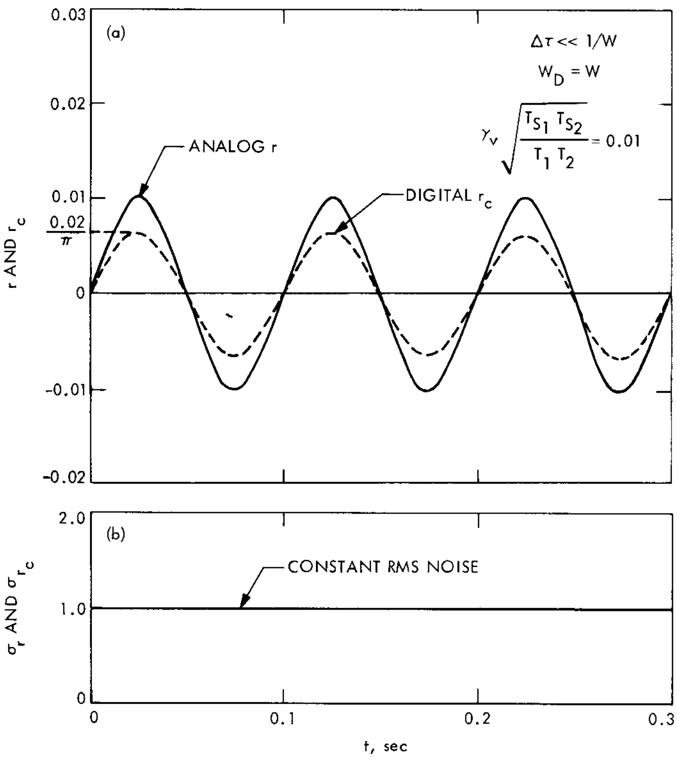

Fig. 1. (a) Normalized analog and digital cross-correlation functions for the case of small correlated amplitude, (b) rms noise on cross-correlation functions of (a)

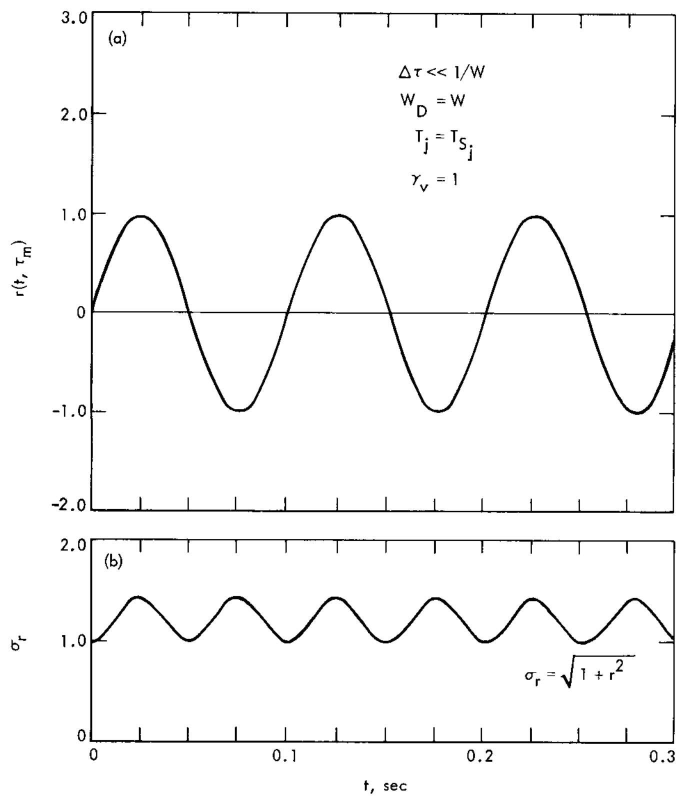

Fig. 2. (a) Normalized analog cross-correlation function for a very strong point source, (b) rms noise on cross-correlation function of (a)

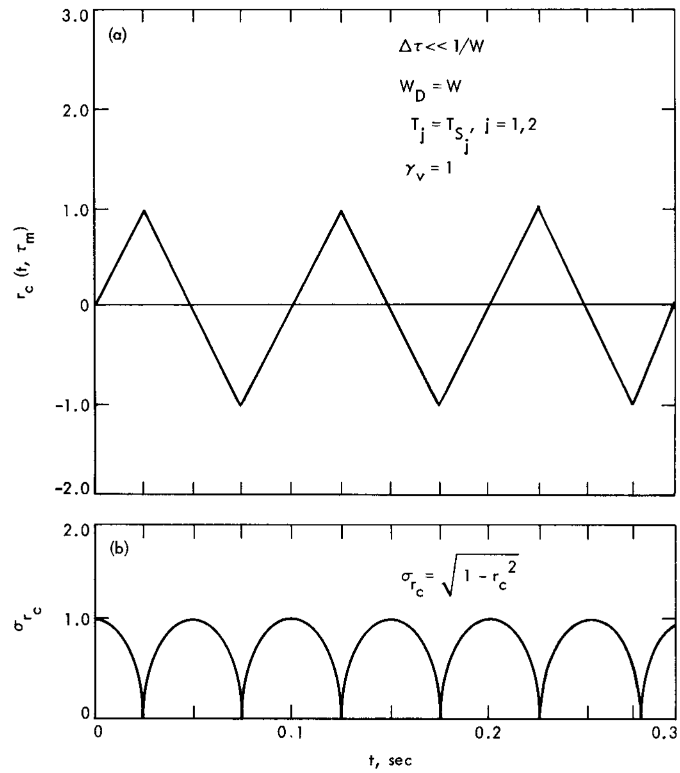

Fig. 3. (a) Normalized digital cross-correlation function for a very strong point source, (b) rms noise on cross-correlation function of (a)