Accuracy of the Signal-to-Noise Ratio Estimator

J. R. Lesh
DSIF Operations Section

In this article the effects of external and internal noise, finite sample size, and transition estimation errors are included in an analysis of the signal-to-noise ratio estimator used in the symbol synchronizer assembly. Expressions for the estimator mean and variance are developed, from which their dependence on the above effects are determined. The results of this study show that the estimator mean depends almost entirely on the external and internal signal-to-noise ratios while the estimator variance depends almost exclusively on the sample size.

I. Introduction

The statistics of the signal-to-noise ratio (SNR) estimation algorithm have been the target of several noteworthy analyses over the past eight years. The first of these was performed by Gilchriest (Ref. 1) in 1964 and resulted in an expression for the probability density function for the estimator. Gilchriest, however, did not take into consideration the tails of the noise distribution, which rendered his result valid for only high signal-to-noise ratios. Two years later Boyd (Refs. 2 and 3) extended Gilchriest's result to a more computable expression and included some additional error sources, but he still retained Gilchriest's basic assumption.

In 1967 Layland (Ref. 4) derived the expression for the mean SNR estimate with the effects of the noise distribution tails included. From this expression he found a significant bias in the estimator at low signal-to-noise ratios from which a set of correction expressions were determined. Layland's result, however, disregarded the effects of symbol transition estimation errors and was only true asymptotically with the sample size. Layland later revised his result (Ref. 5) to include the effects of transition estimation errors, but his resulting expressions (even when a uniform error distribution was assumed) required numerical integration for evaluation.

With this history in mind we shall develop in this article the expressions for the mean and variance of the SNR estimator used in conjunction with the Symbol Synchronizer Assembly (SSA) with the effects of (1) input noise distribution tails, (2) finite sample size, (3) gaussian distributed transition estimation errors, (4) quantization errors, and (5) internal equipment noise all included. Upon evaluation of these equations we will find that the estimator mean depends heavily on the input noise and internally generated noise but is relatively insensitive to changes in the sample size. The variance of the estimator, on the other hand, will be found to be strongly dependent on the sample size and relatively independent of the noise sources.

Before proceeding it is necessary to make a comment regarding notation. In this analysis extensive use of the error function and its complement is necessary. However, one often finds in the literature two definitions of the error function. For this reason the error function recognized by the National Bureau of Standards will be denoted by $\widetilde{\mathrm{erf}}$ and is defined by

$$\widetilde{\mathrm{erf}}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

The other error function (sometimes called the Q-function) will not contain the tilde and is defined by

$$\widetilde{\mathrm{erf}}\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2}\,dt$$

The associated complementary error functions will similarly be denoted.

II. Mean and Variance of the SNR Estimator

Consider the model of the SSA and estimator algorithm shown in Fig. 1. The input data signal $V(t)$ is a binary signal taking on either of the values $+V$ or $-V$ in the intervals t_k to t_{k+1}, $k = 1, 2, \cdots$. This signal is corrupted by additive white gaussian noise $n(t)$ having a zero mean and two-sided spectral density $N_0/2$. The Symbol Transition Estimator uses this composite signal to estimate the values of t_k and t_{k+1}. Using these estimates ($\hat{t}_k$ and $\hat{t}_{k+1}$) the received signal is integrated over the expected symbol duration. After removing the algebraic sign from the integral, the resulting quantity is quantized to produce the random sequence y_k. $\{y_k\}$ is assumed to be an independent sequence.

Two noise sequences, the quantizer noise $n_q(k)$ and the internal equipment noise $n_E(k)$ (produced from crosstalk, computational inaccuracies, etc.), are then added to y_k to produce the sequence Z_k. It is assumed that $n_q(k)$ and $n_E(k)$ are independent zero mean white gaussian sequences having variances σ_q^2 and σ_E^2, respectively. Finally, Z_k and its square are transferred to the SNR estimator where the estimate $\hat{R}$ is produced.

Let us for the moment assume that the first four moments of y_k are known. For notational simplicity, define $E\{y_k\} = \mu$ and $E\{(y_k - \mu)^2\} = \sigma^2$. Then we know that

$$E\{Z_k\} = \mu \tag{1}$$

and

$$\mathrm{var}\{Z_k\} = \sigma^2 + \sigma_q^2 + \sigma_E^2 \stackrel{\Delta}{=} \sigma_Z^2 \tag{2}$$

Next, note that the SNR algorithm is the square of the sample mean divided by twice the sample variance. Both the sample mean and variance are random variables depending on the statistics of Z_k. Then, if the sample size (N) is large enough ($N \geqq 20$ should suffice), we can use the Central Limit Theorem to express the estimated SNR by

$$\hat{R} = \frac{(\mu + \sqrt{\sigma_M^2}\,\xi)^2}{2\,(\sigma_Z^2 + \sqrt{\sigma_{SV}^2}\,\psi)} \tag{3}$$

where ξ and ψ are independent zero mean gaussian random variables with unit variances, σ_M^2 is the variance of the sample mean given by

$$\sigma_M^2 = \frac{1}{N}\,\sigma_Z^2 \tag{4}$$

and σ_{SV}^2 is the variance of the sample variance. The variance of a sample variance is derived in Appendix A and with the help of Eqs. (1) and (2) can be expressed as

$$\sigma_{SV}^2 = \frac{1}{N}\,E\{y^4\} - \frac{4}{N}\,E\{y^3\}\,\mu + \frac{3}{N}\,[E\{y^2\}]^2 - \frac{2\,(2N-3)}{N\,(N-1)}\,\sigma^4 + \frac{2}{N-1}\,(\sigma_q^2 + \sigma_E^2)\,(\sigma_q^2 + \sigma_E^2 + 2\sigma^2) \tag{5}$$

where y represents any of the y_k's.

Returning now to Eq. (3) and expanding the denominator in a geometric series gives

$$\hat{R} = \frac{\mu^2}{2\sigma_Z^2}\left(1 + \frac{\sqrt{\sigma_M^2}}{\mu}\,\xi\right)^2\left(1 - \frac{\sqrt{\sigma_{SV}^2}}{\sigma_Z^2}\,\psi + \frac{\sigma_{SV}^2}{\sigma_Z^4}\,\psi^2 - \cdots\right) \tag{6}$$

However, if the sample variance is a reasonably accurate estimator for the variance, then $\sqrt{\sigma_{SV}^2}/\sigma_Z^2 << 1$ so that we can neglect the higher order terms, and

$$\hat{R} = \frac{\mu^2}{2\sigma_Z^2}\left(1 + \frac{\sqrt{\sigma_M^2}}{\mu}\,\xi\right)^2\left(1 - \frac{\sqrt{\sigma_{SV}^2}}{\sigma_Z^2}\,\psi + \frac{\sigma_{SV}^2}{\sigma_Z^4}\,\psi^2\right) \tag{7}$$

Now, by using the independence of ξ and ψ, we have that the mean SNR estimate is

$$E\{\hat{R}\} = \frac{1}{2NA}\,(1 + A)\,(1 + B) \tag{8}$$

where

$$A = \frac{\sigma_M^2}{\mu^2} \tag{9}$$

$$B = \frac{\sigma_{SV}^2}{\sigma_Z^4} \tag{10}$$

Similarly we find that the variance of the estimator is given by

$$\begin{aligned}\text{var}\{\hat{R}\} = \frac{1}{4N^2 A^2} \{&B(1+B)(1+A)^2 \\ &+ 2A(2+A)(1+B)^2 \\ &+ 2AB(2+A)(1+2B)\}\end{aligned} \tag{11}$$

Equations (8) and (11) are the desired equations for the mean and variance of $\hat{R}$ in terms of the expressions given in Eqs. (1), (2), (4), (5), (9), and (10). However, when all of these equations are substituted into Eqs. (8) and (11), we find that the results depend only on N (the sample size), σ_q^2, $\sigma_{\hat{k}}^2$, and the first four moments of y. We will now consider these quantities.

III. The First Moment of *y*

Referring again to Fig. 1, we see that y_k is given by

$$y_k = \left| \int_{\hat{t}_k}^{\hat{t}_{k+1}} [V(t) + n(t)]\, dt \right| \tag{12}$$

We shall assume that $\hat{t}_k$ differs from t_k by an amount Δ, which is zero mean gaussian random variable having a variance σ_Δ^2. Furthermore, if we assume that the transition estimator loop is in lock and has a sufficiently narrow bandwidth, then the duration of the symbol integration will be the symbol period $T\,(T = t_{k+1} - t_k)$. Thus, we can express Eq. (12) as

$$y_k = \left| \int_{t_k+\Delta}^{t_k+\Delta+T} [V(t) + n(t)]\, dt \right| \tag{13}$$

In order to evaluate the mean of y_k, we must condition on the random variable Δ. However, since the sign of Δ will determine which symbols are integrated during the kth estimated interval, we must also condition on $\Delta \geqq 0$ or $\Delta < 0$. Thus, the expected value of y_k can be expressed as

$$\begin{aligned}E\{y_k\} = &\int_0^\infty E\{y_k/\Delta, \Delta \geqq 0\} f_\Delta\, d\Delta \\ &+ \int_{-\infty}^0 E\{y_k/\Delta, \Delta < 0\} f_\Delta\, d\Delta\end{aligned} \tag{14}$$

where f_Δ is the probability density function of Δ and is given by

$$f_\Delta(\Delta) = \frac{1}{\sqrt{2\pi\sigma_\Delta^2}} \exp\left[-\frac{\Delta^2}{2\sigma_\Delta^2}\right] \tag{15}$$

Now by evaluating the symbol integrals under the appropriate errors (Δ) allows Eq. (14) to be expressed as

$$\begin{aligned}E\{y_k\} = &\int_0^\infty E\{|V_k T + (V_{k+1} - V_k)\Delta + n'|/\Delta, \Delta \geqq 0\} f_\Delta\, d\Delta \\ &+ \int_{-\infty}^0 E\{|V_k T + (V_{k-1} - V_k)\Delta + n'|/\Delta, \Delta < 0\} f_\Delta\, d\Delta\end{aligned} \tag{16}$$

where n' is a zero mean gaussian random variable having a variance $N_0 T/2$.

The conditional expectations in the integrands can be further simplified by noting that

$$E\{g(V_k, V_{k+1})/\Delta, \Delta \geqq 0\} = \sum_{V_k} \sum_{V_{k+1}} E\{g(V_k, V_{k+1})/V_k, V_{k+1}, \Delta, \Delta \geqq 0\}\, P\{V_k, V_{k+1}/\Delta, \Delta \geqq 0\} \tag{17}$$

However, $P\{V_k, V_{k+1}/\Delta, \Delta \geqq 0\} = P\{V_k, V_{k+1}\}$ since the transition estimates do not affect the value of the incoming signal. Recalling now that the V_k's are independent and equally likely to be $+V$ or $-V$ allows us to express Eq. (16) as

$$\begin{aligned}E\{y_k\} = &\frac{1}{4}\sqrt{\frac{N_0 T}{2}} \int_0^\infty \sum_{V_k} \sum_{V_{k+1}} E\left\{\left| \text{sgn}(V_k)\sqrt{2R} + \text{sgn}(V_{k+1} - V_k)\frac{2\Delta}{T}\sqrt{2R} + n'' \right| \Big/ V_k, V_{k+1}, \Delta, \Delta \geqq 0\right\} f_\Delta\, d\Delta \\ &+ \frac{1}{4}\sqrt{\frac{N_0 T}{2}} \int_{-\infty}^0 \sum_{V_k} \sum_{V_{k-1}} E\left\{\left| \text{sgn}(V_k)\sqrt{2R} + \text{sgn}(V_{k-1} - V_k)\frac{2\Delta}{T}\sqrt{2R} + n'' \right| \Big/ V_k, V_{k-1}, \Delta, \Delta < 0\right\} f_\Delta\, d\Delta\end{aligned} \tag{18}$$

where R is the actual signal-to-noise ratio $R = V^2T/N_0$, n'' is a unit normal random variable ($\mathcal{N}(0,1)$) and sgn (x) is defined by

$$\operatorname{sgn}(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

In Appendix B it is shown that if x is a unit normal random variable and α is a constant, then

$$E\{|x+\alpha|\} = \frac{2}{\sqrt{2\pi}} \exp\left[-\frac{\alpha^2}{2}\right] + \alpha\,\widetilde{\operatorname{erf}}\left(\frac{\alpha}{\sqrt{2}}\right) \tag{19}$$

Using this relationship the expectations in the integrands of Eq. (18) can be evaluated yielding

$$\begin{aligned} E\{y_k\} = {} & \frac{1}{4}\sqrt{\frac{N_0T}{2}} \int_0^\infty \sum_{V_k}\sum_{V_{k+1}} \Bigg\{ \frac{2\exp\left\{-R\left[\operatorname{sgn}(V_k) + \operatorname{sgn}(V_{k+1}-V_k)\frac{2\Delta}{T}\right]^2\right\}}{\sqrt{2\pi}} \\ & + \sqrt{2R}\left[\operatorname{sgn}(V_k) + \operatorname{sgn}(V_{k+1}-V_k)\frac{2\Delta}{T}\right] \widetilde{\operatorname{erf}}\left(\sqrt{R}\left[\operatorname{sgn}(V_k) + \operatorname{sgn}(V_{k+1}-V_k)\frac{2\Delta}{T}\right]\right)\Bigg\} f_\Delta\, d\Delta \\ & + \frac{1}{4}\sqrt{\frac{N_0T}{2}} \int_{-\infty}^0 \sum_{V_k}\sum_{V_{k-1}} \Bigg\{ \frac{2\exp\left\{-R\left[\operatorname{sgn}(V_k) + \operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]^2\right\}}{\sqrt{2\pi}} \\ & + \sqrt{2R}\left[\operatorname{sgn}(V_k) + \operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right] \widetilde{\operatorname{erf}}\left(\sqrt{R}\left[\operatorname{sgn}(V_k) + \operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]\right)\Bigg\} f_\Delta\, d\Delta \end{aligned} \tag{20}$$

Now, by summing over the possible values of the V_k's we find that the two integrands are equal and the above expression becomes

$$\begin{aligned} E\{y_k\} = {} & \frac{1}{4}\sqrt{\frac{N_0T}{2}} \int_{-\infty}^\infty \Bigg\{ \frac{4}{\sqrt{2\pi}}\exp[-R] + \frac{4}{\sqrt{2\pi}}\exp\left[-R\left(1-\frac{2\Delta}{T}\right)^2\right] + 2\sqrt{2R}\,\widetilde{\operatorname{erf}}(\sqrt{R}) \\ & + 2\sqrt{2R}\left(1-\frac{2\Delta}{T}\right)\widetilde{\operatorname{erf}}\left[\sqrt{R}\left(1-\frac{2\Delta}{T}\right)\right]\Bigg\} f_\Delta\, d\Delta \end{aligned} \tag{21}$$

By straightforward integration we obtain

$$\begin{aligned} E\{y_k\} = {} & \sqrt{\frac{N_0T}{2}} \Bigg\{ \frac{e^{-R}}{\sqrt{2\pi}} + \frac{T\exp\left[\dfrac{-RT^2}{8R\sigma_\Delta^2 + T^2}\right]}{\sqrt{2\pi(8R\sigma_\Delta^2+T^2)}} + \sqrt{\frac{R}{2}}\,\widetilde{\operatorname{erf}}(\sqrt{R}) \\ & + \sqrt{\frac{R}{2}} \int_{-\infty}^\infty \widetilde{\operatorname{erf}}\left[\sqrt{R}\left(1-\frac{2\Delta}{T}\right)\right] f_\Delta\, d\Delta - \frac{\sqrt{2R}}{T}\int_{-\infty}^\infty \Delta\,\widetilde{\operatorname{erf}}\left[\sqrt{R}\left(1-\frac{2\Delta}{T}\right)\right] f_\Delta\, d\Delta \Bigg\} \end{aligned} \tag{22}$$

A closed form for the last integral can be obtained through integration by parts but the first integral is not quite as easy to evaluate. In Appendix C we prove the identity

$$\int_{-\infty}^\infty \frac{\widetilde{\operatorname{erf}}(a+bx)\exp\left[-\dfrac{x^2}{2\sigma^2}\right]}{\sqrt{2\pi\sigma^2}}\, dx = \widetilde{\operatorname{erf}}\left(\frac{a}{\sqrt{2b^2\sigma^2+1}}\right) \tag{23}$$

Using these results we have

$$E\{y_k\} = \sqrt{N_0 T}\left\{\frac{e^{-R}}{2\sqrt{\pi}} + \frac{T \exp\left[\frac{-RT^2}{8R\sigma_\Delta^2 + T^2}\right]}{2\sqrt{\pi(8R\sigma_\Delta^2 + T^2)}} + \frac{\sqrt{R}}{2}\widetilde{\text{erf}}(\sqrt{R})\right.$$

$$\left. + \frac{\sqrt{R}}{2}\widetilde{\text{erf}}\left(\sqrt{\frac{RT^2}{8R\sigma_\Delta^2 + T^2}}\right) + \frac{4R\sigma_\Delta^2}{T\sqrt{\pi(8R\sigma_\Delta^2 + T^2)}}\exp\left[-\frac{RT^2}{8R\sigma_\Delta^2 + T^2}\right]\right\} \tag{24}$$

It is quite often more practical to express these functions in terms of the variance of the SSA loop phase error (σ_ϕ^2) rather than the variance of the timing error (σ_Δ^2). Since

$$\sigma_\Delta^2 = \frac{T^2\sigma_\phi^2}{4\pi^2} \tag{25}$$

and if we simplify notation by using the substitution

$$D = \sqrt{2R\sigma_\phi^2 + \pi^2} \tag{26}$$

then Eq. (24) becomes

$$E\{y_k\} = \sqrt{N_0 T}\left\{\frac{e^{-R}}{2\sqrt{\pi}} + \frac{D\exp\left[-\frac{R\pi^2}{D^2}\right]}{2\pi\sqrt{\pi}} + \frac{\sqrt{R}}{2}\left[\widetilde{\text{erf}}(\sqrt{R}) + \widetilde{\text{erf}}\left(\frac{\pi\sqrt{R}}{D}\right)\right]\right\} \tag{27}$$

Equation (27) is the desired expression for the first moment of y. To verify this equation, note that as $\sigma_\phi^2 \to 0$, then D goes to π. Making this substitution yields

$$\lim_{\sigma_\phi^2 \to 0} E\{y_k\} = \sqrt{N_0 T}\left\{\frac{e^{-R}}{\sqrt{\pi}} + \sqrt{R}\,\widetilde{\text{erf}}(\sqrt{R})\right\}$$

$$= \sqrt{N_0 T}\left\{\frac{e^{-R}}{\sqrt{\pi}} + \sqrt{R}\,[1 - 2\,\text{erfc}(\sqrt{2R})]\right\}$$

which is precisely Layland's result (Ref. 4).

As a further comment, Anderson and Hurd (Ref. 6) have shown that the SSA loop phase error is given by

$$\sigma_\phi^2 = \frac{\pi^2 \omega_L t_S}{2R\,\widetilde{\text{erf}}(\sqrt{R})} \tag{28}$$

where $\omega_L t_S$ is the fractional loop bandwidth. Thus, if we know the value of $\omega_L t_S$, then Eq. (27) depends only on the actual signal-to-noise ratio R and the noise factor $\sqrt{N_0 T}$. However, in the SNR algorithm the factors containing $\sqrt{N_0 T}$ all cancel, leaving the resulting expressions functions of only R.

IV. The Second Moment of *y*

Determination of the second moment is much simpler than the first since the absolute value is absorbed by the square. We have, therefore,

$$E\{y_k^2\} = \int_0^\infty E\left\{\left|\int_{t_k+\Delta}^{t_k+T+\Delta}[V(t) + n(t)]\,dt\right|^2 \Big/ \Delta, \Delta \geqq 0\right\} f_\Delta\,d\Delta$$

$$+ \int_{-\infty}^0 E\left\{\left|\int_{t_k+\Delta}^{t_k+T+\Delta}[V(t) + n(t)]\,dt\right|^2 \Big/ \Delta, \Delta < 0\right\} f_\Delta\,d\Delta \tag{29}$$

which becomes

$$E\{y_k^2\} = \frac{1}{4}\int_0^{\infty} \sum_{V_k}\sum_{V_{k+1}} [V_k T + (V_{k+1} - V_k)\,\Delta]^2 f_\Delta\, d\Delta$$

$$+ \frac{1}{4}\int_{-\infty}^{0} \sum_{V_k}\sum_{V_{k-1}} [V_k T + (V_{k-1} - V_k)\,\Delta]^2 f_\Delta\, d\Delta + \frac{N_0 T}{2} \tag{30}$$

Again we find that the integrands are the same so that

$$E\{y_k^2\} = \int_{-\infty}^{\infty} (2V\Delta^2 - 2V^2 T\Delta + V^2T^2)\, f_\Delta\, d\Delta + \frac{N_0 T}{2}$$

$$= V^2T^2 + 2V^2\sigma_\Delta^2 + \frac{N_0 T}{2} \tag{31}$$

Now, if we use the relationships given in Eq. (25) and the definition of R, we obtain

$$E\{y_k^2\} = N_0 T\left(R + \frac{R\sigma_\phi^2}{2\pi^2} + \frac{1}{2}\right) \tag{32}$$

Again we see that, when σ_ϕ^2 goes to zero, Eq. (32) reduces to Layland's result (Ref. 4).

V. The Third Moment of *y*

Using the same procedure as in *Section III*, we can express the third moment of y_k by

$$E\{y_k^3\} = \frac{1}{4}\left(\sqrt{\frac{N_0T}{2}}\right)^3 \int_0^{\infty} \sum_{V_k}\sum_{V_{k+1}} E\left\{\left|\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k+1} - V_k)\frac{2\Delta}{T}\right]\sqrt{2R} + n''\right|^3 \Big/ \Delta, \Delta \geq 0\right\} f_\Delta\, d\Delta$$

$$+ \frac{1}{4}\left(\sqrt{\frac{N_0T}{2}}\right)^3 \int_{-\infty}^{0} \sum_{V_k}\sum_{V_{k-1}} E\left\{\left|\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k-1} - V_k)\frac{2\Delta}{T}\right]\sqrt{2R} + n''\right|^3 \Big/ \Delta, \Delta < 0\right\} f_\Delta\, d\Delta \tag{33}$$

In Appendix D we use the properties of the characteristic function to prove the identity

$$E\{|x + \alpha|^3\} = \frac{\sqrt{2}\,(\alpha^2 + 2)}{\sqrt{\pi}} \exp\left[-\frac{\alpha^2}{2}\right] + \alpha(\alpha^2 + 3)\,\widetilde{\mathrm{erf}}\left(\frac{\alpha}{\sqrt{2}}\right) \tag{34}$$

whenever x is a unit normal random variable ($\mathcal{N}(0,1)$) and α is a constant. With this result Eq. (33) becomes

$$E\{y_k^3\} = \frac{1}{4}\left(\frac{N_0T}{2}\right)^{3/2} \int_0^{\infty} \sum_{V_k}\sum_{V_{k+1}} \left\{\left(2R\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k+1} - V_k)\frac{2\Delta}{T}\right]^2 + 2\right)\frac{\sqrt{2}}{\sqrt{\pi}}\right.$$

$$\times \exp\left\{-R\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k+1} - V_k)\frac{2\Delta}{T}\right]^2\right\}$$

$$+ \left(2R\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k+1} - V_k)\frac{2\Delta}{T}\right]^2 + 3\right)\left(\sqrt{2R}\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k+1} - V_k)\frac{2\Delta}{T}\right]\right)$$

$$\left. \times\, \widetilde{\mathrm{erf}}\left(\sqrt{R}\left[\mathrm{sgn}\,(V_k) + \mathrm{sgn}\,(V_{k+1} - V_k)\frac{2\Delta}{T}\right]\right)\right\} f_\Delta\, d\Delta$$

$$+\frac{1}{4}\left(\frac{N_0T}{2}\right)^{3/2}\int_{-\infty}^{0}\sum_{V_k}\sum_{V_{k-1}}\Bigg\{\left(2R\left[\operatorname{sgn}(V_k)+\operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]^2+2\right)\frac{\sqrt{2}}{\sqrt{\pi}}$$

$$\times\exp\left\{-R\left[\operatorname{sgn}(V_k)+\operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]^2\right\}$$

$$+\left(2R\left[\operatorname{sgn}(V_k)+\operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]^2+3\right)\left(\sqrt{2R}\left[\operatorname{sgn}(V_k)+\operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]\right)$$

$$\times\widetilde{\operatorname{erf}}\left(\sqrt{R}\left[\operatorname{sgn}(V_k)+\operatorname{sgn}(V_{k-1}-V_k)\frac{2\Delta}{T}\right]\right)\Bigg\}f_\Delta\,d\Delta \tag{35}$$

As before, by carrying out the summations, we find that the integrands are the same, producing

$$E\{y_k^3\}=\left(\frac{N_0T}{2}\right)^{3/2}\Bigg\{\frac{2R+2}{\sqrt{2\pi}}e^{-R}+\frac{(2R+3)\sqrt{R}}{\sqrt{2}}\widetilde{\operatorname{erf}}(\sqrt{R})$$

$$+\frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left(R+1-\frac{4R\Delta}{T}+\frac{4R\Delta^2}{T^2}\right)\exp\left[-R\left(1-\frac{2\Delta}{T}\right)^2\right]f_\Delta\,d\Delta$$

$$+\sqrt{2R}\int_{-\infty}^{\infty}\left[R+\frac{3}{2}-\frac{3}{T}(2R+1)\Delta+\frac{12R\Delta^2}{T^2}-\frac{8R\Delta^3}{T^3}\right]\widetilde{\operatorname{erf}}\left[\sqrt{R}\left(1-\frac{2\Delta}{T}\right)\right]f_\Delta\,d\Delta\Bigg\} \tag{36}$$

The first integral of this expression can be evaluated using straightforward techniques. The second integral involves error functions and is of the form

$$\sum_{n=0}^{3}\int_{-\infty}^{\infty}a_n\Delta^n\,\widetilde{\operatorname{erf}}\left[\sqrt{R}\left(1-\frac{2\Delta}{T}\right)\right]f_\Delta\,d\Delta$$

For $n = 0$, we can use the result of Appendix C. For $n = 1, 2, 3$, we can with some effort use integration by parts. After performing these operations and using Eqs. (25) and (26), we finally arrive at

$$E\{y^3\}=(N_0T)^{3/2}\Bigg\{\frac{(R+1)}{2\sqrt{\pi}}e^{-R}+\frac{(2R+3)\sqrt{R}}{4}\widetilde{\operatorname{erf}}(\sqrt{R})$$

$$+\frac{1}{2}\sqrt{R}\left(R+\frac{3D^2}{2\pi^2}\right)\widetilde{\operatorname{erf}}\left(\frac{\pi\sqrt{R}}{D}\right)+\frac{D\exp\left[-\dfrac{R\pi^2}{D^2}\right]}{2\pi\sqrt{\pi}}\left(R+\frac{D^2}{\pi^2}\right)\Bigg\} \tag{37}$$

VI. The Fourth Moment of y

The fourth moment of y_k is found by the same procedure used to determine the second moment. After conditioning on Δ and evaluating the integrands one obtains

$$E\{y_k^4\}=\int_{-\infty}^{\infty}\Big\{V^4T^4+3V^2T^3N_0+\frac{3N_0^2T^2}{4}$$
$$-2V^2T^2(2V^2T+3N_0)\Delta$$
$$+6V^2T(2V^2T+N_0)\Delta^2$$
$$-16V^4T\Delta^3+8V^4\Delta^4\Big\}f_\Delta\,d\Delta \tag{38}$$

which, after using the previously used substitutions, becomes

$$E\{y_k^4\}=$$
$$N_0^2T^2\left[R^2+3R+\frac{3}{4}+\frac{3R\sigma_\phi^2}{\pi^2}\left(R+\frac{1}{2}\right)+\frac{3R^2\sigma_\phi^4}{2\pi^4}\right] \tag{39}$$

VII. Quantizing and Internal Noise Variances

In this section we shall consider both the quantizing error variance and the internal equipment noise variance.

These are considered together since in practice one would measure the effect of both simultaneously.

The quantizing errors result from the SSA analog-to-digital converters, which are 12-bit converters having a dynamic range of ± 5 volts. The least-significant digit, therefore, corresponds to ± 2.44 mV. However, the nominal integrated symbol voltage is 1.1 volts. Thus, if we let δ_v represent the quantizing step size, then

$$\delta_v = \frac{VT}{450} \tag{40}$$

Now, from Panter (Ref. 7), we know that for equal quantizing intervals the variance of the quantizing error is given by

$$\sigma_q^2 = \frac{(\delta_v)^2}{12} \tag{41}$$

Therefore, the desired quantizing error variance is

$$\sigma_q^2 = \frac{N_0 TR}{2.43 \times 10^6} \tag{42}$$

For the internal noise variance, let us assume that the internal noise causes the equipment to have an internal signal-to-noise ratio of γ dB. Also recall that the signal voltage (after the integrator and quantizer) is VT. Then, if we let $\widetilde{N}_0$ represent the internal noise one-sided spectral density, then

$$\widetilde{N}_0 = V^2 T^2 (10^{-\gamma/10}) \tag{43}$$

However, the desired variance is the two-sided noise density, so that

$$\sigma_E^2 = \frac{\widetilde{N}_0}{2} = \frac{1}{2} N_0 TR (10^{-\gamma/10}) \tag{44}$$

VIII. Evaluation Results

A computer program was written to evaluate the mean (Eq. 8) and the variance (Eq. 11) of the SNR estimator using the appropriate substitutions from *Section II*, the expressions for the first four moments (*Sections III–VI*), and the equations for the internal error variances. The program was designed to compute the mean and variance over the range of input SNRs (R) from -7 dB to $+42$ dB for specified values of sample size (N), fractional loop bandwidth ($\omega_L t_S$), and equipment SNR (γ). By holding any two of these parameters constant, the dependence on the third quantity could be determined.

When the mean of the estimator was evaluated, it was found to depend heavily on both the input and internal signal-to-noise ratios. At low input SNRs, the estimator behaved essentially as predicted by Layland. However, as the input SNR was raised, the estimator became limited by the internal SNR and asymptotically approached that internal quantity. This effect is illustrated in Fig. 2.

The dependence of the mean on the sample size or the loop bandwidth, however, was found to be much less pronounced. As the sample size was varied, the estimator mean was found to be virtually insensitive to these variations except at extremely low input SNRs. When the fractional loop bandwidth was varied, no significant change in the mean was detected until the bandwidth was made much larger than the values presently available in the SSA.

These effects are illustrated in Figs. 3 and 4. Figure 3 shows the manner in which the mean approaches Layland's asymptotic value of -0.363 dB as the sample size increases and the input SNR is fixed at -7 dB. Figure 4 illustrates the decrease in the estimator mean as the fractional loop bandwidth is increased.

Unlike the mean, the estimator variance was found to depend quite heavily on the sample size and was essentially independent of variations in the signal-to-noise ratio or loop bandwidth. Figure 5 illustrates the dependence of the 1-σ estimation error on the sample size. In Figure 6 we see the dependence of the 1-σ error on the input SNR. Note that as the input SNR varies over five orders of magnitude, the 1-σ error varies only about 30%.

IX. Conclusions

In this article the effects of input and internal noise, transition estimation errors, and finite sample size were considered to derive expressions for the mean and variance of the SNR estimation algorithm of the SSA. These quantities were found to depend on the first four moments of the absolute symbol integrals which in turn were derived. When these expressions were evaluated, we found that the estimator mean depended heavily on the input and internal signal-to-noise ratios, but was virtually insensitive to changes in the sample size or fractional loop bandwidth. The estimator variance, on the other hand, was found to be essentially independent of the signal-to-noise ratios and the loop bandwidths and to vary appreciably only with changes in the sample size.

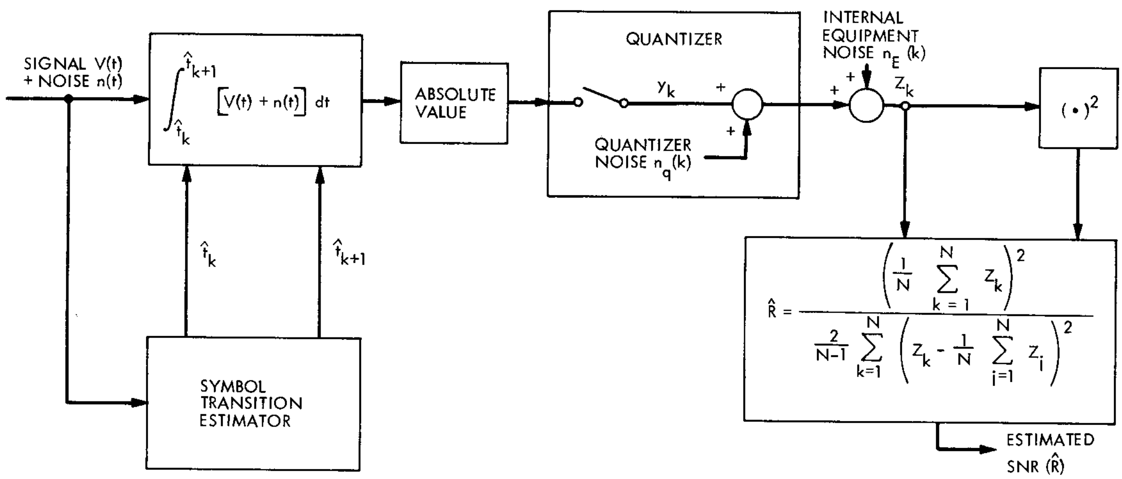

Fig. 1. Model of the SNR estimation process

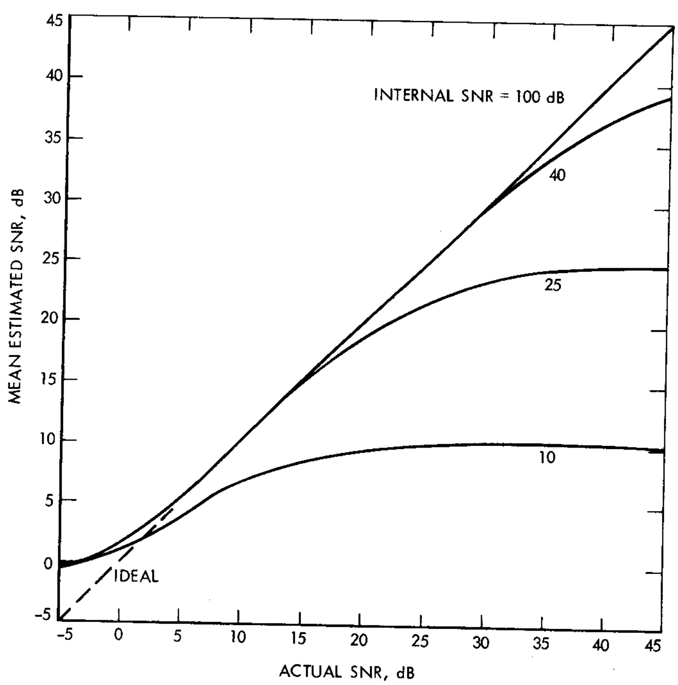

Fig. 2. Mean estimated SNR vs actual SNR

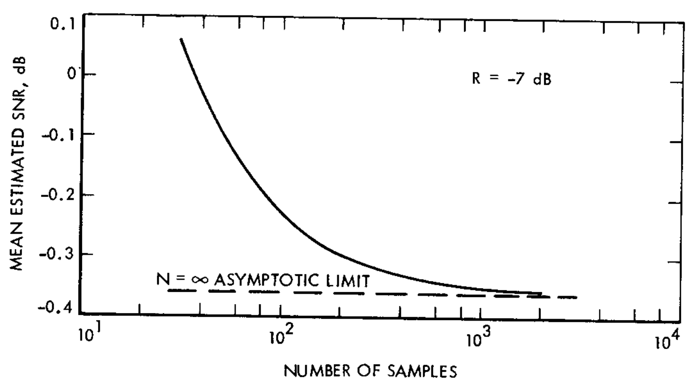

Fig. 3. Dependence of estimator mean on the sample size

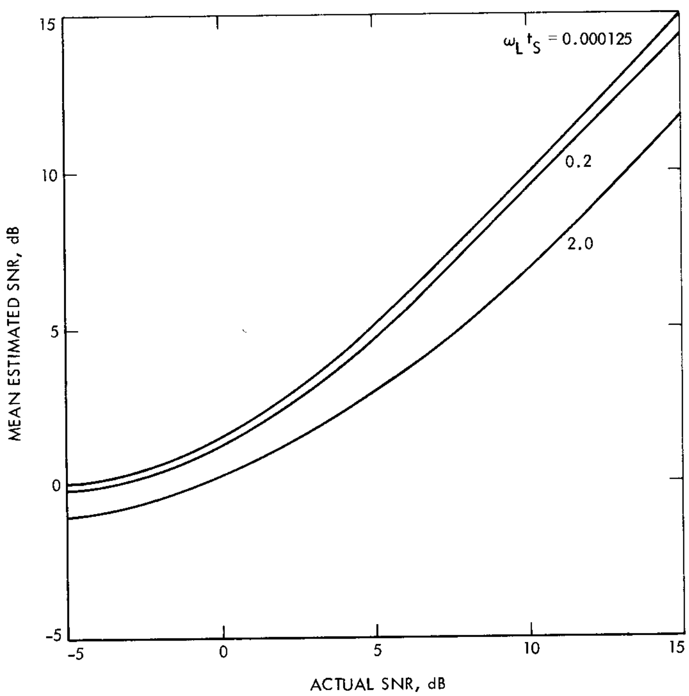

Fig. 4. Dependence of estimator mean on fractional loop bandwidth

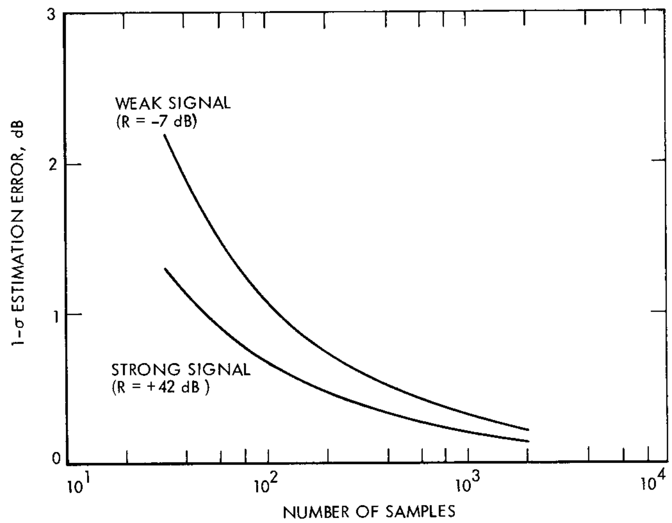

Fig. 5. Dependence of 1-σ estimation error on sample size

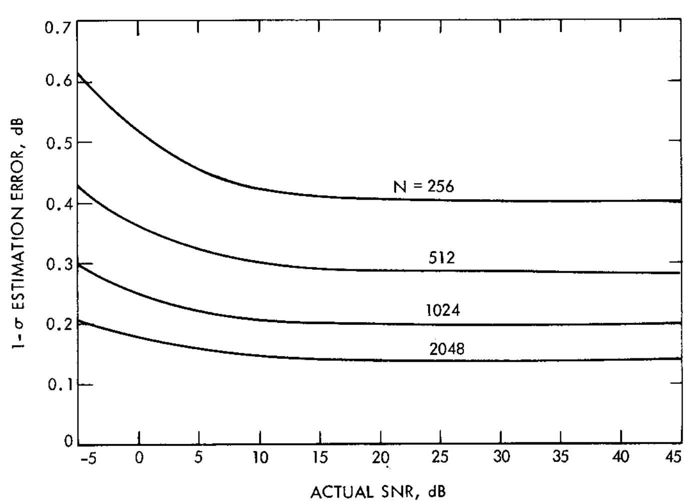

Fig. 6. Dependence of 1-σ estimation error on input SNR for various sample sizes

Appendix A

The Variance of a Sample Variance

Assume we are given a sequence of random variables $\{y_i\}$, which are taken independently from a stationary random process having some arbitrary probability distribution. Assume also that the first four moments of y_i are finite and that the mean and variance of y_i are denoted μ and σ^2, respectively. We form the unbiased variance estimator (W) defined by

$$W = \frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \frac{1}{N}\sum_{j=1}^{N} y_j\right)^2 \qquad \text{(A-1)}$$

where N is the number of samples ($N \geqq 2$). We desire to determine the variance of W.

First we note that since W is an unbiased estimator

$$E\{W\} = \sigma^2 \qquad \text{(A-2)}$$

Now, let us consider

$$E\{W^2\} = E\left\{\left[\frac{1}{N-1}\sum_{i=1}^{N}\left(y_i - \frac{1}{N}\sum_{j=1}^{N} y_j\right)^2\right]^2\right\} \qquad \text{(A-3)}$$

If we expand the inner square, we have

$$E\{W^2\} = \left(\frac{1}{N-1}\right)^2 E\left\{\left(\sum_{i=1}^{N} y_i^2\right)^2 - \frac{2}{N}\sum_{i=1}^{N} y_i^2\left(\sum_{j=1}^{N} y_j\right)^2 + \frac{1}{N^2}\left(\sum_{j=1}^{N} y_j\right)^4\right\} \qquad \text{(A-4)}$$

Now, by expanding the summations and using the independence of the y_i's, the first expectation becomes

$$E\left\{\left(\sum_{i=1}^{N} y_i^2\right)^2\right\} = NE\{y^4\} + N(N-1)[E\{y^2\}]^2 \qquad \text{(A-5)}$$

where y corresponds to any one of the samples. Likewise, the second expectation becomes

$$\begin{aligned} E\left\{\sum_{i=1}^{N} y_i^2\left(\sum_{j=1}^{N} y_j\right)^2\right\} &= NE\{y^4\} + 2N(N-1)E\{y^3\}\mu \\ &\quad + N(N-1)[E\{y^2\}]^2 \\ &\quad + N(N-1)(N-2)E\{y^2\}\mu^2 \end{aligned} \qquad \text{(A-6)}$$

The third expectation is somewhat more complex. However, by considering the fourth power of the summation as the square of the summations squared, we can decompose this expectation into the previously evaluated expectation plus an expectation involving the cross-product terms. The resulting expression is

$$\begin{aligned} E\left\{\left(\sum_{i=1}^{N} y_i\right)^4\right\} &= NE\{y^4\} + 4N(N-1)E\{y^3\}\mu \\ &\quad + 3N(N-1)[E\{y^2\}]^2 \\ &\quad + 6N(N-1)(N-2)E\{y^2\}\mu^2 \\ &\quad + N(N-1)(N-2)(N-3)\mu^4 \end{aligned} \qquad \text{(A-7)}$$

Combining these results and using the definition of the variance yields

$$E\{W^2\} = \frac{E\{y^4\}}{N} - \frac{4E\{y^3\}\mu}{N} + \frac{3[E\{y^2\}]^2}{N} + \frac{(N-2)(N-3)\sigma^4}{N(N-1)} \qquad \text{(A-8)}$$

Finally, we have that the variance of the sample variance is given by

$$\operatorname{var}\{W\} = \frac{1}{N}E\{y^4\} - \frac{4}{N}E\{y^3\}\mu + \frac{3}{N}[E\{y^2\}]^2 - \frac{2(2N-3)\sigma^4}{N(N-1)} \qquad \text{(A-9)}$$

In deriving Eq. (A-9), no properties of the probability distribution of y were used. Consequently, the result will remain valid for any distribution (symmetrical or not) provided that the samples are taken independently.

One additional simplification of Eq. (A-9) can be obtained if one is willing to compute centralized moments instead of ordinary moments. This can be accomplished by making the substitution $y = x + \mu$, where x is a zero mean random variable with the appropriate distribution. After substitution and simplification, one obtains

$$\operatorname{var}\{W\} = \frac{1}{N} E\{x^4\} - \frac{(N-3)}{N(N-1)} \sigma^4 \qquad \text{(A-10)}$$

or equivalently

$$\operatorname{var}\{W\} = \frac{1}{N} E\{(y-\mu)^4\} - \frac{(N-3)}{N(N-1)} \sigma^4 \qquad \text{(A-10a)}$$

As an example of the use of this result let us assume that y is a zero mean gaussian random variable having a variance σ^2. It is a well-known result that for this case the sample variance given by Eq. (A-1) is an $\mathcal{X}$-square distributed random variable with $(N-1)$ degrees of freedom. Since

$$E\{(y-\mu)^4\} = E\{y^4\} = 3\sigma^4 \qquad \text{(A-11)}$$

then we have from Eq. (A-10a)

$$\operatorname{var}\{W\} = \frac{2}{N-1} \sigma^4 \qquad \text{(A-12)}$$

which is precisely the variance of the associated $\mathcal{X}$-square distribution (Ref. 8).

Finally, for the sake of completeness, we should verify that our result (Eq. A-9 or A-10) is indeed a candidate solution for a variance (i.e., it produces a nonnegative variance). This can be easily verified by noting from Eq. (A-10) that

$$N \operatorname{var}\{W\} \geqq E\{x^4\} - \sigma^4 \qquad \text{(A-13)}$$

However, the right-hand side of expression (A-13) is by definition the variance of $\lambda = x^2$. Therefore,

$$\operatorname{var}\{W\} \geqq 0 \qquad \text{(A-14)}$$

Appendix B

Evaluation of $E\{|x+\alpha|\}$

In this appendix we desire to establish the identity

$$E\{|x+\alpha|\} = \frac{2}{\sqrt{2\pi}} \exp\left[-\frac{\alpha^2}{2}\right] + \alpha \, \widetilde{\operatorname{erf}}\left(\frac{\alpha}{\sqrt{2}}\right) \tag{B-1}$$

whenever x is a unit normal ($\mathcal{N}(0,1)$) random variable and α is a constant. To establish this result, let

$$y = |z| \tag{B-2}$$

where $z = x + \alpha$. The probability density function for z is then

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(z-\alpha)^2}{2}\right] \tag{B-3}$$

From Papoulis (Ref. 9) the density function for y is

$$f_Y(y) = [f_Z(y) + f_Z(-y)]\, U(y) \tag{B-4}$$

where $U(y)$ is the unit step function. Then, the expected value of y is expressed by

$$E\{y\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty y \exp\left[-\frac{(y-\alpha)^2}{2}\right] dy + \frac{1}{\sqrt{2\pi}} \int_0^\infty y \exp\left[-\frac{(y+\alpha)^2}{2}\right] dy \tag{B-5}$$

After a change of variables and integrating, we obtain

$$E\{y\} = \frac{2}{\sqrt{2\pi}} \exp\left[-\frac{\alpha^2}{2}\right] + \alpha \operatorname{erf}(\alpha) - \alpha \operatorname{erfc}(\alpha) \tag{B-6}$$

And, finally, with a trivial identity for error functions, we have

$$E\{y\} = E\{|x+\alpha|\} = \frac{2}{\sqrt{2\pi}} \exp\left[-\frac{\alpha^2}{2}\right] + \alpha \, \widetilde{\operatorname{erf}}\left(\frac{\alpha}{\sqrt{2}}\right) \tag{B-7}$$

which is the desired result.

Appendix C

Evaluation of $\int_{-\infty}^{\infty} \widetilde{\mathrm{erf}}\,(a+bx)\,\dfrac{\exp\left[-\dfrac{x^2}{2\sigma^2}\right]}{\sqrt{2\pi\sigma^2}}\,dx$

Consider the integral

$$I = \int_{-\infty}^{\infty} \widetilde{\mathrm{erf}}\,(a+bx)\,\frac{\exp\left[-\dfrac{x^2}{2\sigma^2}\right]}{\sqrt{2\pi\sigma^2}}\,dx \qquad \text{(C-1)}$$

By changing to the nonstandard error function, Eq. (C-1) can be expressed as

$$I = 1 - 2\int_{-\infty}^{\infty} \mathrm{erfc}\,(a\sqrt{2} + b\sqrt{2}\,x)\,\frac{\exp\left[-\dfrac{x^2}{2\sigma^2}\right]}{\sqrt{2\pi\sigma^2}}\,dx \qquad \text{(C-2)}$$

Now, for notational simplicity, let $I = 1 - 2I'$, $a' = a\sqrt{2}$ and $b' = b\sqrt{2}$. Then, using the definition of the complementary error function, we have

$$I' = \frac{1}{2\pi\sqrt{\sigma^2}}\int_{-\infty}^{\infty}\int_{a'+b'x}^{\infty} \exp\left[-\frac{u^2}{2} - \frac{x^2}{2\sigma^2}\right] du\,dx \qquad \text{(C-3)}$$

This expression is a double integration of the exponential function over the (x, u) plane such that x varies over the entire real line and u is restricted to be above the line $u = a' + b'x$ as shown in Fig. C-1. This strongly suggests that a rotation of coordinates could simplify the integration. If we let θ represent the angle between the line $u = a' + b'x$ and the positive x-axis, then we can define a new coordinate system (x', u') by the transformations

$$\left.\begin{aligned} x &= x'\cos\theta - u'\sin\theta \\ u &= x'\sin\theta + u'\cos\theta \end{aligned}\right\} \qquad \text{(C-4)}$$

as shown in Fig. C-2.

Now, recall that if a function $\phi(x, u)$ is to be integrated over a region A and if we define the transformations

$$\left.\begin{aligned} x &= f(x', u') \\ u &= g(x', u') \end{aligned}\right\} \qquad \text{(C-5)}$$

then (Ref. 10)

$$\iint_A \phi(x, u)\,dx\,du = \iint_B \phi\,[f(x', u'), g(x', u')]\,|J|\,dx'\,du' \qquad \text{(C-6)}$$

where $|J|$ is the Jacobian

$$|J| = \det\begin{bmatrix} \dfrac{\partial x}{\partial x'} & \dfrac{\partial x}{\partial u'} \\[2ex] \dfrac{\partial u}{\partial x'} & \dfrac{\partial u}{\partial u'} \end{bmatrix} \qquad \text{(C-7)}$$

and the region B is the region which results from mapping the region A through the transformations $x = f(x', u')$ and $y = g(x', u')$. In our problem (see Fig. C-2)

$$B = \{(x', u') : -\infty \leqq x' \leqq \infty, u' \geqq c\}$$

where c can be determined from the Law of Sines and the fact that

$$\theta = \tan^{-1}(b') \qquad \text{(C-8)}$$

Thus we have that

$$c = \frac{a'}{\sqrt{(b')^2 + 1}} \qquad \text{(C-9)}$$

Using these results and the fact that for a rotation of coordinates the Jacobian equals 1, we can express Eq. (C-3) as

$$I' = \frac{1}{2\pi\sqrt{\sigma^2}}\int_{-\infty}^{\infty}\int_{a'/\sqrt{(b')^2+1}}^{\infty} \exp\left[-\frac{1}{2\sigma^2\,[(b')^2+1]}\{(x')^2\,[(b')^2\sigma^2 + 1] - 2x'u'b'\,(1-\sigma^2) + (u')^2\,[\sigma^2 + (b')^2]\}\right] du'\,dx' \qquad \text{(C-10)}$$

Now, if we interchange the order of integration, complete the square on the variable x' in the exponent, and use the properties of the gaussian density function, we obtain

$$I' = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(b')^2+1}{(b')^2\sigma^2+1}} \int_{a'/\sqrt{(b')^2+1}}^{\infty} \exp\left\{-\frac{1}{2}\left[\frac{(b')^2+1}{(b')^2\sigma^2+1}\right](u')^2\right\} du' \tag{C-11}$$

Now using the substitution

$$\lambda = \sqrt{\frac{(b')^2+1}{(b')^2\sigma^2+1}}\, u' \tag{C-12}$$

we obtain

$$I' = \frac{1}{\sqrt{2\pi}} \int_{a'/\sqrt{(b')^2\sigma^2+1}}^{\infty} \exp\left[-\frac{\lambda^2}{2}\right] d\lambda$$

$$= \operatorname{erfc}\left(\frac{a'}{\sqrt{(b')^2\sigma^2+1}}\right) \tag{C-13}$$

Returning now to Eq. (C-2) with the substitutions for the unprimed values of a and b, we have

$$I = 1 - 2\operatorname{erfc}\left(\frac{a\sqrt{2}}{\sqrt{2b^2\sigma^2+1}}\right) \tag{C-14}$$

so that we arrive at the result

$$I = \int_{-\infty}^{\infty} \widetilde{\operatorname{erf}}\,(a+bx) \frac{\exp\left[-\frac{x^2}{2\sigma^2}\right]}{\sqrt{2\pi\sigma^2}} dx$$

$$= \widetilde{\operatorname{erf}}\left(\frac{a}{\sqrt{2b^2\sigma^2+1}}\right) \tag{C-15}$$

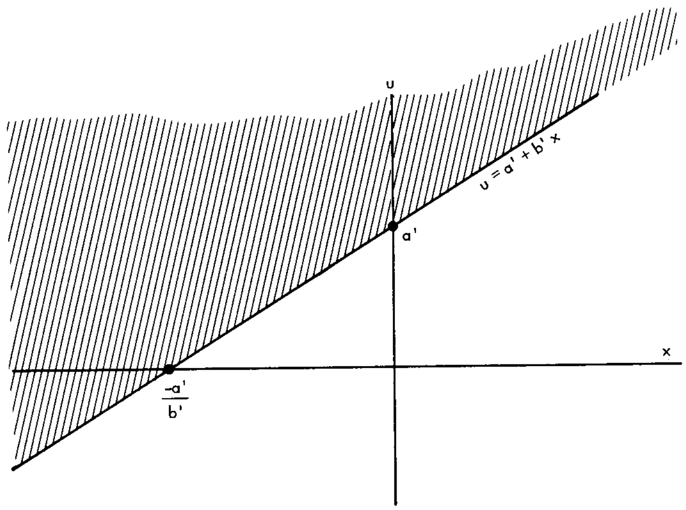

Fig. C-1. Domain of integration for I' (shaded area)

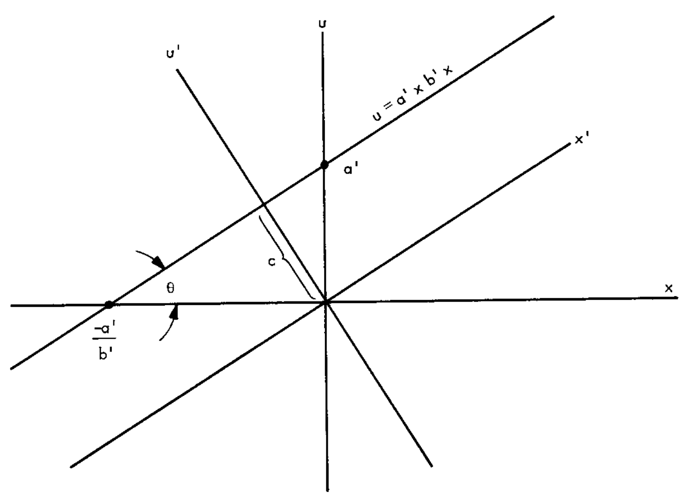

Fig. C-2. Rotated coordinate system

Appendix D

Evaluation of $E\{|x+\mu|^3\}$

Let

$$y = |x + \mu| \tag{D-1}$$

where x is a unit normal random variable ($\mathcal{N}(0, 1)$) and μ is a constant. We wish to compute $E\{y^3\}$.

From Appendix A we found that the probability density function of y was given by

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\left\{\exp\left[-\frac{(y-\mu)^2}{2}\right] + \exp\left[-\frac{(y+\mu)^2}{2}\right]\right\} U(y) \tag{D-2}$$

where $U(y)$ is the unit step function. Then, the characteristic function of y $(\phi_Y(\cdot))$ is given by

$$\begin{aligned}\phi_Y(\omega) &= E\{\exp[i\omega y]\}\\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left[-\frac{(y-\mu)^2}{2}\right](\cos\omega y + i\sin\omega y)\,dy\\ &\quad + \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left[-\frac{(y+\mu)^2}{2}\right](\cos\omega y + i\sin\omega y)\,dy\end{aligned} \tag{D-3}$$

From Erdelyi, et al. (Ref. 11), Eq. (D-3) can be expressed as

$$\begin{aligned}\phi_Y(\omega) = \frac{\exp\left[-\frac{\omega^2}{2}\right]}{2}&\left\{\exp[i\mu\omega]\,\widetilde{\mathrm{erfc}}\left[-\frac{\sqrt{2}}{2}(\mu+i\omega)\right]\right.\\ &\left.+ \exp[-i\mu\omega]\,\widetilde{\mathrm{erfc}}\left[\frac{\sqrt{2}}{2}(\mu-i\omega)\right]\right\}\end{aligned} \tag{D-4}$$

Since we now have the characteristic function, we can determine the moments of y by successive differentiation. Differentiating once we obtain

$$\begin{aligned}\frac{\partial\phi_Y(\omega)}{\partial\omega} &= i\sqrt{\frac{2}{\pi}}\exp\left[-\frac{\mu^2}{2}\right] + \frac{(i\mu-\omega)}{2}\exp\left[-\frac{\omega^2}{2}+i\mu\omega\right]\widetilde{\mathrm{erfc}}\left[-\frac{\sqrt{2}}{2}(\mu+i\omega)\right]\\ &\quad - \frac{(i\mu+\omega)}{2}\exp\left[-\frac{\omega^2}{2}-i\mu\omega\right]\widetilde{\mathrm{erfc}}\left[\frac{\sqrt{2}}{2}(\mu+i\omega)\right]\end{aligned} \tag{D-5}$$

Differentiating again we have

$$\begin{aligned}\frac{\partial^2\phi_Y(\omega)}{\partial\omega^2} &= \frac{-2i\omega}{\sqrt{2\pi}}\exp\left[-\frac{\mu^2}{2}\right]\\ &\quad + \frac{(\omega^2-2i\mu\omega-\mu^2-1)}{2}\exp\left[-\frac{\omega^2}{2}+i\mu\omega\right]\widetilde{\mathrm{erfc}}\left[-\frac{\sqrt{2}}{2}(\mu+i\omega)\right]\\ &\quad + \frac{(\omega^2+2i\mu\omega-\mu^2-1)}{2}\exp\left[-\frac{\omega^2}{2}-i\mu\omega\right]\widetilde{\mathrm{erfc}}\left[\frac{\sqrt{2}}{2}(\mu-i\omega)\right]\end{aligned} \tag{D-6}$$

Finally, a third differentiation yields

$$
\begin{aligned}
\frac{\partial^3 \phi_Y(\omega)}{\partial \omega^3} = \frac{1}{2} \Bigg\{ & \frac{-2i\sqrt{2}}{\sqrt{\pi}} \exp\left[-\frac{\mu^2}{2}\right] + 2(\omega^2 - \mu^2 - 1)\, i \sqrt{\frac{2}{\pi}} \exp\left[-\frac{\mu^2}{2}\right] \Bigg\} \\
& + \left\{ \frac{-\omega^3 - 3i\mu\omega^2 + 3\omega\mu^2 + 3\omega + 3i\mu + i\mu^3}{2} \right\} \exp\left[-\frac{\omega^2}{2} - i\mu\omega\right] \widetilde{\text{erfc}}\left[\frac{\sqrt{2}}{2}(\mu - i\omega)\right] \\
& + \left\{ \frac{-\omega^3 + 3i\mu\omega^2 + 3\omega\mu^2 + 3\omega - 3i\mu - i\mu^3}{2} \right\} \exp\left[-\frac{\omega^2}{2} + i\mu\omega\right] \widetilde{\text{erfc}}\left[\frac{-\sqrt{2}}{2}(\mu + i\omega)\right]
\end{aligned} \tag{D-7}
$$

and, since

$$
\left.\frac{\partial^3 \phi_Y(\omega)}{\partial \omega^3}\right|_{\omega=0} = -iE\{y^3\} \tag{D-8}
$$

we have the result

$$
\begin{aligned}
E\{y^3\} &= E\{|x + \mu|^3\} \\
&= (\mu^2 + 2)\sqrt{\frac{2}{\pi}} \exp\left[-\frac{\mu^2}{2}\right] + (\mu^2 + 3)\,\mu\, \widetilde{\text{erf}}\left(\frac{\mu}{\sqrt{2}}\right)
\end{aligned} \tag{D-9}
$$

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