A Comparison of Cowell's Method and a Variation-of-Parameters Method for the Computation of Precision Satellite Orbits: Phase Three Results

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Additional test cases were run using a precision special perturbations program employing either Cowell's method or a variation-of-parameters method to compute a nearly circular, nearly equatorial orbit using two different perturbative accelerations. The results obtained again indicate that the variation-of-parameters method with a predict-only integrator and Cowell's method with a predict-partial-correct integrator are equally efficient, and both are significantly more efficient than Cowell's method with a predict-correct integrator.

I. Introduction

The primary objective of the second phase of this study was to determine an accurate measure of the improvement to be expected from using the variation-of-parameters method in place of Cowell's method when computing precision satellite orbits. Reference 1 shows that, in the case of the Mariner Mars 1971 Mission A orbit as described in Ref. 2,

(1) The variation-of-parameters method integrating six parameters is not significantly more efficient than Cowell's method with a predict-partial-correct integrator;

(2) The variation-of-parameters method integrating six parameters and Cowell's method with a predict-partial-correct integrator are both significantly more efficient than Cowell's method with a predict-correct integrator. The Central Processing Unit (CPU) times are approximately 20% less, and the total costs are approximately 8% less. These percentages will be even larger for perturbative func-

tions which are more complex than the one used in this study.

The objective of this third and final phase of investigation is to compare the variation-of-parameters method with Cowell's method in the case of a nearly circular, equatorial orbit (an osculating eccentricity and inclination of 0 at t_o). The previous comparisons were made in the case of the eccentric Mariner Mars 1971 Mission A orbit (an osculating eccentricity and inclination at t_0 of approximately 0.633 and 80 deg, respectively). The initial state vector for this phase of study is

$$\mathbf{q}^T = (a, e, i, \omega, \Omega, T)$$
$$= (4643 \text{ km}, 0, 0 \text{ deg}, 328.3937 \text{ deg}, 38.3701 \text{ deg},$$
$$11/19/71 \ 14^h42^m \text{ UTC}) \tag{1}$$

In addition, two perturbative accelerations are used. The first perturbative acceleration $\mathbf{\ddot{r}}_{J_2+NB+SP}$ is the same as the one used previously and includes the effects of the asphericity of the central body (J_2 only), N bodies other than the central body, and solar radiation pressure. The second perturbative acceleration $\mathbf{\ddot{r}}_{J_2}$ contains only the effect of the asphericity of the central body (J_2 only).

II. Discussion

In this final phase of investigation, three of the four processes of orbit prediction compared in phase two of the study (Ref. 1) are compared for the case of a nearly circular orbit that lies nearly in the equatorial plane of Mars. The first process is the variation-of-parameters method with a predict-only, sixth-order, variable-step (ERMX/ERMN $\approx r_a/r$) integrator.[1] In this process the six parameters a_x, a_y, h_x, h_y, h_z, and L are integrated. The parameters a_z and n are determined from the integrated values of a_x, a_y, h_x, h_y, and h_z through the equations (see Ref. 3)

$$a_z = -\frac{1}{h_z}(a_x h_x + a_y h_y) \text{ and } n = \sqrt{\mu}\left(\frac{1-\mathbf{a}\cdot\mathbf{a}}{\mathbf{h}\cdot\mathbf{h}}\right)^{3/2} \tag{2}$$

(Note that the parameter a_z is well determined since $h_z >> 0$ in the case of $i \simeq 0$). The second process is Cowell's method with a predict-correct, tenth-order, variable-step (constant ERMX and ERMN) integrator.

The third process is Cowell's method with a predict-partial-correct, tenth-order, variable-step (constant ERMX and ERMN) integrator.

As in phase two, each of the three processes of orbit prediction was used to generate trajectory data in such a way that no calibration factors were necessary. In addition, the cost and accuracy criteria used are the same as those used in phase two (see Ref. 1).

Sixteen cases were run in this phase. Each case used the same initial state vector (Eq. 1) and one of the three processes of orbit prediction described above. Nine cases used the perturbative acceleration $\mathbf{\ddot{r}}_{J_2+NB+SP}$ and seven cases used the perturbative acceleration $\mathbf{\ddot{r}}_{J_2}$. The standard of comparison for each of the two sets was obtained using process two (Cowell predict-correct) with a very tight local error control (see Section 6.1 in Ref. 2). In the first set ($\mathbf{\ddot{r}}_{J_2+NB+SP}$), three cases were run using each of the three processes (one case is the standard of comparison). The three cases differed only in the proportionality constants used in the local error control. In the second set ($\mathbf{\ddot{r}}_{J_2}$), two cases were run using process one, three cases (one of which is the standard of comparison) were run using process two, and two cases were run using process three.

III. Results and Conclusions

These cases show that the two orbits differ significantly from each other and from the orbit integrated in phases one and two. The nearly circular, equatorial orbit using $\mathbf{\ddot{r}}_{J_2}$ differs from the elliptical Mariner orbit primarily in size and shape (and therefore in size of integration step). The nearly circular equatorial orbit using $\mathbf{\ddot{r}}_{J_2+NB+SP}$ differs from the elliptical Mariner orbit not only in size and shape but in the effect of solar radiation pressure. In the former, the solar radiation pressure plays a dominant role in the stepping procedure by requiring a restart each time the spacecraft passes in and out of the shadow of Mars (every revolution). In the latter, the spacecraft did not enter the shadow. The two nearly circular, equatorial orbits differ from each other in the complexity of the perturbative acceleration (and consequently in the cost of the derivative evaluations) as well as in the presence or absence of restarts due to solar radiation pressure.

Table 1 presents the cost and accuracy data for all three processes of orbit prediction using the perturbative

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[1] ERMX/ERMN = error maximum/error minimum.

acceleration $\ddot{\mathbf{r}}_{J_2 + NB + SP}$ in the nearly circular, equatorial orbit. Table 2 presents similar data for the perturbative acceleration $\ddot{\mathbf{r}}_{J_2}$.

In comparing the accuracy of the variation-of-parameters cases and the Cowell cases, the single time-point comparisons used in phases 1 and 2 are inconsistent (a tighter ERMX does not necessarily yield more accuracy) and must be replaced by comparisons of the error propagation curves given in Figs. 1–4. This inconsistency appears to be due to the insensitivity of the local error control when the orbit is nearly circular. For example, the two cases run using Cowell's method with a predict-correct integrator yield the same accuracy for values of ERMX quite different (see Table 2).

The cases in Table 2 are essentially fixed-step integrations, since they begin with a step size of 30 seconds, immediately double until the local error approaches ERMX, and then continue at that step size. Unfortunately, the final step sizes in the Cowell cases do not appear to be optimally determined, since a tighter ERMX does not necessarily yield more accuracy.

Figures 1–4 exhibit the more systematic error growth in the case of the variation-of-parameters (consistent with phases 1 and 2). In addition, these figures show that the errors for all these cases are roughly the same during the first 20 revolutions. It appears that these cases need to be run for more than 20 revolutions or with a more sensitive local error control in order to show large error differences.

Based upon these tables and figures and the results of phases one and two (see Refs. 1 and 2), the following primary conclusions are made:

(1) The variation-of-parameters method is not significantly more efficient than Cowell's method with a predict-partial-correct integrator regardless of the type of orbit or complexity of the perturbative acceleration.

(2) The variation-of-parameters method with a predict-only integrator and Cowell's method with a predict-partial-correct integrator are both significantly more efficient than Cowell's method with a predict-correct integrator regardless of the type of orbit or complexity of the perturbative acceleration. The CPU times are approximately 20%, 17%, and 16% less, respectively, in the case of the elliptical orbit perturbed by $\ddot{\mathbf{r}}_{J_2 + NB + SP}$, and the nearly circular orbit

perturbed by $\ddot{\mathbf{r}}_{J_2 + NB + SP}$, and the nearly circular orbit perturbed by $\ddot{\mathbf{r}}_{J_2}$. In addition, the total costs are approximately 8%, 12%, and 6% less, respectively. A comparison of these percentages for the cases of the nearly circular orbit shows that an increase in the complexity of the perturbative acceleration results in increases in these percentages. Consequently, these percentages will be even larger for perturbative accelerations that are more complex than the ones used in this study.

(3) The variation-of-parameters method should not be incorporated into the standard production and mission operations versions of DPTRAJ, since the equally efficient Cowell's method with a predict-partial-correct integrator already exists as an option in these versions. However, the variation-of-parameters method should be maintained in the research version of DPTRAJ so that future studies can be conducted as new cost saving ideas arise and as funds and manpower become available.

In addition to these primary conclusions, the following secondary conclusions based on the data in phase three are made:

(1) The present local error control based upon ERMX and ERMN is insensitive in the case of a nearly circular orbit. In this case, a fixed-step integrator should be used with a specialized algorithm for choosing the optimum step size.

(2) Only the perturbative effects consistent with the desired accuracy should be used during an integration. For example, suppose an accuracy is desired of order J_2 in 20 revolutions of a nearly circular orbit. It would cost half as much and be four times as fast to integrate using $\ddot{\mathbf{r}}_{J_2}$ instead of $\ddot{\mathbf{r}}_{J_2 + NB + SP}$ (see Tables 1 and 2).

(3) The variation-of-parameters method as formulated in Ref. 3 can compute satellite orbits having eccentricities and inclinations near or equal to zero as efficiently as Cowell's method.

IV. Summary of Complete Study

Test cases run using a precision special perturbations program employing either Cowell's method or a variation-of-parameters method to compute an elliptical orbit for two widely different eccentricities and inclinations were analyzed to determine which method is more efficient.

The results obtained indicate that the variation-of-parameters method with a predict-only integrator and Cowell's method with a predict-partial-correct integrator are equally efficient, and both are significantly more efficient than Cowell's method with a predict-correct integrator. Either of the former more efficient methods for computing precision satellite orbits offers the potential for reducing the total cost of computations during orbit design and computer execution time during real-time mission operations for future orbiter projects.

References

1. Dallas, S. S., and Rinderle, E. A., "A Comparison of Cowell's Method and a Variation-of-Parameters Method for the Computation of Precision Satellite Orbits: Addendum 1," in *The Deep Space Network Progress Report*, Technical Report 32-1526, Volume VII, pp. 32–36. Jet Propulsion Laboratory, Pasadena, Calif., Feb. 15, 1972.

2. Dallas, S. S., and Rinderle, E. A., "A Comparison of Cowell's Method and a Variation-of-Parameters Methods for the Computation of Precision Satellite Orbits," Section Technical Memorandum 392-66, Sept. 30, 1971 (JPL internal document).

3. Dallas, S. S., and Rinderle, E. A., "A Special Perturbations Theory Using the Variation-of-Parameters Formulation," Section Technical Memorandum 392-31, Apr. 1, 1970 (JPL internal document).

Table 1. Cost versus accuracy using $\dot{r}_{J_2+NB+SP}$

Method	Local error control		Accuracy[a]			Cost		
	ERMX	ERMN	$\|\Delta r\|$,m	$\|\Delta \dot{r}\|$,m/s	CPU time, s	Core time product, kiloword/h	Total cost, dollars	Through-put time, s
1. Cowell predict-correct	10^{-11}	10^{-16}	0.149 0.082	0.00010 0.00005	247	7.50	81.43	514
2. Cowell predict-correct	10^{-9}	10^{-14}	0.465 0.379	0.00029 0.00026	187	6.24	67.47	445
3. Cowell predict-partial-correct	10^{-11}	10^{-16}	0.261 0.220	0.00018 0.00014	202	6.58	71.20	482
4. Cowell predict-partial-correct	10^{-10}	10^{-15}	1.67 1.75	0.00108 0.00116	176	6.00	64.81	451
5. Cowell predict-partial-correct	10^{-9}	10^{-14}	0.512 0.427	0.00032 0.00030	157	5.58	60.12	433
6. Variation-of-parameters, predict-only	10^{-9} (r_a/r)	10^{-13} (r_a/r)	2.203 2.276	0.00143 0.00151	188	6.29	68.15	465
7. Variation-of-parameters, predict-only	5×10^{-9} (r_a/r)	5×10^{-13} (r_a/r)	2.867 3.049	0.00187 0.00202	177	6.03	65.31	454
8. Variation-of-parameters, predict-only	$5/2 \times 10^{-8}$ (r_a/r)	$5/2 \times 10^{-12}$ (r_a/r)	0.766 0.802	0.00051 0.00052	155	5.42	59.00	437

[a]These errors occur in revolution 20 at $t - t_0 = 50$ h 40 min and 52 h, respectively (period of orbit $\simeq 2$ h 40 min)

Table 2. Cost versus accuracy using $\dot{r}_{J_2}$

| Method	Local error control		Accuracy[a]			Cost		
	ERMX	ERMN	$\|\Delta r\|$,mm	$\|\Delta \dot{r}\|$, mm/s	CPU time, s	Core time product, kiloword/h	Total cost, dollars	Through-put time, s
1. Cowell predict-correct	10^{-11}	10^{-16}	5.864 4.302	0.0036 0.0031	62	3.67	35.77	337
2. Cowell predict-correct	10^{-9}	10^{-14}	5.864 4.302	0.0036 0.0031	63	3.79	36.70	363
3. Cowell predict-partial-correct	10^{-11}	10^{-16}	5.983 4.569	0.0037 0.0032	52	3.47	33.47	304
4. Cowell predict-partial-correct	10^{-9}	10^{-14}	5.983 4.569	0.0037 0.0032	52	3.61	34.42	328
5. Variation-of-parameters, predict-only	10^{-9} (r_a/r)	10^{-13} (r_a/r)	0.874 0.328	0.0005 0.0002	60	3.19	32.21	316
6. Variation-of-parameters, predict-only	$5/2 \times 10^{-8}$ (r_a/r)	$5/2 \times 10^{-12}$ (r_a/r)	61.010 40.500	0.0651 0.0597	44	2.83	28.25	300

[a]These errors occur in revolution 20 at $t - t_0 = 50$ h 40 min and 52 h, respectively (period of orbit $\simeq 2$ h 40 min).
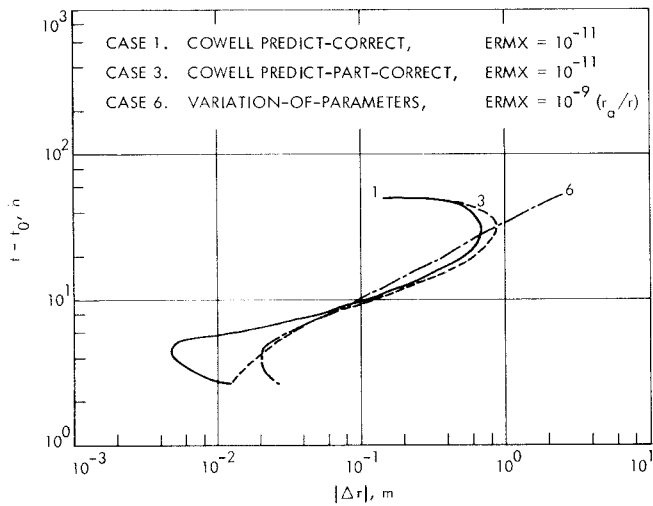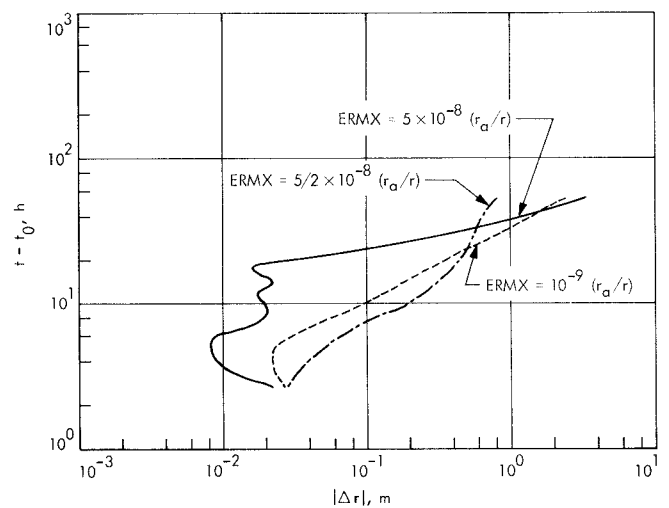
Fig. 1. Error propagation in the variation-of-parameters method and Cowell's method, cases 1, 3, and 6
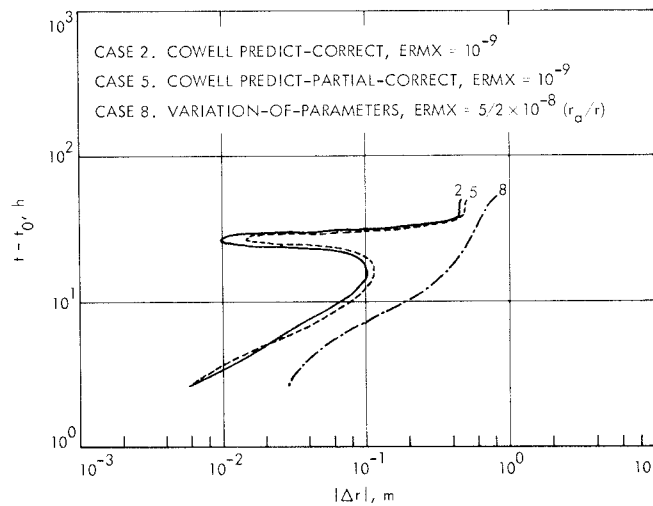
Fig. 2. Error propagation in the variation-of-parameters method and Cowell's method, cases 2, 5, and 8
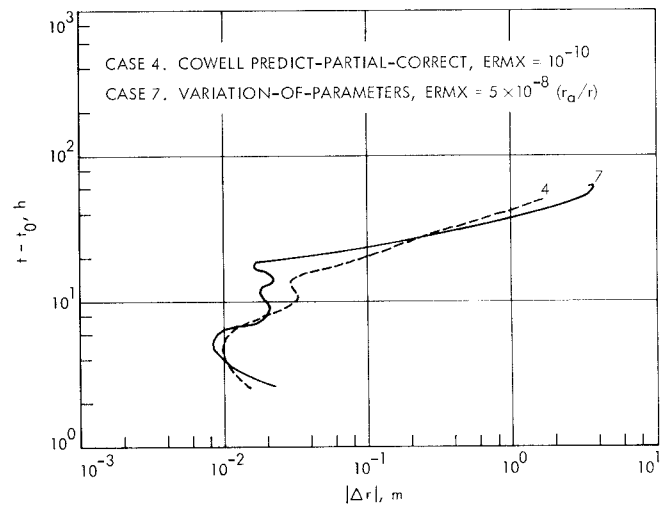
Fig. 3. Error propagation in the variation-of-parameters method and Cowell's method, cases 4 and 7

Fig. 4. Error propagation in the variation-of-parameters method