Optimum Frame Sync Acquisition for Biorthogonally Coded Telemetry

B. K. Levitt
Communications Systems Research Section

An optimum frame sync algorithm for biorthogonally coded telemetry is described. This algorithm takes the coding into account and therefore performs significantly better than algorithms derived for uncoded telemetry, for only a slight increase in implementation complexity.

I. Introduction

As part of the overall task of guaranteeing a given level of telemetry link performance to NASA planetary projects, the problem of acquiring frame synchronization for biorthogonally coded data was investigated. The results of this study can also be applied to the Network Control System (NCS), where master telemetry frames must be stripped to obtain key spacecraft engineering measurements for retransmission to the DSIF stations, and to the Mission Control and Computing Center (MCCC).

In past planetary missions, frame synchronization has been achieved by detecting the periodic peaks of the cross correlation between the known binary frame sync sequence and the decoded bit stream. Because this cross correlation is based on the binary Hamming distance metric, the probability of false sync acquisition is minimized by selecting binary frame sync sequences with highly peaked autocorrelation functions, such as Barker and pseudonoise (PN) sequences.

This approach to the problem of frame sync acquisition is optimum for uncoded telemetry, where successive bit errors in the received bit stream are statistically independent. However, it is not optimum for coded telemetry because it does not take account of the existence of multiple bit error patterns in the decoded data as a result of statistically independent word errors. Despite this incongruency, frame sync acquisition algorithms based on the Hamming distance metric have continued to be used on Mariner missions with telemetry modes employing a (32,6) biorthogonal code.

To alleviate this situation, a frame sync procedure based on a word distance metric is proposed. It is argued heuristically below that this approach is optimum for biorthogonally coded data with regard to minimizing the probability of false sync acquisition. For this design criterion, the superiority of the word distance metric over the Hamming distance metric is demonstrated analytically. Although the frame sync problem is discussed below in the specific context of the Mariner (32,6) biorthogonal code, the results can readily be extended to include any biorthogonal or orthogonal code.

II. Discussion

The problem of acquiring frame synchronization for binary phase-shift-keyed (PSK) telemetry modes employing a (32,6) biorthogonal code is examined below. The scope of the discussion is limited to the special case in which each frame contains N *complete* 6-bit words, including a frame sync prefix of K *complete* words. (The general case wherein the frame sync sequence and the frame itself do not contain integral numbers of words results in a more complicated optimum frame sync acquisition algorithm.) It is implicitly assumed below that word synchronization is correctly established prior to the application of the frame sync acquisition algorithm, so that only ⅙ of the decoded data bits need be considered as possible starting locations for the received frame sync sequences.

Suppose frame synchronization is to be acquired by processing an arbitrary, contiguous span of N decoded words in order to locate the received sync sequence contained therein. If the frame sync decision is actually to be based on n such spans, the problem reduces to the above for a frame length of nN words and a sync sequence composed of nK noncontiguous words. The frame sync sequence will be denoted by the binary $6K$-tuple $\mathbf{s} = (\mathbf{s}_0, \mathbf{s}_1, \cdots, \mathbf{s}_{K-1})$, where each $\mathbf{s}_i$ is a word. The span of N decoded words used to determine frame synchronization will be represented by the binary sequence $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_{N-1})$, where each $\mathbf{r}_i$ is a decoded word.

Define the mth K-word decoded segment

$$\boldsymbol{\rho}_m = (\mathbf{r}_m, \mathbf{r}_{m+1}, \cdots, \mathbf{r}_{m+K-1})$$

where the subscripts are modulo N, and $0 \leq m \leq N-1$. The objective is to determine which of the N segments $\boldsymbol{\rho}_m$ is the most likely received frame sync sequence. In order to compare each $\boldsymbol{\rho}_m$ with the frame sync sequence $\mathbf{s}$, the binary error sequence

$$\mathbf{e}_m = (\mathbf{e}_{0,m}, \mathbf{e}_{1,m}, \cdots, \mathbf{e}_{K-1,m})$$

is formed, wherein the ith error word $\mathbf{e}_{i,m}$ is the bit-by-bit modulo 2 sum of $\mathbf{s}_i$ and $\mathbf{r}_{m+i}$. An appropriate frame sync metric to operate on each error sequence $\mathbf{e}_m$ must now be devised. To this end, the probability distribution of $\mathbf{e}_{m^*}$ will now be examined, where $\boldsymbol{\rho}_{m^*}$ is the actual received frame sync sequence.

When a binary PSK signal is demodulated using a carrier reference derived from the modulated signal, there is a binary phase ambiguity in the detector output: that is, the output of the block decoder can be data or inverted data ($\overline{\text{data}}$) with equal probability. Therefore, in the absence of any errors in $\boldsymbol{\rho}_{m^*}$ due to noise, $\mathbf{e}_{m^*}$ will contain all 0's or all 1's with probability 1/2. This phase ambiguity is resolved when frame sync is established, according to whether $\boldsymbol{\rho}_{m^*}$ resembles $\mathbf{s}$ or $\overline{\mathbf{s}}$. For the (32,6) biorthogonal code, conditioned on data or $\overline{\text{data}}$ at the decoder output, *the* $\mathbf{e}_{i,m^*}$*'s are statistically independent random binary* 6-tuples with probability distributions

$$\Pr[\mathbf{e}_{i,m^*}|\text{data}] = \begin{cases} 1-\varepsilon_w; & \mathbf{e}_{i,m^*} = \mathbf{0} \\ \varepsilon'; & \mathbf{e}_{i,m^*} = \mathbf{1} \\ \dfrac{1}{62}(\varepsilon_w - \varepsilon'); & \mathbf{e}_{i,m^*} \neq \mathbf{0}, \mathbf{1} \end{cases} \qquad (1)$$

$$\Pr[\mathbf{e}_{i,m^*}|\overline{\text{data}}] = \begin{cases} 1-\varepsilon_w; & \mathbf{e}_{i,m^*} = \mathbf{1} \\ \varepsilon'; & \mathbf{e}_{i,m^*} = \mathbf{0} \\ \dfrac{1}{62}(\varepsilon_w - \varepsilon'); & \mathbf{e}_{i,m^*} \neq \mathbf{0}, \mathbf{1} \end{cases} \qquad (2)$$

where $\mathbf{0}$ is the 6-tuple containing all 0's, and $\mathbf{1}$ is similarly defined. In Eqs. (1) and (2), ε_w is the probability that a transmitted 6-bit word is incorrectly decoded, and, in particular, ε' is the probability that it is decoded to its complement, conditioned on the detection of data. Typically, ε' is sufficiently small that it can be neglected; for example, when $\varepsilon_w \sim 0.5$, $\varepsilon'/\varepsilon_w \sim 10^{-5}$, and $\varepsilon'/\varepsilon_w$ decreases monotonically as ε_w decreases. The assumption that ε' is in fact negligible is made in the work that follows.

Because ε_w is generally small, it is evident from Eq. (1) that, when the detector output is data, the probability that $\mathbf{e}_{i,m^*}$ is $\mathbf{0}$ is much greater than the probability that $\mathbf{e}_{i,m^*}$ is a particular mixed pattern of 0's and 1's. A similar statement can be made for the case where the detector output is $\overline{\text{data}}$.

Therefore, conditioned on the detection of data, most of the error words in $\mathbf{e}_{m^*}$ can be expected to be $\mathbf{0}$'s, with a small number of mixed error patterns; for the case of $\overline{\text{data}}$, most of the error words will be $\mathbf{1}$'s, with some mixed patterns. Under the earlier assumption that ε' is negligible, $\mathbf{e}_{m^*}$ cannot contain both $\mathbf{0}$'s and $\mathbf{1}$'s.

As shown in Eqs. (1) and (2), $\mathbf{e}_{i,m^*}$ has an equal probability of being any of the 62 mixed patterns of 0's and 1's. As a particular example,

$$\Pr[\mathbf{e}_{i,m^*} = 000100] = \Pr[\mathbf{e}_{i,m^*} = 011111] \cong \frac{\varepsilon_w}{62} \qquad (3)$$

whether the detector output is data or $\overline{\text{data}}$. Conditioned on the detection of data, a frame sync acquisition algorithm based on the Hamming distance metric will be more likely to select $\boldsymbol{\rho}_{m^*}$ as the received frame sync sequence if a particular sync word error $\mathbf{e}_{i,m^*}$ is 000100 than if it is 011111. However, this is clearly not optimal since these two sync word error patterns are equally likely.

Because sync *word* errors are statistically independent, conditioned on data or $\overline{\text{data}}$ at the decoder output, whereas sync *bit* errors are not, the optimum frame sync acquisition algorithm should be based on a *word* distance metric. Furthermore, from the arguments above, it is evident that this metric should flag those word errors $\mathbf{e}_{i,m}$ within a given error sequence $\mathbf{e}_m$ which are mixed patterns of 0's and 1's, treating all mixed word error patterns equally. Finally, under the assumption that ε' is negligible, the word distance metric should eliminate from consideration those indices m for which $\mathbf{e}_m$ contains both **0**'s and **1**'s. To this end, define the word distance 'metric' $C(\mathbf{e}_m)$ to have the value $K + 1$ if $\mathbf{e}_m$ contains both **0**'s and **1**'s, and to be the number of mixed word error patterns in $\mathbf{e}_m$ otherwise. (Actually $C(\mathbf{e}_m)$ is not a true metric because it does not satisfy the triangle inequality; however, it will still be referred to as a metric in a looser sense below.) Based on this metric, the following frame sync acquisition algorithm can be applied to the decoded sequence $\mathbf{r}$:

(1) For each index m in the range $0 \leq m \leq N - 1$, form the error sequence $\mathbf{e}_m$.

(2) Decide that $\boldsymbol{\rho}_{\hat{m}}$ is the received sync sequence if $\hat{m}$ is the index m that minimizes $C(\mathbf{e}_{\hat{m}})$.

(3) Decide that the decoder output is data if $\mathbf{e}_m$ contains **0**'s; otherwise, decide that the decoder output is $\overline{\text{data}}$.

Using a more rigorous mathematical derivation, it has been verified that this frame sync acquisition algorithm is optimum in the sense that it minimizes the probability of false synchronization, based on the observable $\mathbf{r}$, for a given sync sequence $\mathbf{s}$. And with respect to this algorithm, there is no longer any advantage to selecting a frame sync sequence with good autocorrelation properties relative to the binary Hamming distance metric. Allowing for the possibility of data or $\overline{\text{data}}$ at the decoder output, one can argue that the optimum $\mathbf{s}$ is any sequence in which the K 6-bit sync words $\mathbf{s}_i$ have mutually orthogonal 32-symbol code words.

III. Analysis

It has been argued above that for biorthogonally coded telemetry, a frame sync acquisition algorithm based on a word distance metric achieves the lowest probability of false synchronization. The next question of interest is how much better this optimum algorithm performs than other frame sync acquisition algorithms. As a partial measure of the superiority of the word distance metric algorithm, a union bound argument will now be used to compare its performance with that of a frame sync acquisition algorithm based on the Hamming distance metric. For convenience, these two algorithms will frequently be identified below by the terms "word metric algorithm" and "bit metric algorithm."

First consider the performance of the word metric algorithm using previously defined notation. Suppose $\boldsymbol{\rho}_{m^*}$ is the actual received frame sync sequence: then the word distance metric $C(\mathbf{e}_{m^*})$ has the probability distribution

$$\Pr[C(\mathbf{e}_{m^*}) = \gamma] = \binom{K}{\gamma} \varepsilon_w^{\gamma} (1 - \varepsilon_w)^{K-\gamma}; \qquad 0 \leq \gamma \leq K \qquad (4)$$

where ε_w is the word error rate. For those received segments $\boldsymbol{\rho}_m$ which do not overlap $\boldsymbol{\rho}_{m^*}$, the word errors within the error sequence $\mathbf{e}_m$ each have an equal probability of being any of the 64 binary 6-tuples: the corresponding metric $C(\mathbf{e}_m)$ has the probability distribution

$$\Pr[C(\mathbf{e}_m) = \mu] = \begin{cases} 2\binom{K}{\mu}\left(\frac{62}{64}\right)^{\mu}\left(\frac{1}{64}\right)^{K-\mu}; & 0 \leq \mu \leq K - 1 \\ \left(\frac{62}{64}\right)^{K}; & \mu = K \\ p_e; & \mu = K + 1 \end{cases} \qquad (5)$$

The factor 2 in the first part of Eq. (5) is due to the possibility of data or $\overline{\text{data}}$ at the decoder output. The probability p_e that $C(\mathbf{e}_m)$ will be $K+1$, so that the index m will be eliminated from consideration, can be shown to equal $1 + (62/64)^K - 2(63/64)^K$; however, it does not enter directly into the calculations below. The frame sync acquisition algorithm computes $C(\mathbf{e}_m)$ over the range $0 \leq m \leq N-1$ seeking the index m for which $C(\mathbf{e}_m)$ is minimized. The probability P_{FW} of false synchronization is simply the probability that one or more of the $C(\mathbf{e}_m)$'s, for $m \neq m^*$, is less than $C(\mathbf{e}_{m^*})$. Using the familiar union bounding technique,

$$P_{FW} = \Pr\left[\bigcup_{\substack{m=0\\m\neq m^*}}^{N-1} C(\mathbf{e}_m) < C(\mathbf{e}_{m^*})\right]$$

$$\leq \sum_{\substack{m=0\\m\neq m^*}}^{N-1} \Pr\left[C(\mathbf{e}_m) < C(\mathbf{e}_{m^*})\right] \tag{6}$$

For the $N - 2K + 1$ nonoverlapping indices m defined above, $C(\mathbf{e}_m)$ and $C(\mathbf{e}_{m^*})$ are statistically independent random variables; then Eqs. (4) and (5) can be used to show that

$$P_w \equiv \Pr\left[C(\mathbf{e}_m) < C(\mathbf{e}_{m^*})\right]$$

$$= 2\sum_{\gamma=1}^{K}\binom{K}{\gamma}\varepsilon_w^{\gamma}(1-\varepsilon_w)^{K-\gamma}\sum_{\mu=0}^{\gamma-1}\binom{K}{\mu}\left(\frac{62}{64}\right)^{\mu}\left(\frac{1}{64}\right)^{K-\mu} \tag{7}$$

For the $2K - 2$ overlapping indices m (excluding m^*), the probability that $C(\mathbf{e}_m)$ is less than $C(\mathbf{e}_{m^*})$ depends on the choice of the frame sync sequence $\mathbf{s}$. If the sync words contained within $\mathbf{s}$ have mutually orthogonal code words, as recommended earlier, a particular $\boldsymbol{\rho}_m$ that overlaps $\boldsymbol{\rho}_{m^*}$ should be less likely to resemble $\mathbf{s}$ on a word distance basis than a totally random, nonoverlapping $\boldsymbol{\rho}_m$. Then

$$\Pr\left[C(\mathbf{e}_m) < C(\mathbf{e}_{m^*})\right] \leq P_w \tag{8}$$

for the overlapping indices. Combining Eqs. (6–8) yields the result

$$P_{FW} \leq (N-1)\,P_w \tag{9}$$

The upper bound in Eq. (9) is tight when P_w is small, which occurs when ε_w is small.

Next, consider the performance of the bit metric algorithm. The Hamming weight of the binary $6K$-tuple $\mathbf{e}_m$, denoted by $H(\mathbf{e}_m)$, is defined as the number of 1's in $\mathbf{e}_m$. Then the Hamming distance between $\mathbf{s}$ and a given $\boldsymbol{\rho}_m$ is given by $H(\mathbf{e}_m)$. If the decoder output were known *a priori* to be data, the bit metric algorithm would choose m to minimize $H(\mathbf{e}_m)$. Because there is an equal probability of having data or $\overline{\text{data}}$, a large value of $H(\mathbf{e}_m)$ should be regarded as favorably as a small value of $H(\mathbf{e}_m)$. Therefore, the frame sync acquisition algorithm should choose m to maximize the normalized bit distance $B(\mathbf{e}_m)$, defined according to

$$B(\mathbf{e}_m) \equiv |3K - H(\mathbf{e}_m)| \tag{10}$$

It is easier to calculate the distribution of $B(\mathbf{e}_m)$ indirectly, by first finding the distribution of $H(\mathbf{e}_m)$, and then using the formula

$$\Pr\left[B(\mathbf{e}_m) = \mu\right] = \begin{cases}\Pr\left[H(\mathbf{e}_m) = 3K\right]; & \mu = 0\\ \Pr\left[H(\mathbf{e}_m) = 3K - \mu\right] & \\ \quad + \Pr\left[H(\mathbf{e}_m) = 3K + \mu\right]; & 1 \leq \mu \leq 3K\end{cases} \tag{11}$$

Regarding the data/$\overline{\text{data}}$ ambiguity, it is evident that

$$\Pr\left[H(\mathbf{e}_m) = \gamma \,|\, \overline{\text{data}}\right] = \Pr\left[H(\mathbf{e}_m) = 6K - \gamma \,|\, \text{data}\right] \tag{12}$$

From Eqs. (11) and (12), one can show that the distribution of $B(\mathbf{e}_m)$ conditioned on the presence of data is identical to that conditioned on $\overline{\text{data}}$, so that

$$\Pr\left[B(\mathbf{e}_m) = \mu\right] = \Pr\left[B(\mathbf{e}_m) = \mu \,|\, \text{data}\right] \tag{13}$$

That is, the distribution of $H(\mathbf{e}_m)$ for the case of data is sufficient to determine the distribution of $B(\mathbf{e}_m)$ averaged over the occurrence of data and $\overline{\text{data}}$.

It is generally assumed that the transmitted data (excluding the sync words) can be regarded as a stream of statistically independent, equally likely 1's and 0's. Consequently, the $\boldsymbol{\rho}_m$'s which do not overlap $\boldsymbol{\rho}_{m^*}$ consist of a sequence of K independent, equally likely bits; therefore, for these indices,

$$\Pr\left[H(\mathbf{e}_m) = \eta \,|\, \text{data}\right] = \binom{6K}{\eta}\left(\frac{1}{64}\right)^{K}; \quad 0 \leq \eta \leq 6K \tag{14}$$

Applying Eqs. (11) and (13), it is clear that for these indices,

$$\Pr\left[B(\mathbf{e}_m)=\mu\right]=\begin{cases}\dbinom{6K}{3K}\left(\dfrac{1}{64}\right)^K; & \mu=0\\[2ex] 2\dbinom{6K}{3K-\mu}\left(\dfrac{1}{64}\right)^K; & 1\leq\mu\leq 3K\end{cases}\tag{15}$$

The distribution of $H(\mathbf{e}_{m^*})$ is more difficult to derive; it is computed in the Appendix to have the form

$$\Pr\left[H(\mathbf{e}_{m^*})=\eta \mid \text{data}\right]=\sum_{\gamma=0}^{K}\binom{K}{\gamma}A_{\eta,\gamma}\left(\frac{\varepsilon_w}{62}\right)^{\gamma}(1-\varepsilon_w)^{K-\gamma};\quad 0\leq\eta\leq 5K\tag{16}$$

where the factor $A_{\eta,\gamma}/(62)^{\gamma}$ is the probability of having a total of η bit errors within a sequence of γ incorrectly decoded 6-bit words. Under the assumption that ε' is negligible in Eq. (1), $A_{\eta,\gamma}$ is nonzero only in the range $\gamma\leq\eta\leq 5\gamma$, and this explains the range in Eq. (16). It is shown in the appendix that $A_{\eta,\gamma}$ satisfies the following recursive formula:

$$A_{\eta,0}=\begin{cases}1; & \eta=0\\ 0; & \text{elsewhere}\end{cases}\qquad A_{\eta,1}=\begin{cases}\dbinom{6}{\eta}; & 1\leq\eta\leq 5\\[1ex] 0; & \text{elsewhere}\end{cases}$$

for $\gamma\geq 2$,

$$A_{\eta,\gamma}=\begin{cases}\displaystyle\sum_{\mu=\max(\eta-5,\,\gamma-1)}^{\min(\eta-1,\,5\gamma-5)}A_{\mu,\gamma-1}A_{\eta-\mu,1}; & \gamma\leq\eta\leq 5\gamma\\ 0; & \text{elsewhere}\end{cases}\tag{17}$$

Combining Eqs. (11), (13), and (16), the distribution of $B(\mathbf{e}_{m^*})$ is found to have the form

$$\Pr\left[B(\mathbf{e}_{m^*})=\zeta\right]=\begin{cases}\displaystyle\sum_{\gamma=1}^{K}\binom{K}{\gamma}A_{3K,\gamma}\left(\frac{\varepsilon_w}{62}\right)^{\gamma}(1-\varepsilon_w)^{K-\gamma}; & \zeta=0\\[3ex] \displaystyle\sum_{\gamma=0}^{K}\binom{K}{\gamma}\left(A_{3K-\zeta,\gamma}+A_{3K+\zeta,\gamma}\right)\left(\frac{\varepsilon_w}{62}\right)^{\gamma}(1-\varepsilon_w)^{K-\gamma} & \\ & 1\leq\zeta\leq 3K\end{cases}\tag{18}$$

The probability P_{FB} of false sync acquisition for the bit metric algorithm can now be upperbounded using Eqs. (15) and (18). Making the same assumptions concerning the overlapping and nonoverlapping indices m as in the case of the word metric algorithm, with the proviso that the sync word is a Barker or PN sequence now, it can be shown that

$$P_{FB}\leq(N-1)\,P_B\tag{19}$$

where P_B is the probability that $B(\mathbf{e}_m)$ exceeds $B(\mathbf{e}_{m^*})$ for a nonoverlapping index m:

$$P_B=\Pr\left[B(\mathbf{e}_m)>B(\mathbf{e}_{m^*})\right]$$

$$=\frac{2}{(64)^K}\sum_{\gamma=1}^{K}\binom{K}{\gamma}\left(\frac{\varepsilon_w}{62}\right)^{\gamma}(1-\varepsilon_w)^{K-\gamma}\left[A_{3K,\gamma}\sum_{\eta=0}^{3K-1}\binom{6K}{\eta}\right.$$

$$\left.+\sum_{\alpha=1}^{3K-1}A_{\alpha,\gamma}\sum_{\eta=0}^{\alpha-1}\binom{6K}{\eta}+\sum_{\beta=3K+1}^{6K-1}A_{\beta,\gamma}\sum_{\eta=0}^{6K-1-\beta}\binom{6K}{\eta}\right]\tag{20}$$

Assuming the bounds in Eqs. (9) and (19) are tight, the superiority of the word metric algorithm over the bit metric algorithm is demonstrated by showing that $P_B/P_W > 1$ for any given K and ε_w. For example, Eqs. (7), (17), and (20) can be used to show that for $K = 1$,

$$P_W = \frac{1}{32}\varepsilon_w$$

$$P_B = \frac{331}{(31)(32)}\varepsilon_w$$

$$\Downarrow$$

$$\therefore \frac{P_B}{P_W} = \frac{331}{31} = 10.68, \text{ for all } \varepsilon_w \tag{21}$$

and for $K = 2$,

$$P_W = \frac{4}{(64)^2}\varepsilon_w + \frac{246}{(64)^2}\varepsilon_w^2$$

$$P_B = \frac{44{,}120}{(62)(64)^2}\varepsilon_w + \frac{3{,}281{,}316}{(62)^2(64)^2}$$

$$\Downarrow$$

$$\therefore \frac{P_B}{P_W} = \frac{2{,}735{,}440 + 3{,}281{,}316\,\varepsilon_w}{15{,}376 + 945{,}624\,\varepsilon_w}$$

$$= \begin{cases} 27.87; & \varepsilon_w = 0.1 \\ 111.48; & \varepsilon_w = 0.01 \\ 177.90; & \varepsilon_w \rightarrow 0 \end{cases} \tag{22}$$

For small ε_w, the following approximations are useful:

$$P_W \cong \frac{2K}{(64)^K}\varepsilon_w \tag{23}$$

$$P_B \cong \begin{cases} \dfrac{331}{(31)(32)}\varepsilon_w; & K = 1 \\[2ex] \dfrac{2K}{(62)(64)^K}(324K^4 + 432K^3 + 459K^2 + 246K + 62)\,\varepsilon_w; & K \geq 2 \end{cases} \tag{24}$$

$$\Downarrow$$

$$\lim_{\varepsilon_w \rightarrow 0} \frac{P_B}{P_W} = \begin{cases} \dfrac{331}{31}; & K = 1 \\[2ex] \dfrac{324K^4 + 432K^3 + 459K^2 + 246K + 62}{62}; & K \geq 2 \end{cases} \tag{25}$$

The limit of P_B/P_W as ε_w approaches zero is tabulated below for $1 \leq K \leq 5$. The results indicate that the word metric algorithm greatly outperforms the bit metric algorithm for small word error rates.

K	$\lim_{\varepsilon_w \rightarrow 0} \frac{P_B}{P_W}$
1	1.068×10^1
2	1.779×10^2
3	6.910×10^2
4	1.919×10^3
5	4.343×10^3

IV. Conclusion

The problem of acquiring frame synchronization for biorthogonally coded telemetry was investigated. To minimize the probability of false synchronization, the optimum frame sync acquisition algorithm, which must operate on the decoded data, uses a word distance metric rather than the Hamming distance metric to locate the received frame sync sequence.

The optimum word metric algorithm was developed heuristically for the particular case in which each frame contains N complete words, including a frame sync prefix of K words. With respect to this algorithm, the optimum frame sync sequence is one in which the K sync words have mutually orthogonal code words.

The Mariner and Viking projects use telemetry modes employing a (32,6) biorthogonal code. For this specific case, the performance of the optimum word metric algorithm was compared analytically with the optimum frame sync acquisition algorithm based on the Hamming distance metric. It was demonstrated that the word metric algorithm is superior, particularly for large K and small word error rates.

For biorthogonally coded telemetry modes with noninteger values of N and K, as in the 1973 Mariner Venus Mercury mission, the optimum frame sync acquisition algorithm is generally more complicated than the word metric algorithm considered in this report. Consequently, implementation considerations may dictate the use of a suboptimum algorithm based on the Hamming distance metric in this case.

It is recommended that future flight projects use integer values of N and K on biorthogonally coded telemetry modes so that optimum frame sync procedures can be easily employed. Furthermore, it should be stressed that improved performance is obtained when frame sync decisions are based jointly on several frames of decoded data, the number of frames being limited by the available storage capacity.

Appendix

Bit Error Distribution for (32,6) Biorthogonally Coded Telemetry

Suppose a sequence of K 6-bit words $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_K)$ is transmitted over a communication link employing a (32,6) biorthogonal code. Because of additive channel noise, the resulting decoded sequence of K 6-bit words $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_K)$ does not in general equal the transmitted sequence $\mathbf{s}$. A binary error sequence

$$\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_K)$$

can be formed, where $\mathbf{e}_i$ is the modulo 2 sum of $\mathbf{s}_i$ and $\mathbf{r}_i$: 1's in the error sequence $\mathbf{e}$ correspond to bit errors in the decoded sequence $\mathbf{r}$.

It is assumed that the error words $\mathbf{e}_i$ are statistically independent random sequences with probability distributions

$$\Pr[\mathbf{e}_i] = \begin{cases} 1 - \varepsilon_w; & \mathbf{e}_i = \mathbf{0} \\ \varepsilon_w/62; & \mathbf{e}_i \neq \mathbf{0}, \mathbf{1} \\ 0; & \mathbf{e}_i = \mathbf{1} \end{cases} \qquad \text{(A-1)}$$

where ε_w is the word error rate for the communication link, $\mathbf{0}$ is the 6-tuple containing all 0's, and $\mathbf{1}$ is similarly defined. Note that Eq. (A-1) rules out the possibility that $\mathbf{r}_i$ can be the complement of $\mathbf{s}_i$, corresponding to 6 bit errors in a given word. Also, if a word error is made, the decoded word is equally likely to be any binary 6-tuple containing from 1 to 5 bit errors, excluding the transmitted word and its complement.

Suppose γ of the words in $\mathbf{r}$ are incorrectly decoded: the probability of this event is

$$\binom{K}{\gamma} \varepsilon_w^\gamma (1 - \varepsilon_w)^{K-\gamma}; \quad 0 \leqq \gamma \leqq K$$

The $K - \gamma$ correctly decoded words contain no bit errors: all of the bit errors in $\mathbf{r}$ lie in the subset of K incorrectly decoded words. Let H denote the Hamming weight of $\mathbf{e}$, which is equivalent to the number of bit errors in $\mathbf{r}$. Define $A_{\eta,\gamma}/(62)^\gamma$ to be the probability of having a total of η bit errors in a group of γ incorrectly decoded words: then the probability distribution of H has the form

$$P_r[H = \eta] = \sum_{\gamma=0}^{K} \binom{K}{\gamma} A_{\eta,\gamma} \left(\frac{\varepsilon_w}{62}\right)^\gamma (1 - \varepsilon_w)^{K-\gamma}; \quad 0 \leqq \eta \leqq 5K \qquad \text{(A-2)}$$

Since each word error contains from 1 to 5 bit errors according to Eq. (A-1), $A_{\eta,\gamma}$ is nonzero only in the range $\gamma \leqq \eta \leqq 5\gamma$: this explains the range in Eq. (A-2). The $A_{\eta,\gamma}$'s must now be determined to complete the expression for $\Pr[H = \eta]$.

If no word errors are made, there are no bit errors; therefore

$$A_{\eta,0} = \begin{cases} 1; & \eta = 0 \\ 0; & \eta \neq 0 \end{cases} \qquad \text{(A-3)}$$

The error word $\mathbf{e}_i$ for an incorrectly decoded word $\mathbf{r}_i$ is equally likely to be any binary 6-tuple with the exception of $\mathbf{0}$ and $\mathbf{1}$. Therefore, the probability of having η bit errors in a given incorrectly decoded word is simply the number of binary 6-tuples of Hamming weight η, divided by 62. Consequently

$$A_{\eta,1} = \begin{cases} \binom{6}{\eta}; & 1 \leqq \eta \leqq 5 \\ 0; & \text{elsewhere} \end{cases} \qquad \text{(A-4)}$$

For $\gamma \leqq 2$, the probability of η bit errors within γ incorrectly decoded words is equal to the probability of μ bit errors in $\gamma - 1$ incorrect words, multiplied by the probability of $\eta - \mu$ bit errors within a single incorrect word, summed over the appropriate range of μ:

$$\underset{(\gamma \geqq 2)}{A_{\eta,\gamma}} = \begin{cases} \displaystyle\sum_{\mu=\max(\eta-5,\gamma-1)}^{\min(\eta-1,5\gamma-5)} A_{\mu,\gamma-1} A_{\eta-\mu,1}; & \gamma \leqq \eta \leqq 5\gamma \\ 0; & \text{elsewhere} \end{cases} \qquad \text{(A-5)}$$

The range of μ in Eq. (A-5) follows from the ranges over which $A_{\mu,\gamma-1}$ and $A_{\eta-\mu,1}$ are nonzero. Thus $A_{\eta,\gamma}$ can be found recursively for $\gamma \geqq 2$.