# Performance of Coded, Noncoherent, Hard-Decision MSFK Systems

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*The capacity of noncoherent multifrequency shift keying (MFSK) systems that use a hard decision receiver is determined as a function of the predetection signal-to-noise ratio ($ST/N_0$). For any given predetection signal-to-noise ratio there is an optimum number of frequencies that maximize the system capacity. This optimum number decreases as the predetection signal-to-noise ratio decreases. However, it is shown here that this number is never less than 7. This means that binary frequency shift keying, a commonly used modulation technique at very low data rates, is suboptimum by at least 2.2 dB, compared to the performance obtainable with 7 signals. Similar results are obtained for the computational cut-off $R_{\text{comp}}$, when convolutional coding with sequential decoding is used over such an MFSK channel. These channels are expected to arise in planetary entry missions into thick atmospheres, such as those of Venus and Jupiter.*

## I. Introduction

Noncoherent reception is necessary when the receiver cannot determine the phase of the received signals. This situation is likely to occur in missions that enter thick atmospheres of planets such as Venus, Jupiter, Saturn, and Uranus. Usually this is due to random phase changes that are too large to ignore and too rapid to estimate accurately, i.e., the signal-to-noise ratio (SNR) in the predetection filter, $ST/N_0$, is too low, where $S$ is the rms power of the received signal, $N_0$ is the one-sided noise density and $T$ is the time interval over which the phase is relatively constant albeit unknown. Causes of this type of behavior in planetary entry are turbulence, dispersion, attenuation and residual doppler.

Roughly speaking, the time $T$ corresponds to the inverse bandwidth of the random phase process. The phase variations cannot be tracked by a phase-locked loop of lower bandwidth, while the signal-to-noise ratio in this minimum loop bandwidth is too low. It is well known (Refs. 1, 2, and 3) that communication under the described conditions requires transmission of signals that are orthogonal over a time interval $T$ or less, and their reception by means of a square-law receiver. Performance curves plotting the error probability for detecting one of $M = 2^K$ equiprobable signals (corresponding to a rate of $K/T$ bits/s) as a function of bit signal-to-noise ratio $ST_B/N_0 = ST/KN_0$, have been computed by Lindsey (Ref. 1). Also, in the limit as $T$ and $M$ approach infinity, with $M$ growing exponentially, Turin (Ref. 2) has proved that zero error probability can be attained for all rates up to the capacity of the infinite bandwidth coherent channel $C_\infty = 1.44\ S/N_0$ bits/s. This behavior in the limit is not surprising because allowing $T$ to grow arbitrarily large means that the phase tends to an unknown constant (between $-\pi$ and $\pi$). Thus, it can be estimated with arbitrarily high accuracy by diverting a small fraction, $\varepsilon$, of the power to a phase reference signal, since $\varepsilon ST/N_0 \to \infty$. Consequently, a coherent receiver can be used and it is well known (Ref. 3) that $C_\infty$ can then be achieved with a coded sequence of short duration antipodal signals ($M = 1$) instead of the very special orthogonal signal set.

It is not difficult to show that noncoherent signaling by itself cannot achieve arbitrarily low error rates when $ST/N_0$ is bounded. In fact the error probability increases as the number of signals, $M$, increases. Nevertheless, arbitrarily reliable communication is still possible at a nonzero rate (but less than $C_\infty$) by employing an additional level of coding (concatenating) on the channel created by the orthogonal signals and the noncoherent receiver. Theoretically, error-free transmission is possible at rates up to the capacity of this noncoherent channel.

The purpose of this article is to investigate the capacity of the above noncoherent channel as a function of $M$ and $ST/N_0$, and to draw conclusions pertaining to the design of coded, noncoherent communication systems.

## II. The Noncoherent Multiple Frequency Shift Keying Channel

Multifrequency shift keying (MFSK) refers to the case in which orthogonal signals over time $T$ are harmonics of the frequency $1/T$. Usually the orthogonal signals are modulated onto a high-frequency carrier and it is the phase of the carrier as opposed to the phase of the signals that cannot be tracked. In that case the maximum number of orthogonal signals that can be distinguished in a system of bandwidth $W$ is approximately $M = 2WT$, since both $\sin(2\pi kt/T)$ and $\cos(2\pi kt/T)$ $k = 1, 2, \cdots, WT$ can be used. However, if the phases of the signals are also unknown, then only $M = WT$ signals can be distinguished by the receiver: the sine terms, for instance, must be dropped to avoid confusion with phase-shifted cosines. The model assumes that the carrier phase is statistically independent every $T$ seconds.

Figure 1 is a block diagram of a noncoherent MFSK system. During each interval of time $T$ one of $M = 2^K$ orthogonal signals $x_m(t)$; $m = 1, 2, \cdots, M$ with unit energy is modulated onto a carrier $\cos \omega t$ and arrives in the presence of additive Gaussian noise $n(t)$ as

$$r(t) = \sqrt{2S}\, x_m(t) \cos(\omega t + \theta) + n(t) \tag{1}$$

where $\theta$ is an unknown phase shift uniformly distributed between $-\pi$ and $\pi$.

The optimum receiver (Ref. 4) for this system computes the $M$ dimensional test statistic $r = (r_1, r_2, \cdots, r_M)$ where

$$r_k = \frac{2}{\sqrt{N_0 T}} \left[ \left( \int_0^T r(t)\, x_k(t) \cos \omega t\, dt \right)^2 + \left( \int_0^T r(t)\, x_k(t) \sin \omega t\, dt \right)^2 \right]^{1/2} \tag{2}$$

It is possible to show (Ref. 4) that the conditional density of $r_k$ is

$$P(r_k | x_m(t)) = \begin{cases} r_k \exp\left(-\frac{1}{2} r_k^2\right) & \text{if } k \neq m \\ r_m \exp\left(-\frac{1}{2} r_m^2 - \frac{1}{2}\alpha^2\right) I_0(\alpha r_m) & \text{if } k = m \end{cases} \tag{3}$$

where

$$\alpha^2 = 2\, ST/N_0 \tag{4}$$

and $I_0(\alpha r)$ is the modified Bessel function of the first kind

$$I_0(\alpha r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\alpha r \cos\theta)\, d\theta \tag{5}$$

Now, if no further coding is used, the optimum decoding rule is to declare $x_m$ received when $r_m$ is largest. In Ref. 1 it is shown that the probability of being correct is

$$P_c = \int_0^\infty \left[1 - \exp\left(-\frac{1}{2}x^2\right)\right]^{M-1} \times \exp\left[-\frac{1}{2}(x^2 + \alpha^2)\right] I_0(\alpha x)\, dx \tag{6}$$

and the data rate is

$$R = \frac{\log_2 M}{T} = \frac{K}{T}\ \text{bits/s} \tag{7}$$

It is now well known (Refs. 2, 3, and 4) that

$$\lim_{T\to\infty} P_c = \begin{cases} 1 & \text{if } R < C_\infty = S/N_0 \log_2 e \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

however, the bandwidth $W$ grows exponentially as

$$W = \frac{M}{2T} = \frac{2^{RT}}{2T} \tag{9}$$

Moreover, as already mentioned, letting $T$ grow to infinity assumes a constant phase. This means that coherent communication was possible in the first place, and that $C_\infty$ could have been achieved with binary signals of duration approaching zero as ½$W$, where $W \to \infty$ *independent of* $T$. However, the problem, in general, is not a dearth of bandwidth; rather it is a lack of $ST/N_0$.

Examination of Eq. (6) reveals that $P_c < 1$ if $ST/N_0$ is finite and decreases ultimately as $1/M$, as $M$ increases. Therefore, additional coding is required if the error probability for a given $ST/N_0$ and $M$ is to be reduced further. The maximum rate of transmission at which the error probability can be made arbitrarily small by use of additional coding is, of course, the capacity of the inner, MFSK channel, and the main concern of this article.

## III. Capacity of the *M*-ary, Noncoherent MFSK Channel

When the receiver makes a decision as to which one of $M$ signals is received and discards all other information the result is an $M$-ary symmetric channel with crossover probability $(1 - P_c)/(M - 1)$. The capacity of such a channel is easy to compute (Ref. 5).

$$C = \frac{1}{T}\left[\log_2 M + P_c \log P_c + (1 - P_c)\log_2(1 - P_c) - (1 - P_c)\log_2(M - 1)\right] \text{bits/s} \tag{10}$$

$$= IR \tag{11}$$

where

$$I(M, ST/N_0) = P_c + \frac{P_c \log_2 P_c + (1 - P_c)\log_2(1 - P_c)}{\log_2 M} - \frac{(1 - P_c)\log_2(1 - 1/M)}{\log_2 M} \tag{12}$$

is the information per input bit of the MFSK channel.

Normalizing with respect to $C_\infty = S/N_0 \log_2 e$ yields

$$\frac{C}{C_\infty} = \frac{I \ln 2}{(ST_B/N_0)_{\text{MFSK}}} \tag{13}$$

where

$$(ST_B/N_0)_{\text{MFSK}} \triangleq \frac{ST}{N_0 K} = \frac{S}{N_0 R} \tag{14}$$

is the signal-to-noise ratio per input bit of the MFSK channel.

Figure 2 is a plot of the normalized capacity versus MFSK signal-to-noise ratio for $K = 1, 2, \cdots, 10, 15, 20$ and $K \to \infty$. The $K \to \infty$ curve is obtained from the fact that $P_c \to 1$ if $ST_B/N_0 < \ln 2 = 0.693$, and $P_c \to 0$ otherwise; therefore,

$$\lim_{K\to\infty} \frac{C}{C_\infty} = \begin{cases} 0 & \text{if } (ST_B/N_0)_{\text{MFSK}} < \ln 2 = 0.693 \\ \ln 2/(ST_B/N_0) & \text{otherwise} \end{cases} \tag{15}$$

This reveals, incidentally, that MFSK signaling approaches $C_\infty$ in a nonuniform manner as $K \to \infty$, as opposed to the uniform convergence obtained over the coherent binary input, infinite quantized (no hard decisions) Gaussian channel. The nonuniform convergence is in accord with the threshold effect that is observed in nonlinear receivers.

Figure 3 is a plot of the minimum required energy-to-noise ratio per coded bit $(E_b/N_0)$ coded vs $ST/N_0$ for various $k$,

$$\left(\frac{E_b}{N_0}\right)_{\text{coded}} = \frac{C_\infty}{C}\ln 2; \qquad \frac{ST}{N_0} = K\frac{ST_B}{N_0}$$

Note that the performance with $K = 1$ (2 signals) and $K = 2$ (4 signals) is always worse than with $K = 3$ (8 signals) for all values of $ST/N_0$. This is proved in the appendix.

## IV. Convolutional Coding and Decoding Limit

As a guide to the performance obtainable when a convolutional code is used over the MFSK channel, it is of interest to evaluate the computational limit on the decodable rate $R_{comp}$. Without going into the details treated adequately in Ref. 3, it can be shown that the expected number of computations necessary to decode a convolutional code using a sequential decoding algorithm becomes infinite if the rate exceeds $R_{comp}$. Thus, $R_{comp}$ is an effective measure of the rate achievable with convolutional codes. Note that $R_{comp}$ here is $R_0'$ in eqn. 6.62b of Ref. 3. For the $M$-ary symmetric channel

$$R_{comp} = -\frac{1}{T}\log_2\left[\frac{1}{M}\left(\sqrt{P_c} + (M-1)\sqrt{\frac{1-P_c}{M-1}}\right)^2\right] \tag{16}$$

$$= RI_{comp} \tag{17}$$

where

$$I_{comp} = 1 - \frac{2\log_2\left(\sqrt{P_c} + \sqrt{(2^K-1)(1-P_c)}\right)}{K} \tag{18}$$

Therefore

$$\frac{R_{comp}}{C_\infty} = \frac{I_{comp}\ln 2}{(ST_B/N_0)_{MFSK}} \tag{19}$$

yields the normalized computational limit for sequential decoding per input bit of the MFSK channel, and is plotted in Fig. 4 per various values of $K$.

The curves in Fig. 5 are plots of the minimum required bit energy-to-noise ratio vs $ST/N_0$ for various $K$:

$$\left(\frac{E_b}{N_0}\right)_{comp} = \frac{C_\infty}{R_{comp}}\ln 2$$

Again, as in the capacity case, we see that the use of 2 and 4 signals is everywhere inferior to using 8 signals for the inner MFSK channel.

Figures 6 and 7 are plots of the optimum performance achievable for given $K$ after optimizing over $ST/N_0$, and for given $ST/N_0$ after optimizing over $K$. From Fig. 6 it is evident that improvement with increasing $M$ is very slow for $M > 1000$ ($K > 10$). Since larger values of $M$ are too complex it also means that when $ST/N_0 > 10$ orthogonal signaling must be replaced by some other scheme; such as by partially coherent schemes, or by schemes involving "soft" decisions in the inner MFSK channel.

Investigation of "soft" decision MFSK systems is currently under way for both large and small values of $ST/N_0$. Better performance is to be expected from the fact that more information is retained when soft rather than hard decisions are made.

## V. Conclusions

This article established performance limits theoretically achievable over noncoherent channels perturbed by additive Gaussian noise using orthogonal signals and a hard-decision MFSK receiver. The performance is, not surprisingly, a function of the signal-to-noise ratio $ST/N_0$ in the MFSK correlators. These correlators can be thought of as a predetection filter and $ST/N_0$ as the predetection signal-to-noise ratio. The performance improves as $ST/N_0$ increases provided the bandwidth, as measured by the number of orthogonal signals, can be increased.

The result of greater practical import, however, concerns operation at moderate and low values of the predetection signal-to-noise ratio, $ST/N_0 < 1.0$, where it was found that the best results are to be obtained by using about 8 signals.

## VI. Acknowledgment

The numerical computations were programmed and run on a digital computer by I. Eisenberger, to whom our thanks are addressed.

# References


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# Appendix
# Capacity at Very Small Values of $ST/N_0$

We will show that

$$\frac{C}{C_\infty} \rightarrow \frac{ST}{N_0}\left(\sum_{m=2}^{M} \frac{1}{m}\right)^2 \bigg/ 2(M-1) \qquad \text{as} \quad \frac{ST}{N_0} \rightarrow 0 \tag{A-1}$$

The coefficient multiplying $ST/N_0$ on the right hand side has a maximum value of 0.211 at $M = 7$. However, the maximum is quite broad and is 0.2107 at $M = 8$. Therefore, the optimum mode of communication in a hard-decision MFSK system at very low signal-to-noise ratios is to use 8 signals (a power of 2 is always convenient). The comparative performance of 2, 4, 8, 16, and 32 signals is as 0.1250, 0.1956, 0.2107, 0.1895, 0.1508. In particular the performance of a binary hard decision frequency shift keying (FSK) system is 2.28 dB below that of the optimum 7 or 8 frequency system.

To prove the above relationship we note that

$$P_c \rightarrow \frac{1}{M} \qquad \text{as} \quad ST/N_0 \rightarrow 0$$

Therefore, it is convenient to introduce

$$\varepsilon_M = MP_c - 1 \tag{A-2}$$

Then

$$\begin{aligned} I \ln M &= \frac{1}{M}(1+\varepsilon_M)\ln(1+\varepsilon_M) \\ &\quad + \frac{M-1}{M}\left(1+\frac{\varepsilon_M}{M-1}\right)\ln\left(1+\frac{\varepsilon_M}{M-1}\right) \\ &\approx \frac{\varepsilon_M^2}{2(M-1)} \qquad \text{for } \varepsilon_M << 1 \end{aligned} \tag{A-3}$$

Since $C/C_\infty = I \ln M/(ST/N_0)$ it is necessary to show that

$$\varepsilon_M \approx \frac{ST}{N_0}\sum_{m=2}^{M}\frac{1}{m}$$

In Ref. 1 it is noted that the probability of a correct decision is also given by

$$P_c = 1 - \frac{1}{M}\sum_{j=2}^{M}(-1)^j\binom{M}{j}\exp\left[-\frac{ST}{N_0}\left(1-\frac{1}{j}\right)\right] \tag{A-4}$$

$$= -\frac{1}{M}\sum_{j=1}^{M}(-1)^j\binom{M}{j}\exp\left[-\frac{ST}{N_0}\left(1-\frac{1}{j}\right)\right] \tag{A-5}$$

Therefore

$$\begin{aligned} \varepsilon_M - \varepsilon_{M-1} &= -\sum_{j=1}^{M-1}(-1)^j\binom{M-1}{j}\left(1-\frac{M}{M-j}\right) \\ &\quad \times \exp\left[-\frac{ST}{N_0}\left(1-\frac{1}{j}\right)\right] \\ &\quad - (-1)^M \exp\left[-\frac{ST}{N_0}\left(1-\frac{1}{M}\right)\right] \end{aligned} \tag{A-6}$$

$$\begin{aligned} &= -\frac{1}{M}\sum_{j=1}^{M}(-1)^j\binom{M}{j} \\ &\quad \times \exp\left[-\frac{ST}{N_0}\left(1-\frac{1}{j}\right)\right] \end{aligned} \tag{A-7}$$

$$= \exp\left[-ST/N_0\right]\int_0^{ST/N_0}\exp\left[x\right]P_c(x)\,dx \tag{A-8}$$

$$= \frac{\exp\left[-ST/N_0\right]}{M}\int_0^{ST/N_0}\exp\left[x\right]\left[1+\varepsilon_M(x)\right]dx \tag{A-9}$$

Now, $\varepsilon_M(0) = 0$ and increases with $x$, therefore

$$\begin{aligned} \left(\frac{1-\exp\left[-ST/N_0\right]}{M}\right) &\leq \varepsilon_M - \varepsilon_{M-1} \\ &\leq \left(\frac{1-\exp\left[-ST/N_0\right]}{M}\right)(1+\varepsilon_M) \end{aligned} \tag{A-10}$$

or

$$\frac{ST}{N_0}\frac{1}{M} \leq \varepsilon_M - \varepsilon_{M-1} \leq \frac{ST}{N_0}(1+\varepsilon_M)\frac{1}{M} \tag{A-11}$$

However, from the left hand inequality we have $\varepsilon_M \geq \varepsilon_m$ for any $m \leq M$. Therefore

$$\frac{ST}{N_0}\frac{1}{m} \leq \varepsilon_m - \varepsilon_{m-1} \leq \frac{ST}{N_0}\frac{1}{m}(1+\varepsilon_M); \qquad m \leq M \tag{A-12}$$

Summing from 2 to $M$ yields

$$\frac{ST}{N_0}\sum_{m=2}^{M}\frac{1}{m} \leq \varepsilon_M \leq \left(\frac{ST}{N_0}\sum_{m=2}^{M}\frac{1}{m}\right)(1+\varepsilon_M) \tag{A-13}$$

thus

$$\frac{ST}{N_0}\sum_{m=2}^{M}\frac{1}{m} \leq \varepsilon_M \leq \frac{\dfrac{ST}{N_0}\displaystyle\sum_{m=2}^{M}\frac{1}{m}}{1-\dfrac{ST}{N_0}\displaystyle\sum_{m=2}^{M}\frac{1}{m}} \tag{A-14}$$

provided the denominator on the right hand side is positive. Since the upper and lower bounds approach each other as $ST/N_0 \rightarrow 0$ the desired result is established.

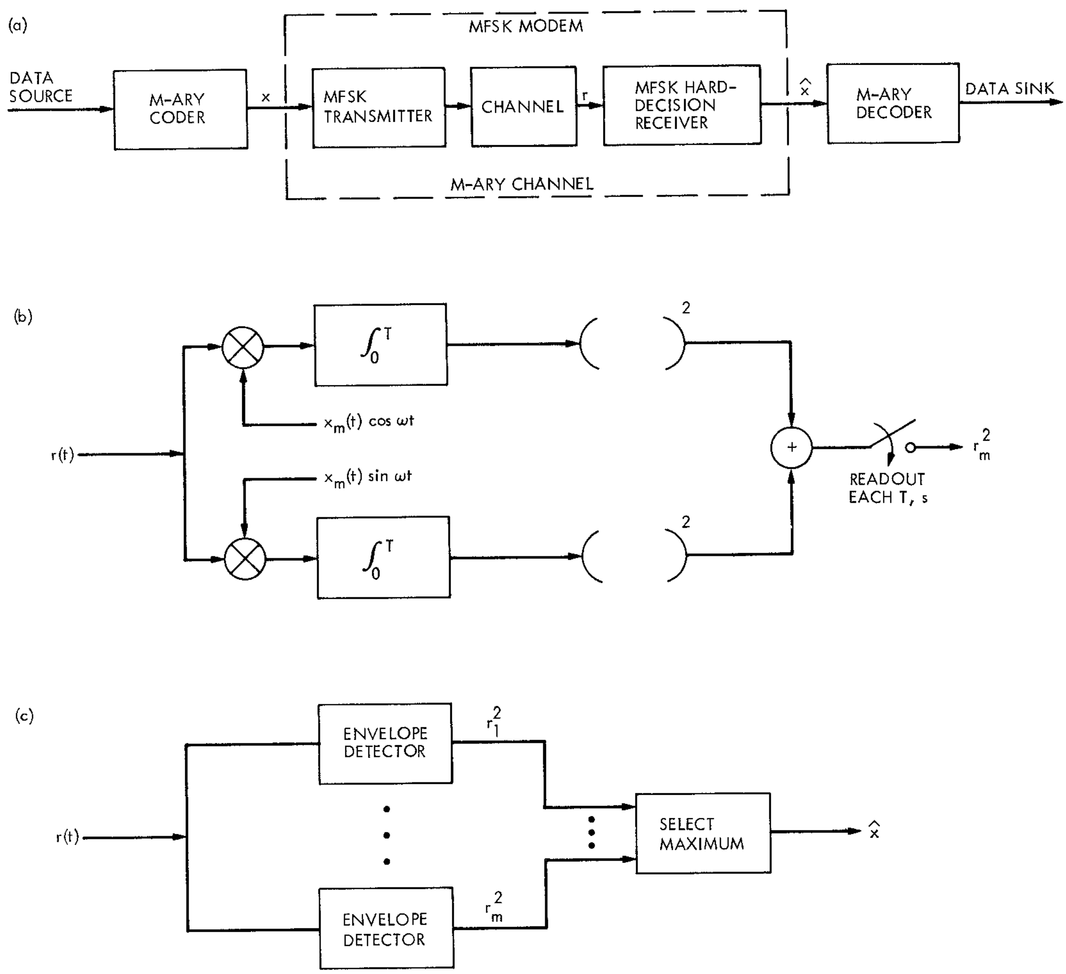


**Fig. 1. Coded hard-decision MFSK system diagram: (a) coder and modem; (b) envelope detector; (c) hard-decision MFSK receiver**

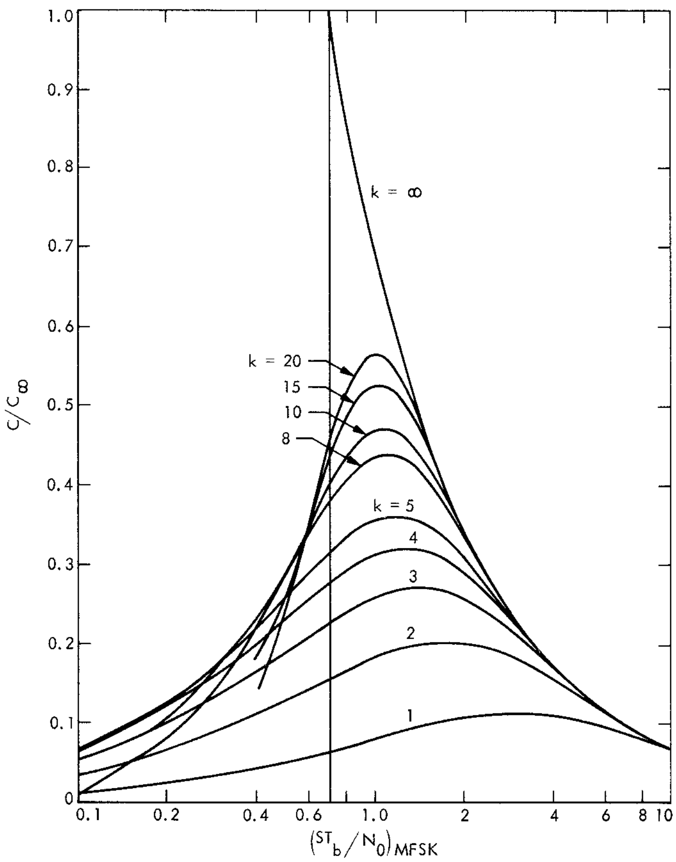


**Fig. 2. Capacity vs MFSK bit SNR**

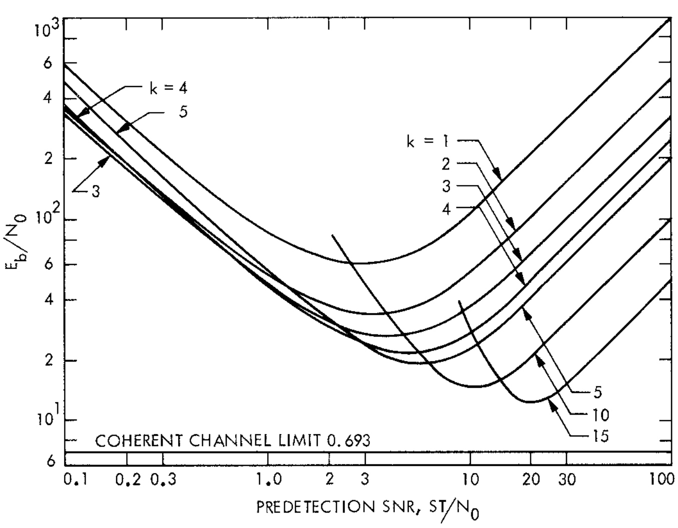


**Fig. 3. Minimum SNR per coded bit vs predetection SNR**

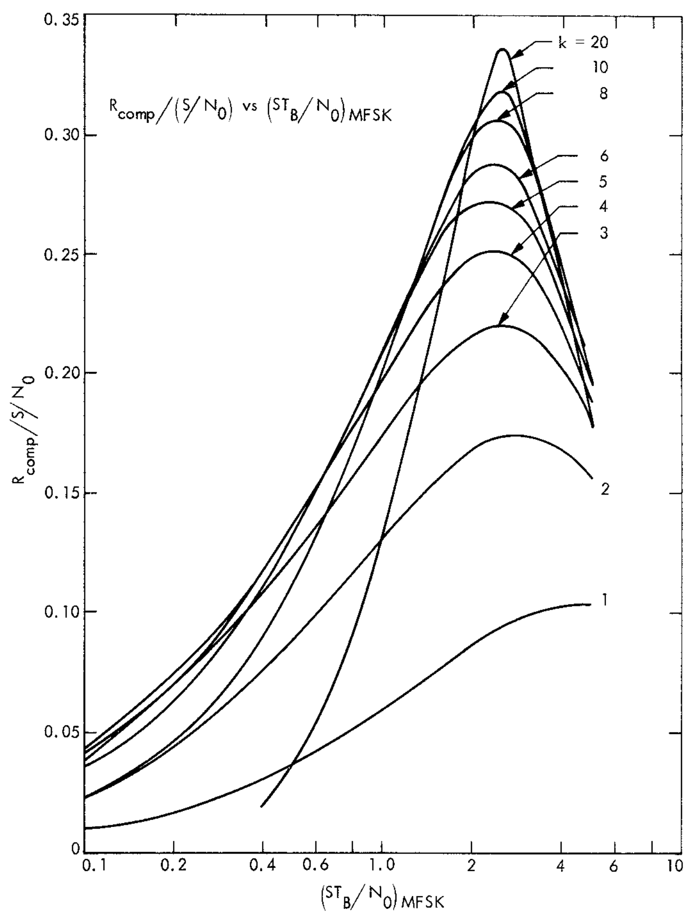


**Fig. 4. $R_{comp}$ vs MFSK bit SNR**

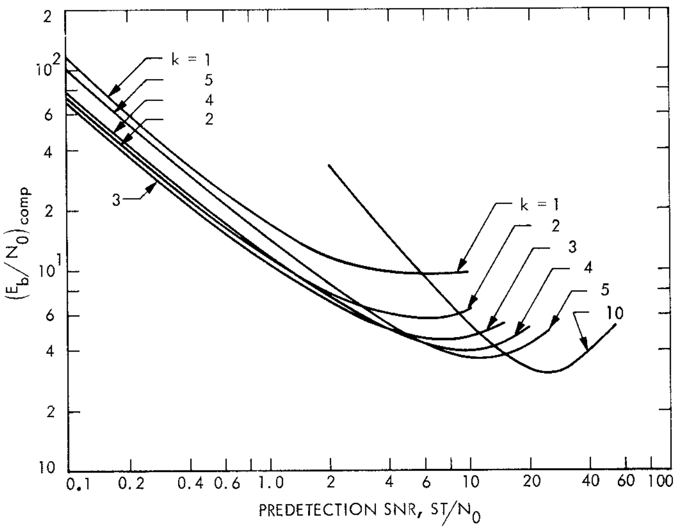


**Fig. 5. $(E_b/N_0)$ comp vs predetection SNR**

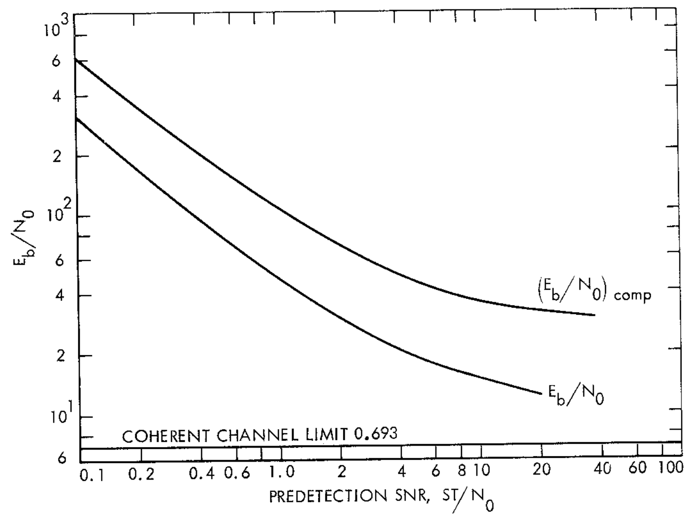


**Fig. 7. Performance vs predetection SNR**

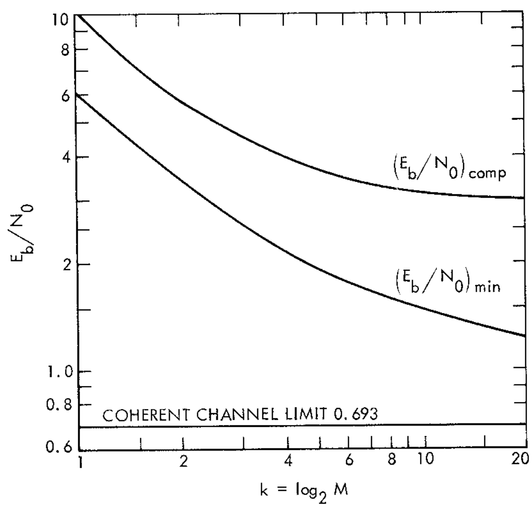


**Fig. 6. Performance vs number of MFSK signals**