Optimal Station Location for Two-Station Tracking

E. R. Rodemich
Communications Systems Research Section

A problem related to the optimal placement of three tracking stations for purposes of two-station tracking is formulated and solved.

I. Introduction

It is known (and easy to see) that three tracking antennas, however they are placed on the globe, cannot provide total coverage of the celestial sphere. However, since most interesting deep-space phenomena (spacecraft and natural astronomical objects) have small declinations, this fact is of little practical importance. For example, the DSN's 64-m antennas cannot "see" certain points on the celestial sphere at declinations of 28 deg or so.

Of course there is typically much overlap in the coverage of the celestial sphere provided by three stations, and this overlap can be used to good advantage, since simultaneous two-station tracking is possible in these doubly-covered regions. Two-station tracking is useful for a variety of reasons. For example, since the diurnal doppler amplitude is proportional to the cosine of the declination, accurate determination of spacecraft declination using doppler is quite difficult at declinations near zero. Two-station tracking can however be used for accurate goniometry at all declinations. Also, accurate tracking of constantly accelerating spacecraft could best be done with two stations. Finally, important astronomical knowledge can be gained from interferometric data.

Thus it is desirable to have as much of the celestial sphere as possible doubly covered. It is easy to see, however, that if no point on the celestial sphere is visible from all *three* stations simultaneously, the total area that is doubly covered is always the same. We therefore arrive at the question answered in this article: What is the maximum overlap possible between the coverages of the celestial sphere provided by *two* of the antennas, given that all *three* cover all celestial declinations that are less than a fixed amount?

More precisely, let us assume that the three antennas each cover a circular cap of angular radius β on the celestial sphere, and that it is required that the three circular caps cover all objects with declinations in the range $[-\alpha, \alpha]$. Then we wish to maximize the region of intersection of two of the caps. Surprisingly, the optimal configuration does *not* always have all three stations centered on the equator.

In the case of the 64-m net, $\beta = 84°$ and α is about 28°. It will turn out that there is a configuration of three $\alpha = 84°$ circular caps covering the band of declinations between $\pm 28°$ with two of the stations separated by only 27° on the globe. (This configuration has all three stations on the equator.) However, the closest pair of stations on the 64-m net (Goldstone and Madrid) are separated by 82°. Thus as far as two-station tracking is concerned, the 64-m net is far from optimally arranged.

II. Solution to the Problem

Three spherical caps of angular radius $\beta\,(0 < \beta < \pi/2)$ are to cover the band on the sphere with latitudes between $-\alpha$ and α $(0 \leqq \alpha < \pi/2)$. We want to maximize the area of the region of intersection of two of the caps.

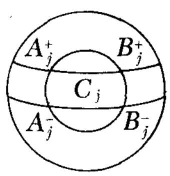
Denote latitude on the sphere by δ, and let ϕ measure longitude west from some reference point. Let the cap C_j have center at (δ_j, ϕ_j). Since each cap covers less than half the equator, following the equator around we see the caps in a cyclic order C_1, C_2, and C_3 such that $0 < \phi_j - \phi_{j-1} \pmod{2\pi} < \pi$. It can be shown that the caps cover the whole band in the same order that they cover the equator: if C_j intersects the band in the region S_j, then the band is the union of six disjoint regions $S_1 - S_2 - S_3$, $S_1 \cap S_2$, $S_2 - S_1 - S_3$, $S_2 \cap S_3$, $S_3 - S_1 - S_2$, and $S_3 \cap S_1$, each of which extends across the band.

C_j contains the points (δ, ϕ) that satisfy the inequality

$$\cos\delta_j \cos\delta \cos(\phi - \phi_j) + \sin\delta_j \sin\delta \geqq \cos\beta$$

The boundary of C_j intersects the boundaries of the band in four points A_j^+, A_j^-, B_j^+, and B_j^-, with coordinates

$$\left.\begin{aligned} A_j^\pm &: \delta = \pm\alpha, \phi = \phi_j + \cos^{-1}\frac{\cos\beta \mp \sin\delta_j \sin\alpha}{\cos\delta_j \cos\alpha} \\ B_j^\pm &: \delta = \pm\alpha, \phi = \phi_j - \cos^{-1}\frac{\cos\beta \mp \sin\delta_j \sin\alpha}{\cos\delta_j \cos\alpha} \end{aligned}\right\} \qquad (1)$$



We have a covering if A_j^+ and A_j^- are in C_{j+1}, and B_j^+ and B_j^- are in C_{j-1}.

Our problem is to maximize the area of $C_1 \cap C_2$. This is equivalent to maximizing the cosine of the angle between the centers of C_1 and C_2, which is

$$\cos\theta_{12} = \cos\delta_1 \cos\delta_2 \cos(\phi_1 - \phi_2) + \sin\delta_1 \sin\delta_2 \qquad (2)$$

There is an extremal covering for this problem, if α and β are such that some coverings exist. Hence it is sufficient to consider only those coverings for which there is no obvious variation that increases $\cos\theta_{12}$.

If we move each cap C_j to a position centered on the equator at $(0, \phi_j)$, we preserve the covering of the band. For, if we denote the points A_j^+, etc., in the new position by bars, $\overline{A_j^+}$ and $\overline{B_j^-}$ are diametrically opposite on the cap, hence

$$\begin{aligned} &\cos^2\alpha \cos[\phi(\overline{A_j^+}) - \phi(\overline{B_j^-})] - \sin^2\alpha = \\ &\quad \cos 2\beta \leqq \cos^2\alpha \cos[\phi(A_j^+) - \phi(B_j^-)] - \sin^2\alpha \end{aligned} \qquad (3)$$

and

$$\phi(\overline{A_j^+}) - \phi(\overline{B_j^-}) \geqq \phi(A_j^+) - \phi(B_j^-)$$

By symmetry, this is equivalent to

$$\phi_j - \phi(\overline{B_j^\mp}) \geqq \frac{1}{2}[\phi_j - \phi(B_j^-) + \phi_j - \phi(B_j^+)]$$

Similarly

$$\phi(\overline{A_{j-1}^\mp}) - \phi_{j-1} \geqq \frac{1}{2}[\phi(A_{j-1}^-) - \phi_{j-1} + \phi(A_{j-1}^+) - \phi_{j-1}]$$

If we started with a covering of the band, C_j and C_{j-1} meet on each boundary of the band, which makes

$$\phi_j - \phi(B_j^\pm) + \phi(A_{j-1}^\pm) - \phi_{j-1} \geqq \phi_j - \phi_{j-1}$$

Hence adding the preceding inequalities, we get

$$\phi_j - \phi(\overline{B_j^\mp}) + \phi(\overline{A_{j-1}^\mp}) - \phi_{j-1} \geqq \phi_j - \phi_{j-1}$$

showing that C_j and C_{j-1} overlap on each boundary of the band. This proves the assertion, that the new configuration covers the band. By Eq. (3),

$$\phi(\overline{A_j^\mp}) - \phi(\overline{B_j^\mp}) = 2\cos^{-1}\frac{\cos\beta}{\cos\alpha}$$

The upper boundary of the band can be covered by three arcs with this change in longitude only if

$$2\cos^{-1}\frac{\cos\beta}{\cos\alpha} \geqq \frac{2\pi}{3}$$

or

$$\cos\beta \leqq \frac{1}{2}\cos\alpha \tag{4}$$

This is a necessary and sufficient condition for the existence of coverings.

One way to increase $\cos\theta_{12}$ is to move the center of C_1 directly toward the center of C_2. The covering is preserved unless A_3^+ or A_3^- goes out of C_1. Hence, for an extremal covering, one of these points must lie on the boundary of C_1, so that $A_3^+ = B_1^+$ or $A_3^- = B_1^-$. Similarly, $B_3^+ = A_2^+$ or $B_3^- = A_2^-$.

By reflecting across the equator, we can make $A_3^- = B_1^-$. Suppose first that the only other one of these relations that holds is $B_3^+ = A_2^+$. Move the center of C_3 along the great circle equidistant from A_2^+ and B_1^-. There is a nearby position where A_2^+ and B_1^- fall inside C_3, unless the center of C_3 is on the great circle arc from A_2^+ to B_1^-. Then, A_2^+ and B_1^- are at the ends of a diameter of C_3, so $\delta_3 = 0$. Since A_2^- and B_1^+ are inside C_3, $\delta_1 > 0$ and $\delta_2 < 0$. If we go to the covering with centers on the equator at $(0, \phi_j)$ as described above, then by Eq. (3), $\cos\theta_{12}$ is increased. Hence we are not at an extremal covering.

This shows that we need only consider coverings with $A_3^- = B_1^-$ and $B_3^- = A_2^-$. Since B_1^+ and A_2^+ are in C_3,

$$\delta_1 \geqq -\delta_3, \qquad \delta_2 \geqq \delta_3 \tag{5}$$

Now we prove a lemma.

Lemma. A covering with $\delta_3 < 0$, $A_3^- = B_1^-$, $B_3^- = A_2^-$ and $\delta_1 > -\delta_3$ cannot be extremal unless $A_1^- = B_2^-$ and $\delta_1 = \delta_2$.

Proof: First suppose $A_1^- \neq B_2^-$. Then C_1 can be rotated about A_3^- in either direction without destroying the covering. If the covering is extremal, $\cos\theta_{12}$ is at a stationary point under this rotation, which implies that the center of C_1 lies on the great circle through A_3^- and the center of C_2.

If $\delta_2 > -\delta_3$, we find likewise that this circle goes through B_3^-. By symmetry, $\delta_2 = \delta_1$ and $\phi_3 - \phi_2 = \phi_1 - \phi_3$. Then by Eq. (3),

$$\begin{aligned}\cos\theta_{12} &= \cos^2\delta_1\cos[2\pi - 2(\phi_1 - \phi_3)] + \sin^2\delta_1\\ &= 1 - 2[\cos\delta_1\sin(\phi_1 - \phi_3)]^2\end{aligned}$$

If we vary δ_1 and δ_2, keeping $\delta_1 = \delta_2$ and the hypotheses of the lemma, $\cos\delta_1\sin(\phi_1 - \phi_3)$, which is the distance of the center of C_1 from the vertical plane through the center of the sphere and the center of C_3, has a relative maximum for the given configuration. Hence $\cos\theta_{12}$ is at a relative minimum, and can be increased by varying δ_1 in either direction.

For $\delta_2 = -\delta_3$, $B_3^+ = A_2^+$, and we can only rotate C_2 upwards about B_3^-. If this does not increase $\cos\theta_{12}$, the great circle through the centers of C_1 and C_2 must pass above B_3^-. Let the highest point on this circle have longitude ϕ_4, and let it cross $\delta = -\alpha$ at ϕ_5 after passing through the center of C_2. Then we have $\phi_5 - \phi_4 = \phi_4 - \phi(A_3^-)$. Since $\delta_1 > \delta_2$, $|\phi_1 - \phi_4| < \phi_2 - \phi_4$, and so $\phi_5 - \phi_2 < \phi_1 - \phi(A_3^-)$. Also, $\delta_1 > \delta_2$ implies $\phi(B_3^-) - \phi_2 > \phi_1 - \phi(A_3^-)$, hence $\phi_5 - \phi_2 < (B_3^-) - \phi_2$. But this makes the circle pass below B_3^-, a contradiction.

Now we know that $A_1^- = B_2^-$. The circle $\delta = -\alpha$ is partitioned by C_1, C_2, and C_3 into three parts. On each part, the variation in ϕ can be found from Eq. (1). Since the total variation is 2π, we get

$$(\phi(A_1^-) - \phi_1) + (\phi(A_2^-) - \phi_2) + (\phi(A_3^-) - \phi_3) = \pi$$

or

$$\cos^{-1}\frac{\cos\beta + \sin\delta_1\sin\alpha}{\cos\delta_1\cos\alpha} + \cos^{-1}\frac{\cos\beta + \sin\delta_2\sin\alpha}{\cos\delta_2\cos\alpha} + \cos^{-1}\frac{\cos\beta + \sin\delta_3\sin\alpha}{\cos\delta_3\cos\alpha} = \pi \tag{6}$$

Then $\phi_1 - \phi_2 = \phi(A_1^-) - \phi_1 + \phi(A_2^-) - \phi_2 = \pi - (\phi(A_3^-) - \phi_3)$, which is fixed when we vary δ_1 and δ_2 subject to Eqs. (5) and (6). Differentiating Eq. (6), we get

$$\frac{d\delta_2}{d\delta_1} = -\frac{\sin\alpha + \sin\delta_1\cos\beta}{\cos^2\delta_1\sqrt{1-u_1^2}} \bigg/ \frac{\sin\alpha + \sin\delta_2\cos\beta}{\cos^2\delta_2\sqrt{1-u_2^2}} < 0 \tag{7}$$

where $u_j = (\cos\beta + \sin\delta_j\sin\alpha)/(\cos\delta_j\cos\alpha)$. Hence the range of δ_1 satisfying Eqs. (5) and (6) is an interval with $\delta_2 > \delta_1 = |\delta_3|$ at one end, $\delta_1 > \delta_2 = |\delta_3|$ at the other end. Differentiate Eq. (2) with respect to δ_1:

$$\begin{aligned}\frac{d\cos\theta_{12}}{d\delta_1} &= \\ &[-\cos\delta_1\sin\delta_2\cos(\phi_1 - \phi_2) + \sin\delta_1\cos\delta_2]\frac{d\delta_2}{d\delta_1}\\ &- \sin\delta_1\cos\delta_2\cos(\phi_1 - \phi_2) + \cos\delta_1\sin\delta_2\end{aligned}$$

By using Eq. (7) and the relation

$$\cos(\phi_1 - \phi_2) = \cos[(\phi(A_1^-) - \phi_1) + (\phi(A_2^-) - \phi_2)]$$
$$= u_1 u_2 - \sqrt{1-u_1^2}\sqrt{1-u_2^2}$$

this can be reduced to

$$\frac{d\cos\theta_{12}}{d\delta_1} = (\text{positive quantity}) \cdot (\sin\delta_2 - \sin\delta_1)$$

It follows that $\cos\theta_{12}$ has a maximum at $\delta_1 = \delta_2$, which proves the lemma.

Now we continue with considering all coverings for which $A_3^- = B_1^-$ and $B_3^- = A_2^-$. There are essentially three classes:

class (I): A_2^+ and B_1^+ interior to C_3

class (II): $A_2^+ = B_3^+$, B_1^+ interior to C_3

class (III): $A_2^+ = B_3^+$ and $B_1^+ = A_3^+$

For a covering in class (I), the center of C_3 may be moved along its meridian to bring A_2^- and B_1^- inside C_3, unless the center is already at the closest point to A_2^- and B_1^-. Then A_3^- and B_3^- are at the ends of a diameter of C_3, which occurs for

$$\delta_3 = -\sin^{-1}(\sin\alpha/\cos\beta) \tag{8}$$

The hypotheses of the lemma are satisfied, so $A_1^- = B_2^-$ and $\delta_2 = \delta_1$. Equation (6) becomes

$$2\cos^{-1}\frac{\cos\beta + \sin\delta_1\sin\alpha}{\cos\delta_1\cos\alpha} + \cos^{-1}\frac{\cos\beta + \sin\delta_3\sin\alpha}{\cos\delta_3\cos\alpha} = \pi$$

Transpose the last term on the left and take the cosine of each side. By the use of some trigonometric formulas, the result is

$$2\left(\frac{\cos\beta + \sin\delta_1\sin\alpha}{\cos\delta_1\cos\alpha}\right)^2 + \frac{\cos\beta + \sin\delta_3\sin\alpha}{\cos\delta_3\cos\alpha} = 1 \tag{9}$$

From Eq. (2),

$$\cos\theta_{12} = \sin^2\delta_1 - \cos^2\delta_1 \cdot \frac{\cos\beta + \sin\delta_3\sin\alpha}{\cos\delta_3\cos\alpha} \tag{10}$$

This gives a possible extremal covering.

For a covering in class (II), the center of C_3 can be varied along the great circle equidistant from A_1^- and B_2^+. If $\delta_3 > 0$, varying in the direction of decreasing latitude brings all the points A_1^-, B_2^+, and B_2^- inside C_3. Hence $\delta_3 \leqq 0$. If $\delta_3 = 0$, $\delta_2 = 0$ and $\delta_1 > 0$. Then moving the center of C_1 to $(0, \phi_1)$ gives a covering for which $\cos\theta_{12}$ is larger. If $\delta_3 < 0$, $\delta_1 > \delta_2 = -\delta_3$. The lemma applies, and gives a contradiction. Hence there are no extremal coverings in class (II).

For a covering in class (III), $\delta_1 = \delta_2 = -\delta_3$. We can reflect the covering in the equator to make $\delta_1 \geqq 0$. Then we have a covering if C_1 and C_2 overlap on the lower boundary of the band. The condition for this is

$$2\cos^{-1}\frac{\cos\beta + \sin\delta_1\sin\alpha}{\cos\delta_1\cos\alpha} + \cos^{-1}\frac{\cos\beta - \sin\delta_1\sin\alpha}{\cos\delta_1\cos\alpha} \geqq \pi \tag{11}$$

The left side of Eq. (11) is a decreasing function of $\delta_1 \geqq 0$, so we have an interval $0 \leqq \delta_1 \leqq \delta_{10}$ to consider, where equality occurs in Eq. (11) at $\delta = \delta_{10}$.

Put

$$u = (\cos\beta + \sin\delta_1\sin\alpha)/(\cos\delta_1\cos\alpha)$$

$$v = (\cos\beta - \sin\delta_1\sin\alpha)/(\cos\delta_1\cos\alpha)$$

and

$$\gamma = \phi_1 - \phi_3$$
$$= (\phi_1 - \phi(B_1^-)) + (\phi(A_3^-) - \phi_3)$$
$$= \cos^{-1}u + \cos^{-1}v$$

By symmetry, $\phi_2 - \phi_1 + 2\gamma = 2\pi$, and

$$\cos\theta_{12} = \cos^2\delta_1\cos(2\pi - 2\gamma) + \sin^2\delta_1$$
$$= 1 - 2\cos^2\delta_1\sin^2\gamma$$

Let

$$R = (1 - \cos\theta_{12})/2$$
$$= \cos^2\delta_1\sin^2\gamma$$
$$= \cos^2\delta_1[u\sqrt{1-v^2} + v\sqrt{1-u^2}]^2 \tag{12}$$

Eliminating radicals,

$$[R + \cos^2\delta_1(2u^2v^2 - u^2 - v^2)]^2 = 4u^2v^2\cos^4\delta_1 \times (1-u^2)(1-v^2) \tag{13}$$

In the range of δ_1 of interest, u^2 and v^2 are less than 1, and Eq. (13) has real roots for R. The right side of Eq. (13) becomes negative when we pass $\delta_1 = \beta - \alpha$, where u becomes greater than 1. Those points (δ_1, R) with $0 \leqq \delta_1 \leqq \beta - \alpha$, which satisfy Eq. (13), form one connected curve in the $\delta_1 R$ plane. If we multiply Eq. (13) by $\cos^4 \alpha \cos^2 \delta_1$, it can be put in the form

$$\begin{aligned}\cos^4 \alpha \cos^2 \delta_1 R^2 + 4R\,[(\cos^2 \beta - \sin^2 \alpha + \sin^2 \alpha \cos^2 \delta_1) \\ - \cos^2 \alpha \cos^2 \delta_1 (\cos^2 \beta + \sin^2 \alpha \\ - \sin^2 \alpha \cos^2 \delta_1)] + 16 \cos^2 \beta \sin^2 \alpha \\ \times (\cos^2 \delta_1 - \cos^4 \delta_1) = 0 \qquad (14)\end{aligned}$$

where the left side is a quadratic function of $\cos^2 \delta_1$.

The points of the $\delta_1 R$ plane with $0 \leqq \delta_1 \leqq \beta - \alpha$ which satisfy Eq. (14) lie on a curve with its end points on $\delta_1 = 0$, at $R = 0$ and $R = R_0 = 4 \cos^2 \beta (\cos^2 \alpha - \cos^2 \beta)/\cos^4 \alpha$. Since Eq. (14) is quadratic in $\cos^2 \delta_1$, each value of R can correspond to at most two values of δ_1. This implies that the curve passes through $0 < R < R_0$ just once, monotonically.

An extremal covering minimizes R. R_0 is the value for a covering with $\delta_1 = 0$. Since there is no local minimum on $(0, R_0)$, only $\delta_1 = 0$ or $\delta_1 = \delta_{10}$ can give extremal coverings.

This gives the following three coverings to consider:

(1) $\delta_1 = \delta_2 = \delta_3 = 0$. Here $\cos \theta_{12} = 1 - 2R_0 = 1 - 8 \cos^2 \beta (\cos^2 \alpha - \cos^2 \beta)/\cos^4 \alpha$

(2) $\delta_1 = \delta_2 = -\delta_3 > 0$. We need equality in Eq. (11). Then Eq. (9) applies with $\delta_3 = -\delta_1$, or $2u^2 + v = 1$. Then from Eq. (12),

$$\cos \theta_{12} = 1 - 2 \cos^2 \delta_1 (1 - u^2)$$

$$\cos \theta_{12} = 1 - 2 \cos^2 \delta_1 + 2 (\cos \beta + \sin \delta_1 \sin \alpha)^2/\cos^2 \alpha \qquad (15)$$

The condition of Eq. (9) is

$$2 (\cos \beta + \sin \delta_1 \sin \alpha)^2 + \cos \delta_1 \cos \alpha (\cos \beta - \sin \delta_1 \sin \alpha) = \cos^2 \delta_1 \cos^2 \alpha \qquad (16)$$

We also have the covering found in class (I):

(3) $\delta_3 = -\sin^{-1} (\sin \alpha/\cos \beta)$, $\delta_1 = \delta_2 > -\delta_3$. Here δ_1 satisfies Eq. (9) and $\cos \theta_{12}$ is given by Eq. (10).

For any α, β, the optimal covering must be one of these three. The covering (2) is better than (1) if

$$\Delta = \frac{4 \cos^2 \beta (\cos^2 \alpha - \cos^2 \beta)}{\cos^4 \alpha} - \cos^2 \delta_1 + \frac{(\cos \beta + \sin \delta_1 \sin \alpha)^2}{\cos^2 \alpha} > 0$$

where δ_1 satisfies Eq. (16). Equation (16) leads to a 4th-degree equation in $\sin \delta_1$. If we eliminate $\sin \delta_1$ between this equation, and the equation $\Delta = 0$, we get a relation between α and β that holds on the curve separating the regions where (1) or (2) is better. This equation has the following parametric solution:

$$\left.\begin{aligned} y &= \frac{9x^3 - 6x^2 + x}{10x^3 - 13x^2 + 8x - 1} \\ t &= x(y-1) \\ \sin \alpha &= \sqrt{t} \\ \cos \beta &= \cos \alpha \sqrt{t/y} \end{aligned}\right\} \qquad (17)$$

The curve is generated by $3 \leqq x \leqq 3 + \sqrt{8}$.

When the covering (3) exists, $\sin \alpha \leqq \cos \beta$, which already implies that (2) is better than (1). Hence the region of (α, β) for which (3) is extremal lies in the region where (2) is better than (1). Whenever (3) exists, the value of δ_1 for (2) is greater than $\sin^{-1} (\sin \alpha/\cos \beta)$. This implies that (2) is not extremal, for varying δ_3 toward 0 in (2) brings the points $B_1^{\pm}, A_2^{\pm}$ inside C_3. Hence (3) is extremal when it exists, and the boundary of the region of (α, β) for which this occurs is the curve on which Eq. (15) is satisfied by $\delta_1 = \sin^{-1} (\sin \alpha/\cos \beta)$, or

$$2 (\cos^2 \beta + \sin^2 \alpha)^2 + \cos \alpha (\cos^2 \beta - \sin^2 \alpha)^{3/2} = \cos^2 \alpha (\cos^2 \beta - \sin^2 \alpha)$$

Rationalizing, we get a 4th-degree equation for $\cos^2 \beta$ in terms of α. This equation has the parametric solution

$$\left.\begin{aligned} y &= \frac{(1 - x^2)(3x + 5)}{26x^3 + 52x^2 + 42x + 8} - \frac{(1 - x)^2 \sqrt{9x^2 + 14x + 9}}{26x^3 + 52x^2 + 42x + 8}, \qquad 0 \leqq x \leqq 1 \\ \sin \alpha &= \sqrt{xy} \\ \cos \beta &= \sqrt{y} \end{aligned}\right\} \qquad (18)$$

In Fig. 1, the region $\cos \beta \leqq 1/2 \cos \alpha$ is shown, divided into the regions where each configuration is optimal.

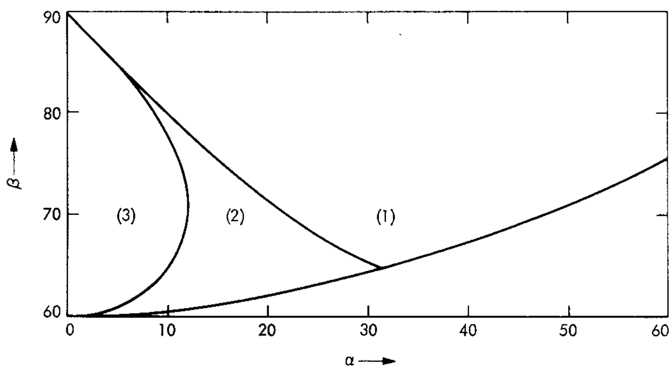


Fig. 1. The regions where each configuration is optimal