Frame Synchronization Performance Analysis for MVM'73 Uncoded Telemetry Modes

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This paper describes a practical frame synchronization (sync) acquisition and maintenance algorithm based on the Hamming distance metric, which is a generalization of the scheme developed for the 1973 Mariner mission to Venus and Mercury (MVM'73). For the special case of uncoded phase-shift-keyed data received over the binary symmetric channel, formulas are derived for computing an upper bound on the probability of false sync acquisition, the mean time to sync acquisition, and the mean time to the subsequent loss of sync, as a function of the bit error rate, frame length, sync word length, and algorithm parameters. These formulas are directly applicable to the uncoded MVM'73 telemetry modes, and a numerical example relating to the 117.6 kilobits/s real time TV mode is included.

I. Introduction

The problem of establishing frame synchronization (sync) usually involves the identification of received sync words periodically interspersed with random data (Ref. 1). For uncoded binary signals received over the additive white Gaussian noise channel, the frame sync words are often located by finding segments of the received data that are highly correlated with the transmitted sync word (Ref. 2). Massey (Ref. 3) has in fact shown that the optimum sync word search requires the addition of a correction term to this correlation rule. If hard decisions are made on the received data prior to the acquisition of frame synchronization, the problem is reduced to detecting the sync words in uncoded data received over the binary symmetric channel: in this case, the optimum frame sync decision is based on the Hamming distance metric.

This article examines the performance of a practical frame sync acquisition and maintenance algorithm for uncoded phase-shift-keyed (PSK) data received over the binary symmetric channel. Because the detected PSK data necessarily have a binary phase ambiguity that cannot be resolved until frame sync is established, the algorithm bases its sync decision on a modified Hamming distance rule. Instead of making a hard sync decision over a single frame of received data, the algorithm scans the detected bit stream, searching for a sequence whose Hamming distance from the sync word satisfies a threshold test. When the bit error rate is high and the threshold is stringent, the reliability of the sync decision is improved at the expense of delaying that decision for several frames. For additional reliability, a hard sync decision is made only when two received sequences one frame apart both satisfy the threshold test.

The frame sync algorithm described herein is not exceptionally innovative, and variations on the same approach have been used on past Mariner missions. It is in fact a two-threshold generalization of the particular acquisition and maintenance algorithm that was developed for use with the uncoded and biorthogonally coded telemetry modes of the Mariner 10 spacecraft (MVM'73). The principal purpose of this article is to document the performance analysis of this algorithm for uncoded PSK data received over the binary symmetric channel. Markov models are employed in this analysis to derive formulas for computing an upper bound on the probability of false sync acquisition, as well as the mean time to sync acquisition, and the mean time to the subsequent loss of sync, for arbitrary bit error rates, frame lengths, sync word lengths, and threshold values. These formulas are directly applicable to the uncoded MVM'73 telemetry modes, and, as a numerical example below, they are used to determine optimum threshold values as a function of the channel bit error rate for the high-rate (117.6 kilobits/s) real time TV data mode. It should be noted here that the author has extended the algorithm performance analysis to include the MVM'73 telemetry modes employing a (32, 6) biorthogonal code, although the work has not been published to date. However, numerical results for the 22.05 kilobits/s coded TV telemetry mode have been used to select proper algorithm threshold values for the MVM'73 mission.

II. Frame Sync Algorithm

The operation of the acquisition and maintenance modes of the frame sync algorithm are detailed in the flow diagrams of Figs. 1 and 2. A brief description of the algorithm now follows to establish some notation that will be needed later for the performance analysis.

Assume that the transmitted information is organized into M-bit frames composed of an L-bit sync word followed by $M-L$ statistically independent, equally likely data bits. The detected bit stream contains independent bit errors: a particular bit will be incorrect with probability ϵ.

Let the received bit stream be represented by the binary sequence $r_1, r_2, r_3 \cdots$, and define an arbitrary L-tuple $\boldsymbol{\rho}_m = (r_m, r_{m+1}, \cdots, r_{m+L-1})$. The frame sync objective is to identify which of the $\boldsymbol{\rho}_m$'s are received frame sync words. When a binary PSK signal is demodulated using a carrier reference derived from the modulated signal, there is an inherent binary phase ambiguity in the detector output: that is, the received bit stream can be inverted data ($\overline{\text{data}}$) with probability ½. Consequently, for low bit error rates, if a particular $\boldsymbol{\rho}_m$ is in fact a received sync word, its Hamming distance d_m from the transmitted sync word is equally likely to be near 0 or L. As indicated in Figs. 1 and 2, the algorithm threshold test reflects this property.

Suppose the receiver is out of sync, and the acquisition mode is initiated. The overlapping L-bit received sequences $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_3, \cdots$ are examined in succession, and the corresponding Hamming distances $d_1, d_2, d_3 \cdots$ from the sync word are computed. The algorithm uses two threshold tests: $d_m \leqq T_1$, and $d_m \geqq L - T_1$ (the latter indicating the detection of $\overline{\text{data}}$). If it finds a sequence $\boldsymbol{\rho}_{\hat{m}}$ which satisfies either threshold test, and if the sequence $\boldsymbol{\rho}_{\hat{m}+M}$ one frame ahead satisfies the *same* threshold test, frame sync is established. That is, a hard decision is made that $\boldsymbol{\rho}_{\hat{m}}$ is a received sync word, the data polarity is deduced according to which of the two threshold tests was satisfied by $\boldsymbol{\rho}_{\hat{m}}$ and $\boldsymbol{\rho}_{\hat{m}+M}$, and the maintenance mode is entered.

If the communication link consisted only of a binary symmetric channel, there would be no need for a frame sync maintenance mode. If bit synchronization could be maintained perfectly once frame sync was acquired, a simple bit counter could maintain frame sync indefinitely. Unfortunately, in practice, interface buffers within the overall communication link typically result in random deletions and insertions of bits in the received data stream. It is the function of the frame sync maintenance mode to detect these occurrences, and to signal the resulting loss of frame sync.

Referring to the flow diagram in Fig. 2, it is seen that the data polarity decision made when frame sync was established confines the maintenance mode to one of the two threshold tests, $d_m \leqq T_2$ and $d_m \geqq L - T_2$. (Note that the acquisition and maintenance modes use different thresholds in general.) In the acquisition mode, it was decided that $\boldsymbol{\rho}_{\hat{m}}$ was a received frame sync word, since $d_{\hat{m}}$ and $d_{\hat{m}+M}$ satisfied the same T_1 threshold test. The maintenance mode examines only presumed received sync words of the form $\boldsymbol{\rho}_{\hat{m}+kM}$, where k is a positive integer. If $T_2 \geqq T_1$ (see discussion below), $d_{\hat{m}}$ and $d_{\hat{m}+M}$ must satisfy the T_2 threshold test; consequently, the maintenance mode can start with $\rho_{\hat{m}+2M}$, as indicated in the diagram. When two consecutive sequences that should be received sync words fail the threshold test, the receiver is declared to be out of sync, and the algorithm reverts to the acquisition mode.

The following design objectives influence the selection of the thresholds T_1 and T_2. For convenience, use the following terminology:

$Pr[FS]$: probability of false sync acquisition.

MTS: mean time to sync acquisition, in frames (expected value of $\hat{m}/M$), conditioned on a correct sync decision (assuming $Pr[FS] << 1$).

$MTLS$: mean time to loss of sync following correct sync acquisition, in frames, neglecting bit insertions or deletions.

Ideally, we would like $Pr[FS]$ and MTS to be small, while $MTLS$ is large. However, for fixed values of ε, L, and M, $Pr[FS]$ decreases while MTS increases monotonically as T_1 decreases, and the reverse condition also holds. So the selection of T_1 involves a tradeoff between $Pr[FS]$ and MTS. Typically T_1 is made relatively small, favoring $Pr[FS]$, so that when a frame sync decision is made, it is likely to be correct. Fortunately, it will be evident in the numerical examples below that T_1 does not have to be too large for MTS to be near the minimum value of ½.

With regard to the selection of T_2, it should be noted that following the acquisition of frame sync, there are two situations that would cause the maintenance mode to decide that sync is lost. If one or more bits are erroneously deleted or inserted into the received data stream, thereafter, the received sync words would not coincide with the L-bit sequences examined by the maintenance mode. If the sync word is a Barker (Ref. 1) or a Neuman-Hofman (Ref. 4) sequence, its correlation properties would ensure that the Hamming distances of the L-bit received sequences from the sync word are likely to be near $L/2$. In this case, a threshold T_2 on the order of $L/4$ would be small enough to signal the loss of sync. On the other hand, if the channel state is such that no bit deletions or insertions are occurring, the maintenance mode will still declare a loss of sync, falsely, if two consecutive received sync words contain more than T_2 bit errors. It is this circumstance, relating only to the binary symmetric channel model, that is measured by the parameter $MTLS$ defined above. (To include the random deletions and insertions in the channel model, a multi-state Markov approach would be required; this complication is avoided in this article.) To maximize $MTLS$, T_2 should be as large as possible; a threshold near $L/4$ is compatible with this objective. In conclusion, T_2 should be near $L/4$, and T_1 should be closer to zero.

III. Algorithm Performance

The following formulas relating to the performance of the frame sync algorithm for uncoded PSK data received over the binary symmetric channel are derived in the Appendix:

$$Pr[FS] \leq$$
$$(M-1)\frac{\gamma^2}{4} + \min\left[A, (M-1)A\frac{\gamma^2}{2} + \frac{3}{2}(L-1)\gamma^2\right] \qquad (1)$$

where it is assumed that $M >> L$, and

$$\gamma \equiv 2^{-(L-1)} \sum_{\ell=0}^{T_1} \binom{L}{\ell} \qquad (2)$$

$$\eta(T) \equiv 1 - \sum_{\ell=0}^{T} \binom{L}{\ell} \varepsilon^{\ell} (1-\varepsilon)^{L-\ell} \qquad (3)$$

$$A \equiv \frac{\eta(T_1)[3 - 2\eta(T_1)]}{[1-\eta(T_1)]^2} \qquad (4)$$

$$MTS = \frac{M+1}{2M} + A \text{ frames;} \qquad Pr[FS] << 1 \qquad (5)$$

$$MTLS = \frac{1 + \eta(T_2) - \eta^2(T_2)}{\eta^2(T_2)} \text{ frames} \qquad (6)$$

Parameters γ and $\eta(T)$ have physical interpretations:

γ: the probability that a particular L-bit received sequence, composed of independent, equally likely 1's and 0's, satisfies either acquisition threshold test (probability of a "false alarm").

$\eta(T)$: the probability that a particular received sync word (or complemented sync word if data is detected) has more than T bit errors, and fails the corresponding threshold test (probability of a "miss").

A numerical example will serve to demonstrate the behaviour of $Pr[FS]$, MTS, and $MTLS$ as a function of threshold values for fixed L, M, and ε. The 1973 Mariner 10 mission to Venus and Mercury has a high-rate (117.6 kilobits/s), uncoded PSK telemetry mode for transmitting real time TV data to Earth. This mode has a frame length $M = 7056$ bits, and a sync word length $L = 31$ bits (a pseudo-noise or PN sequence). As the spacecraft-to-Earth range varies during the mission, so does the bit error rate ε; it is anticipated that during the primary mission, ε will not exceed $1/20$. On this mission, the acquisition and maintenance thresholds are identical, and are denoted by T. For bit error rates in the range $1/20 \leq \varepsilon \leq 1/500$, the frame sync algorithm performance is summarized in Table 1. A single TV picture is composed of 700 frames of data; one of the design objectives in selecting T was that $MTLS$ exceed 700, so that frame sync would be likely to be maintained over a TV picture. Subject to this constraint, as well as the threshold trade-offs discussed earlier, the recommended values of T are circled in Table 1. It is evident from the table that if T is adjusted for changes in ε, MTS will be near $1/2$, $MTLS$ will exceed 700, while $Pr\,[FS]$ is less than 2×10^{-6}, for ε less than $1/20$.

IV. Summary

This paper has considered the problem of acquiring and maintaining frame synchronization in uncoded PSK data received over the binary symmetric channel. To this end, a practical algorithm was described that computes the Hamming distances of L-bit segments of the received bit stream from the L-bit sync word, and applies these distances to a threshold test.

In the acquisition mode, the algorithm ensures a reliable frame sync decision by requiring that two consecutive L-bit received sequences one frame apart satisfy a stringent threshold test. The maintenance mode is designed to flag the loss of frame sync due to random deletions or insertions of bits in the received data stream; at the same time, its threshold test allows sufficient errors in the received sync words to pass so that the loss of sync is not likely to be declared falsely.

The performance of this algorithm was analyzed with the aid of Markov models of the threshold test operation. An upper bound for the probability of false sync acquisition, the mean time to sync acquisition, and the mean time to loss of sync were computed as a function of the bit error rate, frame length, sync word length, and algorithm threshold values.

This algorithm is a generalization of the scheme developed to provide frame synchronization on the Mariner 10 spacecraft-to-Earth telemetry link. As a numerical example, the derived formulas were used to predict the algorithm performance for the spacecraft's high-rate, real-time TV data mode.

References

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Table 1. Frame sync algorithm performance for $M = 7056$, $L = 31$ (note that $Pr\,[FS]$ is actually an upper bound)

ε	T	$Pr\,[FS]$	MTS, frames	$MTLS$, frames
1/20	5	7×10^{-5}	0.51	7×10^{4}
	4	2×10^{-6}	0.55	3179
	3	6×10^{-8}	0.72	236
1/30	4	2×10^{-6}	0.51	9×10^{4}
	3	4×10^{-8}	0.56	2839
	2	6×10^{-10}	0.78	155
1/40	4	2×10^{-6}	0.50	1×10^{6}
	3	4×10^{-8}	0.52	2×10^{4}
	2	5×10^{-10}	0.63	597
1/50	3	4×10^{-8}	0.51	9×10^{4}
	2	4×10^{-10}	0.57	1822
	1	3×10^{-12}	0.96	69
1/100	3	4×10^{-8}	0.50	2×10^{7}
	2	4×10^{-10}	0.51	8×10^{4}
	1	2×10^{-12}	0.62	704
	0	5×10^{-15}	1.73	17
1/200	2	4×10^{-10}	0.50	4×10^{6}
	1	2×10^{-12}	0.53	9064
	0	3×10^{-15}	1.03	54
1/500	2	4×10^{-10}	0.50	8×10^{8}
	1	2×10^{-12}	0.51	3×10^{5}
	0	2×10^{-15}	0.70	292

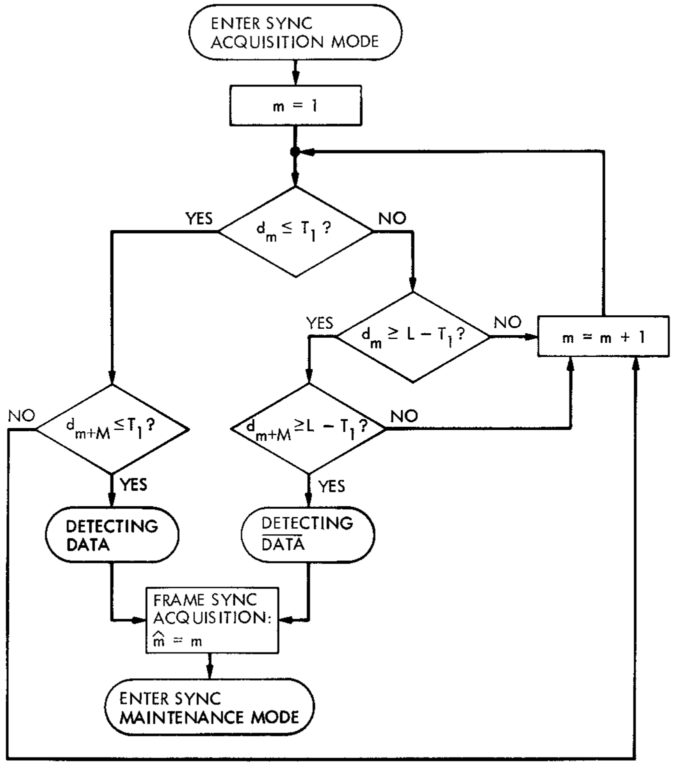

Fig. 1. Frame sync algorithm: acquisition mode flow diagram

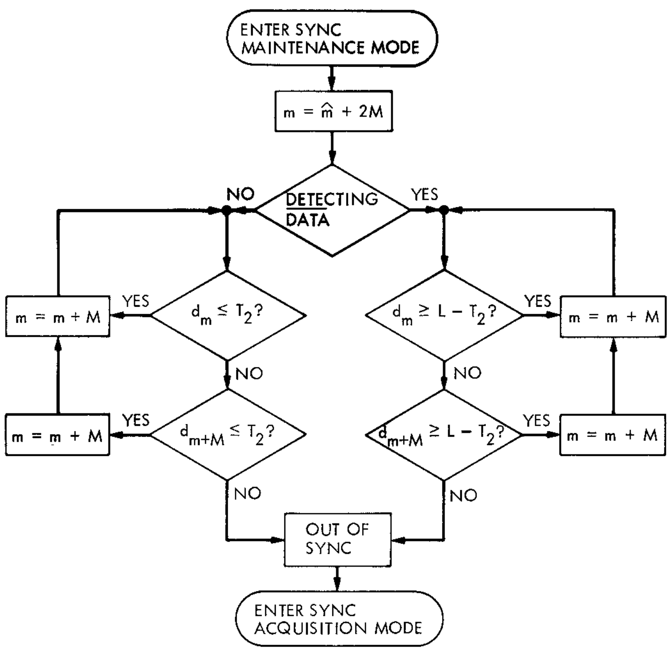

Fig. 2. Frame sync algorithm: maintenance mode flow diagram (assumption: $T_1 \leq T_2$)

Appendix
Derivation of Algorithm Performance Formulas

I. Probability of False Sync Acquisition

Assume that the received sync words are ρ_{m*}, ρ_{m*+M}, ρ_{m*+2M}, $\cdots$, where m^* is uniformly distributed:

$$Pr\,[m^*] = \frac{1}{M}; \qquad 1 \leq m^* \leq M \tag{A-1}$$

Define the following events, conditioned on a particular sync index m^*:

$$P \equiv \bigcup_{m=1}^{m^*-1} (\hat{m} = m)$$

$$R_k \equiv (\hat{m} \neq m^* + kM)$$

$$V_k \equiv \bigcup_{m=m^*+kM+1}^{m^*+kM+L-1} (\hat{m} = m)$$

$$W_k \equiv \bigcup_{m=m^*+kM+L}^{m^*+(k+1)M-L} (\hat{m} = m)$$

$$Y_k \equiv \bigcup_{m=m^*+(k+1)M-L+1}^{m^*+(k+1)M-1} (\hat{m} = m)$$

$$X_k \equiv \bigcup_{m=m^*+kM+1}^{m^*+(k+1)M-1} (\hat{m} = m) = V_k \cup W_k \cup Y_k$$

where $\hat{m}$ denotes the frame sync acquisition decision.

There are two approaches to bounding the probability of false sync acquisition (FS) used in this paper. The first is based on the events $\{X_k\}$:

$$Pr\,[FS\,|\,m^*] = Pr\,[P \cup (R_0 \cap X_0) \cup (R_0 \cap R_1 \cap X_1) \cup \cdots]$$

By the familiar union bounding technique,

$$Pr\,[FS\,|\,m^*] \leq p + \sum_{k=0}^{\infty} \mu_k \tag{A-2}$$

where

$$p \equiv Pr\,[P]$$

$$\mu_k \equiv Pr\,[R_0 \cap R_1 \cap \cdots \cap R_k \cap X_k]$$

To compute p, (still conditioned on m^*), the union bound is again applied:

$$p = Pr\left[\bigcup_{m=1}^{m^*-1} (\hat{m} = m)\right] \leq \sum_{m=1}^{m^*-1} Pr\,[\hat{m} = m\,|\,m \neq m^*]$$

Under the assumption that a Barker or Neuman-Hofman frame sync word is used, it will be assumed that all of the received sequences ρ_m for $1 \leq m \leq m^* - 1$, including those that overlap ρ_{m*}, can be treated as uniformly random binary L-tuples. Then

$$\begin{aligned} Pr\,[\hat{m} = m\,|\,m \neq m^*] &= Pr\,\{[(d_m \leq T_1) \cap (d_{m+M} \leq T_1)] \\ &\qquad \cup [(d_m \geq L - T_1) \\ &\qquad \cap (d_{m+M} \geq L - T_1)]\,|\,m \neq m^*\} \\ &= \{Pr\,[d_m \leq T_1\,|\,m \neq m^*] \\ &\qquad \times Pr\,[d_{m+M} \leq T_1\,|\,m \neq m^*]\} \\ &\qquad + \{Pr\,[d_m \geq L - T_1\,|\,m \neq m^*] \\ &\qquad \times Pr\,[d_{m+M} \geq L - T_1\,|\,m \neq m^*]\} \end{aligned}$$

and each of the four terms above equals $\gamma/2$, where

$$\gamma \equiv Pr\,[(d_m \leq T_1) \cup (d_m \geq L - T_1)\,|\,m \neq m^*]$$

$$= 2^{-L}\left[\sum_{\ell=0}^{T_1}\binom{L}{\ell} + \sum_{\ell=L-T_1}^{L}\binom{L}{\ell}\right] = 2^{-(L-1)}\sum_{\ell=0}^{T_1}\binom{L}{\ell}$$

$$\therefore\ Pr\,[\hat{m} = m\,|\,m \neq m^*] = \frac{\gamma^2}{2}, \qquad \text{independent of } m \tag{A-3}$$

and

$$p \leq (m^* - 1)\frac{\gamma^2}{2}$$

Since Eq. (1) implies that

$$\overline{m}^* = \frac{1}{2}(M + 1) \tag{A-4}$$

it follows that

$$\overline{p}^{m*} \leq (M - 1)\frac{\gamma^2}{4} \tag{A-5}$$

The computation of bounds on μ_k is somewhat more involved. Note that X_k is independent of $R_0, R_1, \cdots, R_{k-2}$, but is correlated with R_{k-1} and R_k, for $k \geqq 2$. Since

$$Pr[E_1 \cap E_2] \leqq \min\{Pr[E_1], Pr[E_2]\}$$

for arbitrary events E_1 and E_2, it follows that

$$\mu_k \leqq \begin{cases} \min\{Pr[R_0 \cap R_1 \cap \cdots \cap R_k], Pr[X_k]\}; & k = 0, 1 \\ \min\{Pr[R_0 \cap R_1 \cap \cdots \cap R_k], \\ \quad Pr[R_0 \cap R_1 \cap \cdots \cap R_{k-2}]\, Pr[X_k]\}; & k \geqq 2 \end{cases}$$

As in the computation of p,

$$Pr[X_k] \leqq (M-1)\frac{\gamma^2}{2}; \qquad k \geqq 0$$

$$\therefore \mu_k \leqq \begin{cases} \min\left[P_k, (M-1)\frac{\gamma^2}{2}\right]; & k = 0, 1 \\ \min\left[P_k, P_{k-2}(M-1)\frac{\gamma^2}{2}\right]; & k \geqq 2 \end{cases}$$

where

$$P_k \equiv Pr[R_0 \cap R_1 \cap \cdots \cap R_k]; \qquad k \geqq 0$$

We want to compute an upper bound for

$$\sum_{k=0}^{\infty} \mu_k$$

It will be seen later that it is difficult to determine a general expression for P_k, whereas

$$\sum_{k=0}^{\infty} P_k$$

has a simple closed form expression. Consequently, we will use the bound

$$\sum_{k=0}^{\infty} \mu_k \leqq \min\left[\sum_{k=0}^{\infty} P_k, (M-1)\frac{\gamma^2}{2}\left(2 + \sum_{k=0}^{\infty} P_k\right)\right] \qquad \text{(A-6)}$$

The Markov model of Fig. A-1 is used to calculate

$$\sum_{k=0}^{\infty} P_k$$

Recall that $\rho_{\hat{m}}$ is declared to be a received sync word if and only if it is the first sequence ρ_m satisfying one of two threshold tests

$$d_m \leqq T_1 \qquad \text{or} \qquad d_m \geqq L - T_1$$

and $\rho_{\hat{m}+M}$ satisfies the same test as $\rho_{\hat{m}}$. And

$$P_k = Pr[\hat{m} \neq m^* + \ell M; 0 \leqq \ell \leqq k]$$

Consider a particular received sync word $\rho_{m^*+\ell M}$: conditioned on the detection of (noninverted) data,

$$Pr[d_{m^*+\ell M} \geqq L - T_1 \,|\, \text{data}] = \sum_{j=L-T_1}^{L} \binom{L}{j} \varepsilon^j (1-\varepsilon)^{L-j}$$

$$= \varepsilon^{L-T_1} \sum_{j=0}^{T_1} \binom{L}{j} \varepsilon^{T_1-j} (1-\varepsilon)^j$$

If T_1 is much smaller than L, as recommended earlier, and ε is not unusually high, the probability above is negligible in comparison with

$$Pr[d_{m^*+\ell M} \leqq T_1 \,|\, \text{data}]$$

and

$$Pr[T_1 < d_{m^*+\ell M} < L - T_1 \,|\, \text{data}]$$

Similarly, conditioned on the detection of $\overline{\text{data}}$,

$$Pr[d_{m^*+\ell M} \leqq T_1 \,|\, \overline{\text{data}}] = \varepsilon^{L-T_1} \sum_{j=0}^{T_1} \binom{L}{j} \varepsilon^{T_1-j} (1-\varepsilon)^j$$

which can be neglected. The conclusion is that although there are two sync acquisition threshold tests with respect to the received sync words, only one of these tests is pertinent, conditioned on the detection of data or $\overline{\text{data}}$, and the probability that a particular received sync word fails this pertinent test is given by

$$\eta(T_1) = 1 - \sum_{\ell=0}^{T_1} \binom{L}{\ell} \varepsilon^{\ell} (1-\varepsilon)^{L-\ell} \qquad \text{(A-7)}$$

independent of the data/$\overline{\text{data}}$ conditioning.

Now, referring to Fig. A-1, suppose the receiver is in the 'bad' state, B. With probability $\eta(T_1)$, ρ_{m^*} fails the pertinent threshold test, and the receiver remains in state B. With probability $1 - \eta(T_1)$, ρ_{m^*} passes this test, and the receiver transfers to the 'good' state, G, which is really a conditional sync acquisition state. If ρ_{m^*+M} subsequently satisfies this threshold test, the receiver is in sync, with $\hat{m} = m^*$. However, if ρ_{m^*+M} fails the test, $\hat{m}$ cannot equal m^* or $m^* + M$, and the receiver reverts to state B. Therefore, with respect to this model,

$P_k = Pr\,[\text{not in SYNC after } (k+2) \text{ transitions from B}]$

$$\Downarrow$$

$$P_k = \eta(T_1)\,P_{k-1} + \eta(T_1)\,[1-\eta(T_1)]\,P_{k-2}; \qquad k \geqq 0 \tag{A-8}$$

This recursion formula for P_k has initial conditions

$$P_{-1} = P_{-2} = 1$$

It yields a complicated expression for larger values of P_k. However, summing both sides of Eq. A-8,

$$\sum_{k=0}^{\infty} P_k = \eta(T_1) + \eta(T_1)\sum_{k=0}^{\infty} P_k + 2\eta(T_1)\,[1-\eta(T_1)] + \eta(T_1)\,[1-\eta(T_1)]\sum_{k=0}^{\infty} P_k$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} P_k = \frac{\eta(T_1)\,[3-2\eta(T_1)]}{[1-\eta(T_1)]} \equiv A \tag{A-9}$$

From Eqs. A-5, A-6, and A-9, the probability of false sync acquisition, averaged over m^*, is given by

$$Pr\,[FS] \leqq (M-1)\frac{\gamma^2}{4} + \min\left[A, (M-1)\frac{\gamma^2}{2}(A+2)\right] \tag{A-10}$$

The second bounding approach involves the events $\{V_k\}$, $\{W_k\}$, and $\{Y_k\}$

$$\begin{aligned} Pr\,[FS \mid m^*] = {} & Pr\,[P \cup (R_0 \cap V_0) \cup (R_0 \cap W_0) \cup (R_0 \cap Y_0) \\ & \cup (R_0 \cap R_1 \cap V_1) \cup (R_0 \cap R_1 \cap W_1) \\ & \cup (R_0 \cap R_1 \cap Y_1) \cup \cdots] \\ \leqq {} & p + \sum_{k=0}^{\infty} \alpha_k + \sum_{k=0}^{\infty} \beta_k + \sum_{k=0}^{\infty} \zeta_k \end{aligned} \tag{A-11}$$

where

$$\alpha_k \equiv Pr\,[R_0 \cap R_1 \cap \cdots \cap R_k \cap V_k]$$

$$\beta_k \equiv Pr\,[R_0 \cap R_1 \cap \cdots \cap R_k \cap W_k]$$

$$\zeta_k \equiv Pr\,[R_0 \cap R_1 \cap \cdots \cap R_k \cap Y_k]$$

To bound α_k, note that V_k is independent of $R_0, R_1, \cdots, R_{k-2}$ for $k \geqq 2$, and

$$Pr\,[V_k] \leqq (L-1)\frac{\gamma^2}{2}; \qquad k \geqq 0$$

$$\therefore \alpha_k \leqq \begin{cases} \min\left[P_k, (L-1)\dfrac{\gamma^2}{2}\right]; & k = 0, 1 \\ \min\left[P_k, P_{k-2}(L-1)\dfrac{\gamma^2}{2}\right]; & k \geqq 2 \end{cases}$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} \alpha_k \leqq \min\left[A, (L-1)\frac{\gamma^2}{2}(A+2)\right] \tag{A-12}$$

With regard to β_k, it is noted that W_k is independent of $R_0, R_1, \cdots, R_k$ for $k \geqq 0$, and

$$Pr\,[W_k] \leqq (M-2L+1)\frac{\gamma^2}{2}$$

$$\therefore \beta_k \leqq P_k(M-2L+1)\frac{\gamma^2}{2}$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} \beta_k \leqq A(M-2L+1)\frac{\gamma^2}{2} \tag{A-13}$$

As for ζ_k, it is evident that Y_k is independent of $R_0, R_1, \cdots, R_{k-1}$ for $k \geqq 1$, and

$$Pr\,[Y_k] \leqq (L-1)\frac{\gamma^2}{2}; \qquad k \geqq 0$$

$$\therefore \zeta_k \leqq \begin{cases} \min\left[P_0, (L-1)\dfrac{\gamma^2}{2}\right]; & k = 0 \\ \min\left[P_k, P_{k-1}(L-1)\dfrac{\gamma^2}{2}\right]; & k \geqq 1 \end{cases}$$

$$\Downarrow$$

$$\sum_{k=0}^{\infty} \zeta_k \leqq \min\left[A, (L-1)\frac{\gamma^2}{2}(A+1)\right] \tag{A-14}$$

Combining Eq. A-5 and Eqs. A-11 through A-14,

$$\begin{aligned} Pr\,[FS] \leqq (M-1)\frac{\gamma^2}{4} + \min\Bigg\{ & \left[2 + (M-2L+1)\frac{\gamma^2}{2}\right]A, \\ & \left[1 + (M-L)\frac{\gamma^2}{2}\right]A + (L-1)\frac{\gamma^2}{2}, \\ & A(M-1)\frac{\gamma^2}{2} + \frac{3}{2}(L-1)\gamma^2 \Bigg\} \end{aligned} \tag{A-15}$$

Comparing Eqs. A-10 and A-15, and noting that we generally have $L << M$, the final bound is

$$Pr\,[FS] \leqq (M-1)\frac{\gamma^2}{4} + \min\left[A, (M-1)A\frac{\gamma^2}{2} + \frac{3}{2}(L-1)\gamma^2\right] \tag{A-16}$$

II. Mean Time to Sync Acquisition

If T_1 is relatively small, the probability that two incorrect L-bit received sequences one frame apart satisfy the acquisition threshold test (false alarm) is negligibly small. In this case, $Pr\,[FS] << 1$, and with high probability $\hat{m} = m^* + kM$ for some integer k. Since the probability γ of a false alarm is negligible, it is the probability $\eta(T_1)$ of a miss that is the dominant factor in making k nonzero. Therefore, the Markov model of Fig. A-1 can be used to compute the mean time to sync

$$MTS = \frac{\overline{\hat{m}}}{M} = \frac{\overline{m}^*}{M} + k \qquad \text{(A-17)}$$

To this end, define

$b \equiv$ number of transitions from state B to SYNC

$g \equiv$ number of transitions from state G to SYNC

If we start in state B, and the first transition depends on ρ_{m^*},

$$\overline{k} = \overline{b} - 2 \qquad \text{(A-18)}$$

From the model of Fig. A-1,

$$\overline{b} = \eta(T_1)(\overline{b} + 1) + [1 - \eta(T_1)](\overline{g} + 1)$$

$$\overline{g} = \eta(T_1)(\overline{b} + 1) + [1 - \eta(T_1)]$$

$$\Downarrow$$

$$b - 2 = \frac{\eta(T_1)[3 - 2\eta(T_1)]}{[1 - \eta(T_1)]^2} \equiv A \qquad \text{(A-19)}$$

From Eqs. A-4 and A-17 through A-19, it follows that

$$MTS = \frac{M+1}{2M} + A; \qquad Pr\,[FS] << 1 \qquad \text{(A-20)}$$

III. Mean Time to Loss of Sync

It is assumed below that the frame sync acquisition mode has made a correct sync decision, and random bit deletions and insertions are neglected. That is, the maintenance mode examines only received sync words. Since $\rho_{\hat{m}}$ and $\rho_{\hat{m}+M}$ satisfy the stringent acquisition threshold test, if $T_2 \geqq T_1$ as recommended earlier, they must also satisfy the maintenance threshold test. Consequently, the frame of received data between $\rho_{\hat{m}}$ and $\rho_{\hat{m}+M}$ is in sync, and the time to loss of sync is at least one frame. In general, if $\rho_{\hat{m}+kM}$ and $\rho_{\hat{m}+(k+1)M}$ are the first two consecutive received sync words to fail the maintenance threshold test, the time to loss of sync, measured by the number of good data frames accepted by the maintenance mode, is $k - 1$. Accordingly, the mean time to loss of sync is given by

$$MTLS = \overline{k} - 1 \qquad \text{(A-21)}$$

Using the Markov model of Fig. A-2, define

$g \equiv$ number of transitions from G to SYNC LOST

$b \equiv$ number of transitions from B to SYNC LOST

Starting in state G, the first transition reflects the outcome of the T_2 threshold test applied to ρ_{m+2M}; then the time to loss of sync is $g - 1$, so that

$$\overline{k} = \overline{g} \qquad \text{(A-22)}$$

From the model, it can be seen that

$$\overline{g} = [1 - \eta(T_2)](1 + \overline{g}) + \eta(T_2)(1 + \overline{b})$$

$$\overline{b} = [1 - \eta(T_2)](1 + \overline{g}) + \eta(T_2)$$

$$\Downarrow$$

$$\overline{g} = \frac{1 + \eta(T_2)}{\eta^2(T_2)} \qquad \text{(A-23)}$$

$$\therefore MTLS = \frac{1 + \eta(T_2) - \eta^2(T_2)}{\eta^2(T_2)} \qquad \text{(A-24)}$$

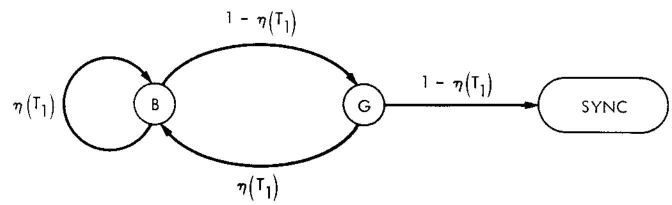

Fig. A-1. Markov model relating to examination of received sync words in acquisition mode

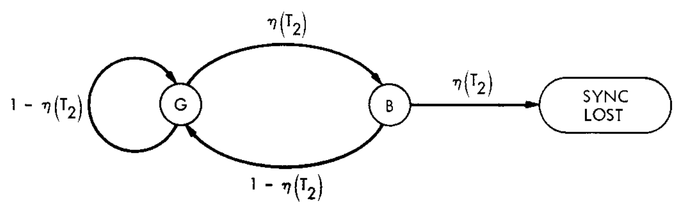

Fig. A-2. Markov model for examination of received sync words in maintenance mode