# A Global Model of the Earth's Ionosphere: The Nighttime Ionosphere

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*The evaluation of an ionospheric model started in Volume XI of the DSN Progress Report series (TR 32-1526) is continued. This article discusses the nighttime ionosphere and considers specifically the dawn and dusk situation in which a transition from electron production by the Sun's radiation to a cessation of this production gradually occurs. The strict nighttime ionosphere is governed by electron attachment and diffusion. The transition period depends very much on the declination angle of the Sun and the latitude of the geographical location on Earth where knowledge of the ionospheric electron distribution is desired. Simple and concise expressions for the electron distribution in the upper ionosphere are derived. Calibrations for range corrections on a global scale are therefore possible. The ionospheric model presented contains six empirical parameters. These parameters are to be considered as functions of geographical location and must be determined by measurement. Future work will concentrate on the determination of these parameters and the implementation of the model to obtain range corrections. Furthermore, a sensitivity analysis of the model with respect to the six parameters will also be performed.*

## I. Introduction

An evaluation of an ionospheric model was presented earlier in Ref. 1. This article continues the evaluation by considering the nighttime ionosphere and, specifically, the dawn and dusk situation in which a transition gradually occurs from electron production by the Sun's radiation to a cessation of this production. The strict nighttime ionosphere is governed by electron attachment and diffusion. The transition period is dependent on the declination angle of the Sun and the latitude of the geographical location on Earth where knowledge of the ionospheric electron distribution is desired. The simple and concise expressions derived here for electron distribution in the upper ionosphere make calibrations for range corrections on a global scale possible.

Because of the absence of solar ultraviolet radiation, no electron production occurs during the night. What does occur is a loss of electrons due to attachment and recombination. Reference 2 gives some values for the total loss coefficients for the F2 layer[1] depending on temperature.

[1]The F2 layer possessing the highest electron concentration affects the high frequencies of the DSN ($\nu \approx 2 \cdot 10^9$ Hz) almost exclusively.

They are

$$B = 1.5 + 10^{-5} \exp\left(-\frac{z-300}{20}\right) \text{ sec}^{-1} \text{ at } 700°\text{K}$$

$$B = 6.5 + 2 \cdot 10^{-5} \exp\left(-\frac{z-300}{29}\right) \text{ sec}^{-1} \text{ at } 1000°\text{K}$$

where $z$ is in kilometers. The electron losses are due to reactions such as

$$e + O_2^+ = O^* + O^*$$

$$e + NO^+ = N^* + O^*$$

etc. Here the * signifies possibly excited states. We expect, therefore, a slow decrease in the electron density during the night. However, there is another effect which also governs the electron distribution. Positive ions and neutral molecules diffuse through each other due to gravity. The diffusion equation determines the density changes of the positive ions. But since the electrons faithfully follow the positive ions (otherwise large electrostatic forces would be engendered pulling electrons and ions quickly together), the influence of diffusion on the positive ions is the same as that on the electrons. Therefore the electron distribution is influenced by both loss and diffusion during the night. To be sure, diffusion also takes place during daytime, but the electron production is so large that diffusion constitutes only a minor part of the overall picture and hence had been neglected previously.

In the next section we shall give a derivation of the nighttime ionosphere and link it with the daytime ionosphere both at dawn and dusk.

## II. The Ionosphere at Night

We begin by writing the pertinent equation for the electron density as approximately applicable at high altitudes (> 300 km) as a function of height and time (Refs. 3 and 4), viz,

$$\frac{\partial N}{\partial t} = \alpha e^{z}\left(\frac{\partial^2 N}{\partial z^2} + \frac{3}{2}\frac{\partial N}{\partial z} + \frac{1}{2}N\right) - \beta\, e^{-z} N \tag{1}$$

The first term on the right-hand side (RHS) is responsible for the diffusion and the second term for electron attachment. $\beta$ is the attachment coefficient in sec$^{-1}$, $z = h/H$ where $h$ is the actual altitude and $H$ the scale height given by

$$H = kT/mg \tag{2}$$

and $m$ is the average mass of the gas molecules. Furthermore, the diffusion constant $\alpha$ is given by (Ref. 4)

$$\alpha = 2g\frac{\sin^2 I}{H\nu} \tag{3}$$

where $g$ is the Earth's gravitational acceleration, $I$ the magnetic dip angle, and $\nu$ the collision frequency of the electrons with the gas molecules (charged and uncharged). In order to proceed with the solution, we make the substitutions

$$\left.\begin{aligned} x &= e^{-z} \\ N(z,t) &= e^{-\gamma t}\, n(x) \end{aligned}\right\} \tag{4}$$

and obtain

$$n_{xx} - \frac{1}{2x}n_x + \left(\frac{1}{2x^2} - \frac{\beta}{\alpha} + \frac{\gamma}{\alpha x}\right)n = 0 \tag{5}$$

If it is realized that Eq. (5) is just Schrödinger's equation for a hydrogen atom, then the well-known general solution of Eq. (5) is given by

$$n(x) = x \exp\left[-\sqrt{\frac{\beta}{\alpha}}\,x\right] L_n^{(1/2)}\left(2\sqrt{\frac{\beta}{\alpha}}\,x\right) \tag{6}$$

while

$$\gamma = \gamma_n = \sqrt{\alpha\beta}\left(2n + \frac{3}{2}\right) \tag{7}$$

so that finally

$$N(z,t) = \sum_n c_n \exp\left[-\gamma_n t - z - \sqrt{\frac{\beta}{\alpha}}\,e^{-z}\right] L_n^{(1/2)}\left(2\sqrt{\frac{\beta}{\alpha}}\,e^{-z}\right) \tag{8}$$

where the $c_n$ are arbitrary coefficients to be determined by the initial conditions, and the $L_n^{(1/2)}$ are generalized Laguerre polynomials defined by

$$L_n^{(1/2)}(x) = x^{-1/2}\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+1/2}\right) \tag{9}$$

Equation (8) constitutes the general solution for the electron distribution of the F layer in the upper atmosphere. The coefficients $c_n$ are as yet quite arbitrary and have to be determined by initial conditions. Since Eq. (1) is of first order in time, a specification of an initial condition is quite sufficient to determine the solution as far as the time domain is concerned. This is somewhat disturbing since it is known that the nighttime ionosphere is asymmetric and, furthermore, a given initial condition (at $t = 0$, say) leads inevitably to a distribution at $t > 0$, which is totally determined by this very initial condition. But eventually the night ends and the rising Sun will start ionizing the upper atmosphere and therefore raise the electron density. The model as expressed by Eq. (8) does vitiate this condition. In order to overcome this drawback, we are going to determine the coefficients $c_n$ in Eq. (8) first in order to see exactly what the situation is. The boundary conditions which can be expressed as initial conditions for Eq. (8) are those which link expression (8) for the nighttime electron density profile to the dawn–dusk profile at a given time $t = t_0$, say. From Eq. (15) of Ref. 1 and putting $\lambda_0 = \pi/2$, the expression for this quantity is given by (replacing $(r - R)/H$ in Eq. 15 of Ref. 1. by $z$)[2]:

$$N_e^{(0)}(z) = N_{e\,\max} \exp \frac{1}{2} \left\{ 1 + \frac{h_{\max}}{H} - z + \int_{\infty}^{z-(h_{\max}/H)} dx\, e^{-x} \left( 1 - \frac{(Hz+R)^2}{(Hx+R+h_{\max})^2} \right)^{-1/2} \right\} \tag{10}$$

We must therefore equate Eq. (10) with Eq. (8) at $t = t_0$ in order to determine the coefficients $c_n$.

From the orthogonality relations of the Laguerre polynomials

$$\int_0^{\infty} dx \sqrt{x}\, e^{-x} L_n^{(1/2)}(x)\, L_m^{(1/2)}(x) = \delta_{nm}\, \Gamma\left(\frac{3}{2}\right) \binom{n+1/2}{n} \tag{11}$$

we obtain first

$$\sum_m c_m \exp\left[ -\gamma_n t_0 - z - \sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right] L_m^{(1/2)}\left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) = N_e^{(0)}(z) \tag{12}$$

Multiplying this equation by

$$\exp\left[ -\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right] \left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right)^{1/2} L_n^{(1/2)}\left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) d\left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right)$$

we now obtain

$$\begin{aligned} &\sum_m c_m \exp\left[ -\gamma_n t_0 - 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right] L_m^{(1/2)}\left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) L_n^{(1/2)} \\ &\quad \times \left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) \left( 2\frac{\beta}{\alpha}\, e^{-z} \right)^{1/2} d\left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) \\ &= N_e^{(0)}(z) \exp\left[ z - \sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right] \left( 2\frac{\beta}{\alpha}\, e^{-z} \right)^{1/2} L_n^{(1/2)} \\ &\quad \times \left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) d\left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) \end{aligned} \tag{13}$$

We integrate Eq. (13) on both sides from $z = 0$ to $z = \infty$ and substitute $2\sqrt{\beta/\alpha}\, e^{-z} = x$ on the left-hand side (LHS) and obtain

$$\begin{aligned} &\sum_m c_m \exp\left[ -\gamma_n t_0 \right] \int_0^{2\sqrt{\beta/\alpha}} dx\, x^{1/2} e^{-x} L_m^{(1/2)}(x)\, L_n^{(1/2)}(x) = \\ &\left( 2\frac{\beta}{\alpha} \right)^{3/2} \int_0^{\infty} dz\, N_e^{(0)}(z) \exp\left[ -\frac{z}{2} - \sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right] L_n^{(1/2)} \\ &\times \left( 2\sqrt{\frac{\beta}{\alpha}}\, e^{-z} \right) \end{aligned} \tag{14}$$

The orthogonality relation (11) cannot yet be applied since the upper limit of the integral is not $\infty$. It turns out, however, that $\sqrt{\beta/\alpha} \approx 10^4$ (Ref. 5) so that relation (11) can be safely employed, yielding

$$\Gamma\left(\frac{3}{2}\right) \binom{n+1/2}{n} \exp\left[ -\gamma_n t_0 \right] c_n = \text{integral on RHS of Eq. (14)} \tag{15}$$

Let us now evaluate the foregoing analysis. With the coefficients $c_n$ determined via Eq. (15), the nighttime ionospheric profile Eq. (8) is completely determined and is seen to decay with progressing time. However, the daytime ionosphere is time-independent (Eq. 15 of Ref. 1). So it appears at first sight that any point on Earth on the night side would eventually be without ionosphere and any point located on the day side would enjoy a stable ionosphere. This would in fact be true if the Earth's rotation is not taken into account.[3] Remember, so far we have worked in the Sun-fixed coordinate system in which the x-axis is always pointed toward the Sun.

To digress a little on this important point, we introduce Figs. 1 and 2, which are analyzed in Ref. 6. They display

[2] It has been assumed that the scale height $H$ is the same for the daytime ionosphere near dawn and dusk as that of the nighttime ionosphere.

[3] Disregarding lateral drifts which would surely set in after a while; but then our simple model would also break down.

the measured integral of the electron profile (total electron content) which can be theoretically calculated from Eq. (8) at night and from Eq. (15) of Ref. 1 at daytime. We see the gradual decline of the electron density during the night as indeed suggested by Eq. (8), then a sharp increase during dawn. In the further analysis we might as well take $t_0 = 0$ in Eq. (15), which amounts to start counting time at sunset. The duration of the night as a function of the Sun's declination $\delta_\odot$ and the station's latitude $\phi$ (measured from the Earth's equator) has been evaluated in Appendix A. It is an intricate function of the variables involved and the reader is urged to read the Appendix. In any case, let the duration of the night be denoted by $t_N$. It is then true that Eq. (8) holds, together with the initial condition, as expressed by Eq. (15) until the time elapsed is $t_N$. Then dawn starts and ultraviolet radiation of the Sun will immediately produce electron–ion pairs and the electron density will start to increase. We shall quickly determine another time needed and that is the time elapsed between total night and full illumination during which the ionosphere builds up to its dawn value. We do this with the aid of Figs. 3 and 4. In Fig. 3 we have depicted a cross section through Earth together with the ionosphere. The coordinate system used is that of the Sun-fixed system of Fig. 1 (Ref. 1). The ionosphere is characterized by the height of the maximum electron density $h_{max}$ and the scale height $H$ as it prevails at that location. A point fixed on Earth as it rotates toward dawn must traverse the angle $\beta = \alpha_1 - \alpha_2$ in the Sun-fixed coordinate system and must subsequently be transformed to the proper, that is, geographical coordinate system (see Appendix B). During the transition time as determined in Appendix B (Eq. B-4), the electron concentration will gradually increase to the value prevalent at dawn (Ref. 1). Before we go ahead and evaluate this transition period we must go back to Eq. (13). Equation (13) in conjunction with Eq. (14) looks quite formidable. However, the coefficients (Eq. 13) rapidly decrease in size and the infinite series Eq. (13) can be brought to a halt rather quickly. To see this we have to know the parameters $\alpha$ and $\beta$, where $\alpha$ is given by Eq. (3) and $\beta$ is the electronic attachment coefficient as explained earlier. Although the parameters $\alpha$ and $\beta$ are subject to debate as far as their exact values are concerned, an order of magnitude evaluation reveals that

$$\left.\begin{aligned} \sqrt{\alpha\beta} &\approx 10^4 \sec^{-1} \\ \sqrt{\frac{\alpha}{\beta}} &\approx 2 \cdot 10^4 \end{aligned}\right\} \tag{16}$$

From Eq. (8), then, it is clear that the leading term ($n = 0$) has a decay rate amounting to just about 4 hours. This in fact seems to be correct, taking Figs. 1 and 2 into account. Figures 1 and 2, taken at different dates, show the slow decrease of the electron concentration during nighttime and the comparatively fast increase at sunrise. We must realize that the data displayed in Figs. 1 and 2 represent the *total* electron content. That is to say, the integral of the electron concentration along the ray path which in the figures shown is the zenith or vertical ray path and therefore is quite indicative of the model used in this report and Ref. 1. There are, of course, much more data gathered than would be possible to display here. The reader should realize, however, that the general trend is more or less the same.

From Eq. (15) and the values given by Eqs. (16), we see that it suffices to retain the two terms $n = 0$ and $n = 1$ in the series expansion (8). This expression is then valid until sunrise which happens according to the two appendices at

$$t = t_{\text{night}} = \text{Eq. (B-4)} \tag{17}$$

of Appendix B.

At this time the transition toward total daylight occurs. From a mathematical point of view, the buildup of the ionospheric plasma due to the Sun's ultraviolet radiation is a rather difficult problem. We are going to shortcut it by using intuitive reasoning, since for moderate latitudes the time of twilight in the upper ionosphere is comparatively short (< ½ hour). What we are proposing to do is the following:

(1) Use Eq. (1) of this report with appropriate alterations as determined below.

(2) Augment it by a source term of ionization similar to the one given in Eq. (9) of Ref. 1.

(3) Increase this source term to full strength (from zero) during the transition time $\tau_T$ (Eq. B-7).

(4) Decrease the diffusion term to zero during that same time span.

(5) Adjust the solution during the transition time $\tau_T$ to the solution Eq. (8) at $t = t_{\text{night}}$ and to Eq. (15) of Ref. 1 at $t = t_{\text{night}} + \tau_T$, where $t_{\text{night}}$ is given in Appendix B, Eq. (B-4).

Accordingly we will proceed in the following manner: We take Eq. (1) again but the diffusion coefficient $\alpha$ and the absorption coefficient $\beta$ will be time-dependent in the transition region:

$$\left.\begin{aligned}\widetilde{\alpha} &= \alpha \exp\left[-\frac{t - t_{\text{night}}}{\tau}\right] \\ \widetilde{\beta} &= \beta \exp\left[-\frac{t - t_{\text{night}}}{\tau}\right]\end{aligned}\right\} \tag{18}$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the values of $\alpha$ and $\beta$ during the transition time. Equation (1), augmented by expressions (18), reads

$$\exp\left[\frac{t - t_{\text{night}}}{\tau}\right]\frac{\partial N}{\partial t} = \alpha e^{z}\left\{\frac{\partial^2}{\partial z^2} + \frac{3}{2}\frac{\partial}{\partial z} + \frac{1}{2}\right\}N - \beta e^{-z}N \tag{19}$$

On the other hand the ionosphere will start to experience ionization when $t = t_{\text{night}}$. This ionization will increase until $t = t_{\text{night}} + \tau_T$, when the ionosphere will experience the full ionization as discussed on previous pages. Putting $(t - t_{\text{night}})/\tau = \eta$, we represent the ionization by[4]

$$S = e^{\eta}e^{-\tau_T/\tau}\,S_0 \tag{20}$$

where $S_0$ is considered to be the ion production at dusk, so that $S = e^{-\tau_T/\tau}\,S_0$ i.e., negligible at $t = t_{\text{night}}$ (start of dawn) and $S = S_0$ *at* $t = t_{\text{night}} + \tau_T$ (completion of dawn). Equation (19), augmented by Eq. (20), reads

$$\left(\frac{e^{\eta}}{\tau}\frac{\partial}{\partial\eta} - \exp\left[2\eta - \frac{\tau_T}{\tau}\right]S_0\right)N(\eta,z) = \alpha e^{z}\left(\frac{\partial^2}{\partial z^2} + \frac{3\partial}{2\partial z} + \frac{1}{2}\right)N(\eta,z) - \beta e^{-z}N(\eta,z) \tag{21}$$

Separation of variables reveals that the spatial part of the electron density is governed by the following equation (after the substitution $z = e^{-x}$):

$$\overline{n}_{xx} - \frac{1}{2x}\overline{n}_x + \left(\frac{1}{2x^2} - \frac{\beta}{\alpha} - \frac{a}{\alpha x}\right)\overline{n} = 0 \tag{22}$$

where $a$ is the separation parameter. But Eq. (22) is just Eq. (5) with $-\gamma$ replaced by $a$, so that we are able to immediately write the general solution as

$$\overline{n}(x) = \overline{n}_n(x) = x\exp\left[-\sqrt{\frac{\beta}{\alpha}}\,x\right]L_n^{(1/2)}\left(2\sqrt{\frac{\beta}{\alpha}}\,x\right) \tag{23}$$

with

$$a = a_n = -\sqrt{\alpha\beta}\left(\frac{3}{2} + 2n\right) = -\gamma_n \tag{24}$$

and $n$ an arbitrary positive integer. The equation for the time-dependent part becomes[5]

$$\left(e^{\eta}\frac{\partial}{\tau\partial\eta} - \exp\left[2\eta - \frac{\tau_T}{\tau}\right]S_0\right)T_n(\eta) = -\gamma_n \tag{25}$$

with the general solution

$$T_n(\eta) = \exp\left[\eta - \frac{\tau_T}{\tau}\right]S_0\tau + e^{-\eta}\gamma_n\tau + A_n \tag{26}$$

where $A_n$ is an arbitrary integration constant.

The general solution for the differential equation (21) is then obtained using the completeness of the orthogonal functions $\overline{n}_n(x)$. With arbitrary constants $d_n$, we may write

$$N(\eta,z) = \sum_n d_n\,T_n(\eta)\,\overline{n}_n(x) \tag{27}$$

Solution (27) is only valid in the transition region $t_{\text{night}} < t < t_{\text{night}} + \tau_T$. It must be continuous both at $t = t_{\text{night}}$ $(\eta = 0)$ and at $t = t_{\text{night}} + \tau_T$ $(\eta = \tau_T/\tau)$. In Eq. (15) we have seen that the series (8) with $c_n$ from Eq. (15) yields just the daytime ionosphere at dawn (Eq. 10). Since in this simplified model the dawn and dusk ionospheres are the same, we have from Eq. (27):

$$\sum_n d_n T_n\left(\frac{\tau_T}{\tau}\right)\overline{n}_n(x) = \sum_n c_n\overline{n}_n(x) \tag{28}$$

As can be verified from Eqs. (23) and (6), the spacial functions are the same in both equations; $c_n$ in Eq. (28) is given by Eq. (15) and is therefore already determined.

On the other hand, we must also have

$$\sum_n d_nT_n(0)\,\overline{n}_n(x) = \sum_n c_n\exp\left[-\gamma_n t_{\text{night}}\right]\overline{n}_n(x) \tag{29}$$

a condition which states that the electron concentration at the beginning of dawn must be equal to the electron density at night given by the RHS of Eq. (29). It follows

[4] $S_0$ is considered a constant for simplicity.

[5] The total electron density is given by $N = T_n(\eta)\overline{n}(x)$; see more details below.

from Eqs. (28) and (29) that

$$\left.\begin{aligned} d_n T_n\left(\frac{\tau_T}{\tau}\right) &= c_n \\ d_n T_n(0) &= c_n \exp\left[-\gamma_n t_{\mathrm{night}}\right] \end{aligned}\right\} \tag{30}$$

so that

$$d_n = \frac{c_n}{T_n\left(\dfrac{\tau_T}{\tau}\right)} \tag{31}$$

and for the integration constant we obtain from Eqs. (29), (30), and (31)[6]:

$$\frac{1}{\tau}A_n = (1-\exp[-\gamma_n t_{\mathrm{night}}])^{-1}\left[S_0\left(\exp[-\gamma_n t_{\mathrm{night}}] - \exp\left[-\frac{\tau_T}{\tau}\right]\right) + \gamma_n\left(-1+\exp\left[-\gamma_n t_{\mathrm{night}} - \frac{\tau_T}{\tau}\right]\right)\right] \tag{32}$$

In theory then, the problem of a global coverage of the ionosphere is solved, although in practice more has to be done. To summarize, the electron density as a function of altitude in daytime is given by Eq. (15) of Ref. 1; the nighttime electron density is given by Eq. (8) of this report in conjunction with Eq. (15), an equation which determines the coefficients $c_n$ of the series (8). Furthermore, the transition region, i.e. dusk and dawn, are described by Eq. (27) in conjunction with Eqs. (30) which establish the necessary boundary conditions. Before we go ahead and simplify the nighttime and dusk–dawn electron distributions, let us enunciate the parameters needed for a concise description of the ionosphere in its various regimes:

(1) *Daytime ionosphere.* There are three parameters: $N_{e\,\max}$, the maximum electron density; $h_{\max}$, the altitude at which $N_e = N_{e\,\max}$; and $H$, the scale height of the ionosphere, a measure of how fast the density tapers off toward zero at higher altitudes.

(2) *Nighttime ionosphere.* Due to the initial condition which determines the coefficients $c_n$ of the nighttime solution, the three parameters mentioned above carry over into the night. In addition to these we have two more parameters: $\alpha$ (Eq. 3), the diffusion coefficient; and $\beta$, the attachment coefficient.

(3) *Transition region ionosphere.* Because of the fact that the solution for the electron density has to satisfy boundary conditions at both the onset of dawn and the completion of dawn (full Sun), it depends also on the parameters $N_{e\,\max}$, $h_{\max}$, $H$, $\alpha$, and $\beta$. In addition, a parameter $\tau$ had been introduced; essentially $\tau$ is the life time of the diffusion in the transition region. The diffusion is blotted out more and more by the increase in ionization.

Six parameters describe the global ionosphere. They may in principle be considered empirical parameters to be adjusted by means of measured values of the total electron content and other quantities (see also Ref. 1).

As mentioned previously, the first two terms $n = 0$ and $n = 1$ in the series expansions for the electron density (Eqs. 8 and 27) are sufficient for the analysis, the reason being the rapid convergence of the series.

With the help of Ref. 7, together with Eqs. (15) and (10), it is easy to see that

$$c_0 = \frac{1}{4\sqrt{\pi}} N_{e\,\max} \exp\left[\frac{1}{2}+\frac{h_{\max}}{2H}\right]\left(\frac{\beta}{\alpha}\right)^{3/2} \times \int_0^\infty \exp\left[-z+\phi(z)-\sqrt{\frac{\beta}{\alpha}}\,e^{-z}\right]dz \tag{33}$$

and

$$c_1 = \frac{1}{4\sqrt{\pi}} N_{e\,\max} \exp\left[\frac{1}{2}+\frac{h_{\max}}{2H}\right]\left(\frac{\beta}{\alpha}\right)^{3/2} \times \int_0^\infty \exp\left[-z+\phi(z)-\sqrt{\frac{\beta}{\alpha}}\,e^{-z}\right] \times \left\{\frac{3}{2}-2\sqrt{\frac{\beta}{\alpha}}\,e^{-z}\right\}dz \tag{34}$$

where

$$\phi(z) = \frac{1}{2}\int_\infty^{z-(h_{\max}/H)} dx \times e^{-x}\left(1-\frac{Hz+R}{(Hx+R+h_{\max})^2}\right)^{1/2} \tag{35}$$

[6]Note that without the source term $S_0$ condition (6) would lead to *negative* values for $A_n$ and therefore negative electron densities, or, in other words, the boundary conditions could not be satisfied in complete agreement with physical intuition.

In Appendix C an approximate solution for Integral (35) is found with the result that

$$\phi(z) = -\sqrt{\frac{2\pi}{4}}\sqrt{z + \frac{R}{H}}\exp\left[-z + \frac{h_{\max}}{H}\right] \tag{36}$$

For those values of $z$ for which the integrand in both Eqs. (33) and (34) is significantly different from zero, $z \ll R/H$ and the $z$ dependence of the square root in Eq. (36) may be neglected. In this case the integrals can be performed directly with the result:

$$c_0 = \frac{1}{4\sqrt{\pi}}\left(\frac{\beta}{\alpha}\right)^{3/2} N_{e\max}\exp\left[\frac{1}{2}\left(1 + \frac{h_{\max}}{H}\right)\right] \times \left[\sqrt{\frac{2\pi}{4}}\sqrt{\frac{R}{H}}\exp\left[\frac{h_{\max}}{H}\right] + \sqrt{\frac{\beta}{\alpha}}\right]^{-1} \tag{37}$$

$$c_1 = c_0\left\{\frac{3}{2} - \frac{2\sqrt{\frac{\beta}{\alpha}}}{\sqrt{\frac{2\pi}{4}}\sqrt{\frac{R}{H}}\exp\left[\frac{h_{\max}}{H}\right] + \sqrt{\frac{\beta}{\alpha}}}\right\} \tag{38}$$

Prevailing values of $\beta/\alpha$ and $R/H$ give $c_1 \approx 0.2\ c_0$, suggesting again a rapid convergence of the series (8).

## III. Summary and Conclusions

On the preceding pages a model of the ionosphere has been developed. Reference 1 dealt with the *daytime* ionosphere and this article considered the *nighttime* ionosphere. The complete model as given in this report is therefore *global* in the sense that given a station location, the universal time, and elevation angle and the six parameters describing the ionosphere, range and range rate corrections can be evaluated. The model described here is comprehensive and as simple as possible. Before expatiating on this model, we like to point out other efforts undertaken with the same goal envisioned. Reference 8 also gives a global model of the ionosphere. However, this model is not derived from first principles nor is it generally applicable to nighttime situations and, in particular, it fails to incorporate the twilight transition. The merits or demerits of the model given in Ref. 8 are not known, nor, it must be confessed, has this model been tested at this time.

Before concluding this report we must mention a number of items which have been glossed over to some extent. One of them is the choice for $S_0$ in Eq. (32). The dimension of $S_0$ is sec$^{-1}$ and is linked to expression (9) of Ref. 1. Expression (9) of Ref. 1 has the dimension erg-cm$^{-2}$-sec$^{-1}$ or energy flux. It follows that $S_0$ of Eq. (32) is given by

$$S_0 = \sigma_i \frac{S_\infty}{h\bar{\nu}} \tag{39}$$

where $S_\infty$ is the solar ultraviolet energy flux, $h\bar{\nu}$ the average energy per photon, and $\sigma_i$ the ionization cross section. Values of $S_0 \approx 10^5$ sec$^{-1}$ are suggested by the work of Chapman (Ref. 9).

Another parameter that must be known is $\tau$, the time constant of the transition between night and day. This, as we have seen earlier, is one of the parameters to be adjusted from measurements. Figure 1, for instance, shows the quick rise of the total electron content at 5 a.m. local time. It is not difficult to estimate this rise, which is just $2.8 \cdot 10^{13}$ m$^{-2}$-sec$^{-1}$. Translated into the time domain, we have approximately $\tau = 2.8 \cdot 10^{13}\ [m^{-2}\text{-sec}^{-1}] \times H^{-1}\ [m^{-1}] \times N_{e\max}^{-1}\ [m^{-3}] \approx 10^{-4}$ sec$^{-1}$, assuming a scale height $H$ of 40 km and a maximum electron density of $10^6$ cm$^{-3}$. Of course, the values of the six parameters listed in the previous section, values which determine the characteristics of the ionosphere, are to be considered slowly varying functions of both the geographical coordinates and the local time and as such must be determined empirically.

Future work will concentrate on the determination of these parameters and the implementation of the model to obtain range corrections. Furthermore, a sensitivity analysis of the model with respect to the six parameters will also be performed.

# References


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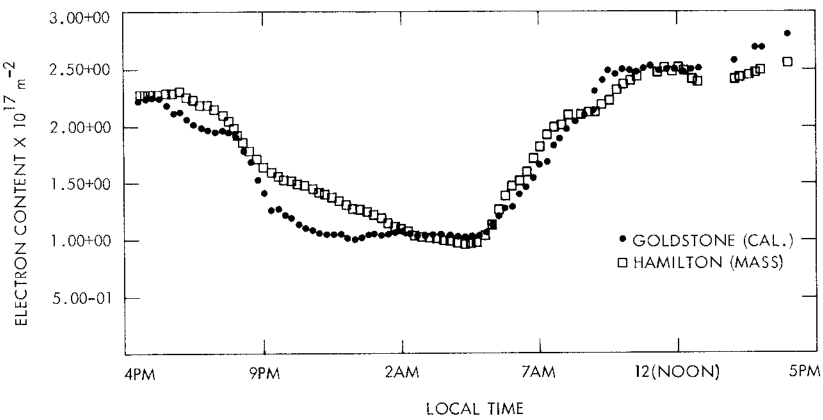


**Fig. 1. Total zenith electron content from Faraday rotation (Aug. 7, 1971)**

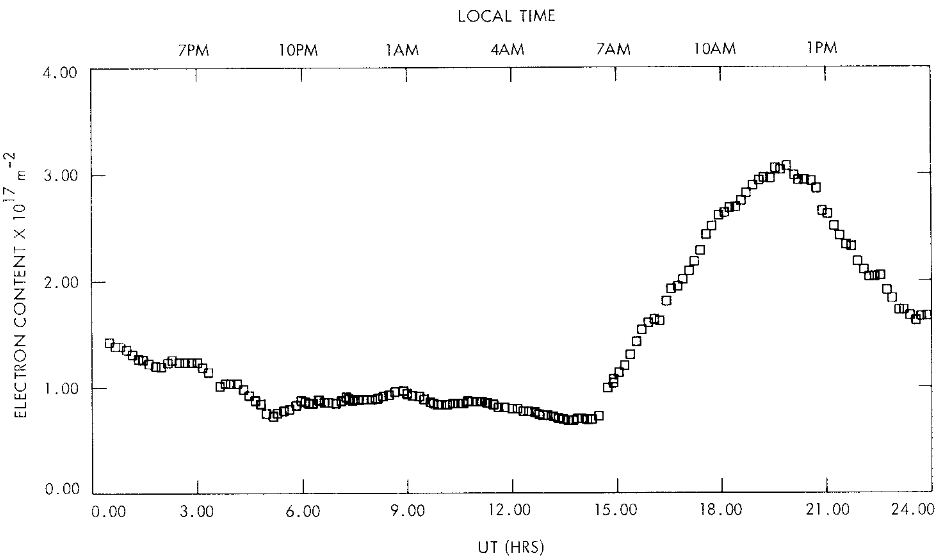


**Fig. 2. Total zenith electron content from Faraday rotation (Goldstone data, Dec. 30, 1971)**

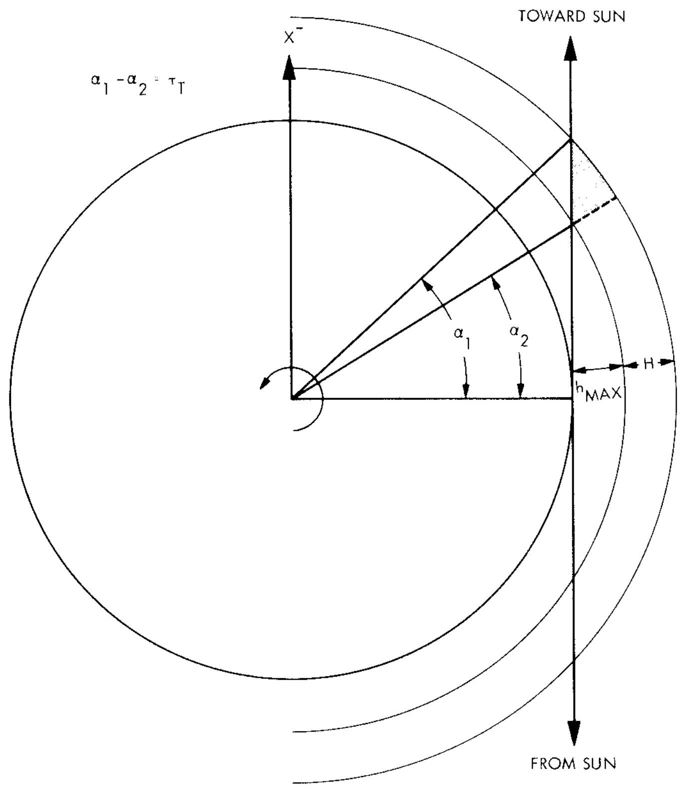


Fig. 3. The geometry for twilight when $\delta_\odot = 0$ (ionospheric height and thickness exaggerated)

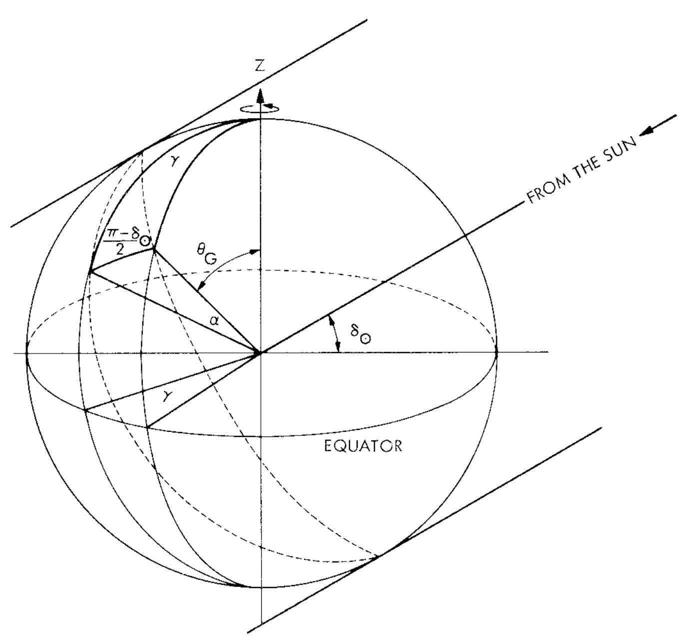


Fig. 4. The general geometry of twilight

# Appendix A

## Analytic Determination of the Length of the Night at Any Point on the Earth's Surface

### I. Introduction

This appendix is self-contained and can be read without reference to the body of the report.

A simple model for the motion of the Sun relative to Earth is stated. Expressions for station coordinates relative to the center of Earth are developed. Derivation of a function $T$, where $T = 0$ denotes twilight, $T > 0$ denotes daylight, and $T < 0$ denotes night, is undertaken. The solution to the general equation is discussed. Finally, simplifying assumptions are introduced which permit the controlling equation to be solved in closed form. It is shown that the duration of the day at any desired location can be obtained by the subtraction of two arc cosines.

### II. Approximate Right Ascension and Declination of the Sun

The right ascension and declination of the Sun, as measured from the inertial x-axis (Fig. A-1), can be obtained exactly or can be approximated by the following formulas:

$$\delta_{\odot} = 23.5 \sin (\dot{\alpha}_{\odot}[t - \text{March } 22]) \tag{A-1}$$

$$\alpha_{\odot} = \dot{\alpha}_{\odot}[t - \text{March } 22] \tag{A-2}$$

where

$$\dot{\alpha}_{\odot} = 360/365 \text{ deg/day}$$

Therefore, assuming a circular orbit for the path of Earth, the rectangular coordinates of a vector from the center of Earth to the Sun $\mathbf{R}_{\odot}$ are given by

$$\mathbf{R}_{\odot} = \begin{bmatrix} X_{\odot} \\ Y_{\odot} \\ Z_{\odot} \end{bmatrix} = R_{\odot} \begin{bmatrix} \cos \delta_{\odot} & \cos \alpha_{\odot} \\ \cos \delta_{\odot} & \sin \alpha_{\odot} \\ \sin \delta_{\odot} & \end{bmatrix} \tag{A-3}$$

where $R_{\odot}$ is the mean distance of the Earth from the Sun, i.e., $\mathbf{R}_{\odot} = 1$ AU. More exact coordinates can be obtained if desired.

### III. Station Coordinates

Consider a station location on Earth with a given geodetic latitude $\phi$, east longitude $\lambda_E$ measured relative to the Greenwich meridian, and elevation $H$ above the adopted geoid[7] (Fig. A-2).

The rectangular coordinates of the station relative to the inertial system are given by (Ref. 10)

$$\mathbf{R} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -G_1 \cos \phi \cos \theta \\ -G_1 \cos \phi \sin \theta \\ -G_2 \sin \phi \end{bmatrix} = \begin{bmatrix} -G_1 \cos \phi \cos (\theta_g + \lambda_E + \dot{\theta} t) \\ -G_1 \cos \phi \sin (\theta_g + \lambda_E + \dot{\theta} t) \\ -G_2 \sin \phi \end{bmatrix} \tag{A-4}$$

where

$$G_1 \equiv \frac{a_e}{[1 - (2f - f^2) \sin^2 \phi]^{1/2}} + H$$

$$G_2 \equiv \frac{(1 - f)^2 a_e}{[1 - (2f - f^2) \sin^2 \phi]^{1/2}} + H$$

and

$a_e \equiv$ radius of Earth

$f \equiv$ geometric flattening ($f = 1/298.3$)

$\theta_g \equiv$ Greenwich sidereal time at midnight[8] for a particular date

$\dot{\theta} \equiv$ constant sidereal rotation rate ($4.3752695 \times 10^{-3}$ rad/min)

$t \equiv$ number of minutes past midnight

Hence, for a given date (e.g., March 22, 1972), $\theta_g$ can be determined; then $t$ corresponds to the hours, minutes, and seconds, for example, to

March 22, 1972, 15 hrs, 22 min, 13 sec

where 15 hrs, 22 min, 13 sec are converted to minutes.

---

[7] The station may be thought of as a point in space with $\phi$, $\lambda_E$ orienting the point while the distance from the Earth's surface is fixed by $H$.

[8] This angle is easily computed as a function of time from a polynomial. Specifically, $\theta_g = 99.69098 + 36000^{\circ}.7689 + 0^{\circ}.00038708\, T_u^2$, where $T_u = (\text{JD} - 2415020.0)/36525$ and JD is the Julian date.

## IV. Twilight Function

From Section II the coordinates of the Sun have been determined in the inertial system. Similarly from Section III the coordinates of the station have been determined in the inertial system. The geometry that exists between the station and the Sun is illustrated in Fig. A-3.

From the figure the fundamental vector relationship is

$$\mathbf{R}' = \mathbf{R} + \mathbf{R}_{\odot} \tag{A-5}$$

Twilight/dusk will occur when $\beta \equiv \pi/2$. This condition can be represented by

$$\mathbf{R}' \cdot \mathbf{R} = 0$$

Eliminating $\mathbf{R}'$ yields

$$(\mathbf{R} + \mathbf{R}_{\odot}) \cdot \mathbf{R} = 0 \tag{A-6}$$

or

$$\mathbf{R}_{\odot} \cdot \mathbf{R} = -\mathbf{R} \cdot \mathbf{R} = -R^2 \tag{A-7}$$

where

$$R^2 = G_1^2 \cos^2\phi + G_2^2 \sin^2\phi$$

Expanding the dot product yields

$$X_{\odot} X + Y_{\odot} Y + Z_{\odot} Z = -R^2 \tag{A-8}$$

Substituting for the station coordinates results in

$$\begin{aligned} &G_1 X_{\odot} \cos\phi \cos(\theta_g + \lambda_E + \dot{\theta} t) \\ &+ G_1 Y_{\odot} \cos\phi \sin(\theta_g + \lambda_E + \dot{\theta} t) \\ &+ G_2 Z_{\odot} \sin\phi = -R^2 \end{aligned} \tag{A-9}$$

Finally, inserting the polar coordinates of the Sun, i.e., use of Eq. (A-3), yields the generalized twilight function $T$, namely,

$$\begin{aligned} T \equiv\ & G_1 \cos\phi \cos\delta_{\odot} \cos\alpha_{\odot} \cos(\theta_g + \lambda_E + \dot{\theta} t) \\ &+ G_1 \cos\phi \cos\delta_{\odot} \sin\alpha_{\odot} \sin(\theta_g + \lambda_E + \dot{\theta} t) \\ &+ G_2 \sin\phi \sin\delta_{\odot} + R^2/R_{\odot} = 0 \end{aligned} \tag{A-10}$$

where daylight occurs for $T > 0$, while $T < 0$ implies night. The general solution of Eq. (A-10) will be discussed next, followed by a simplified model that achieves a closed-form solution to the problem under discussion.

## V. General Solution of Twilight Function (Duration of Night)

Suppose that the model of the Sun's motion is taken as in Section II (obviously more exact models (Ref. 10) can be used). The result of the substitution yields (in functional notation)

$$\begin{aligned} T \equiv\ & G_1 \cos\phi \cos\delta_{\odot}\{t - t_E\} \cos\alpha_{\odot}\{t - t_E\} \\ &\times \cos\left[\theta_{g_{t_E}} + \lambda_E + \dot{\theta}(t - t_E)\right] \\ &+ \cdots + G_2 \sin\phi \sin\delta_{\odot}\{t - t_E\} + \frac{R^2}{R_{\odot}} \\ &= 0 \end{aligned} \tag{A-11}$$

where $t_E$ = March 22, and $t$ is the time elapsed since that date, and $\theta_{g_{t_E}}$ is the Greenwich sidereal time at $t_E$. In functional notation

$$T = T(\phi, \lambda_E, a_e, f, \theta_{g_{t_E}}, \dot{\theta}, \delta_{\odot}(t), \alpha_{\odot}(t), t) \tag{A-12}$$

therefore starting at March 22 and solving Eq. (A-11) for successive dates will yield the dawn to dusk history during the year for the specified station.

## VI. Simplified Model

Let it be assumed that a certain station location is selected so that $\phi$, $\lambda_E$ are known; furthermore, at a given time of the year, suppose that $\alpha_{\odot}$ and $\delta_{\odot}$ are selected from suitable tables or from Section II, and held fixed. More exactly for the day in question, average values of $\alpha_{\odot}$ and $\delta_{\odot}$ are computed as follows:

$$\left.\begin{aligned} \bar{\alpha}_{\odot} &= \frac{1}{2}\left[\alpha_{\odot}(t'_E) + \alpha_{\odot}(t'_E + 24)\right] \\ \bar{\delta}_{\odot} &= \frac{1}{2}\left[\delta_{\odot}(t'_E) + \delta_{\odot}(t'_E + 24)\right] \end{aligned}\right\} \tag{A-13}$$

where $t'_E$ is the date under consideration in hours.

Under these assumptions, and collecting all constant terms,

$$T \cong A\cos[\alpha_s + \dot{\theta} t] + B\sin[\alpha_s + \dot{\theta} t] + C = 0 \tag{A-14}$$

where

$$A = G_1 \cos\phi \cos\bar{\delta}_\odot \cos\bar{\alpha}_\odot$$

$$B = G_1 \cos\phi \cos\bar{\delta}_\odot \sin\bar{\alpha}_\odot$$

$$C = G_2 \sin\phi \sin\bar{\delta}_\odot + R^2/R_\odot$$

$$\alpha_s = \theta_g + \lambda_E$$

The only free parameter in Eq. (A-14) which for a fixed date can be used to satisfy that $\beta$ (Fig. A-3) be $\pi/2$ is $t$. Hence, the intent will be to solve for $t$. Divide Eq. (A-14) by $\sqrt{A^2 + B^2}$ and let

$$\left.\begin{aligned} \cos\sigma &= \frac{A}{\sqrt{A^2+B^2}} \\ \sin\sigma &= \frac{B}{\sqrt{A^2+B^2}} \end{aligned}\right\} \tag{A-15}$$

where obviously $\sigma$ is a known constant angle. The result is

$$\cos\sigma\cos(\alpha_s + \dot{\theta}t) + \sin\sigma\sin(\alpha_s + \dot{\theta}t) = \frac{-C}{\sqrt{A^2+B^2}}$$

or

$$\cos(\sigma - \alpha_s + \dot{\theta}t) = \frac{-C}{\sqrt{A^2+B^2}} \tag{A-16}$$

Solving for $t$ yields

$$t = \left[\tan^{-1}\left(\frac{B}{A}\right) - \alpha_s - \cos^{-1}\left(\frac{-C}{\sqrt{A^2+B^2}}\right)\right]\frac{1}{\dot{\theta}} \tag{A-17}$$

Eliminating $A$, $B$, $C$ results in

$$t_{td} = \left[\bar{\alpha}_\odot - \alpha_s - \cos^{-1}\left(-\frac{G_2\sin\phi\sin\bar{\delta}_\odot + \dfrac{R^2}{R_\odot}}{G_1\cos\phi\cos\bar{\delta}_\odot}\right)\right]\frac{1}{\dot{\theta}} \tag{A-18}$$

where the values of $t_t$ or $t_d$ are obtained for the two zeros of Eq. (A-18). The duration of the night is therefore given by

$$\Delta T = t_d - t_t = t_N \tag{A-19}$$

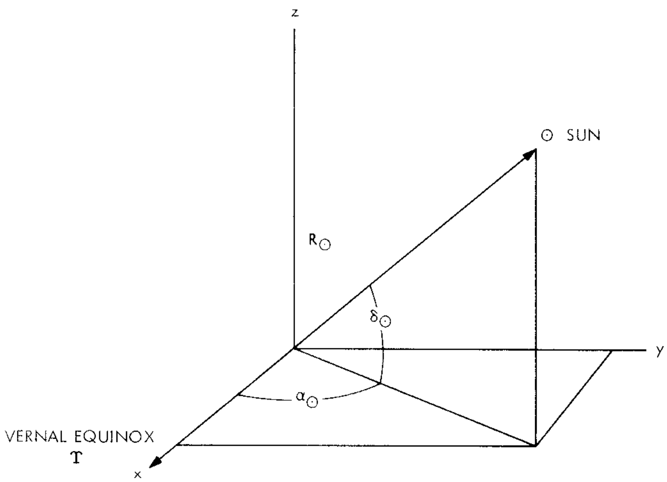


Fig. A-1. Sun's coordinates

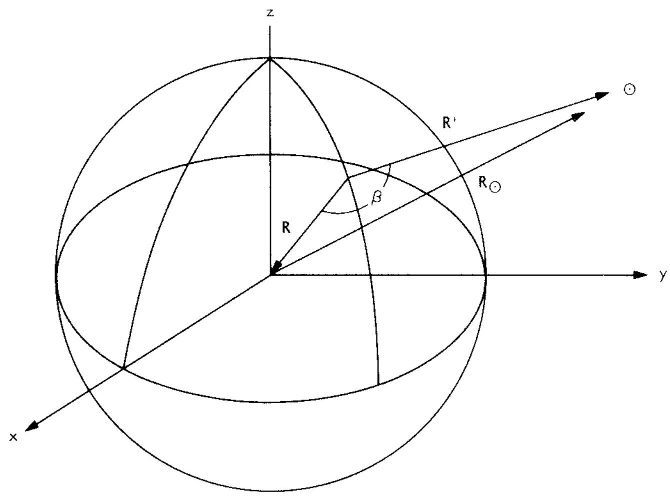


Fig. A-3. Vector triangle between center of Earth, Sun, and station

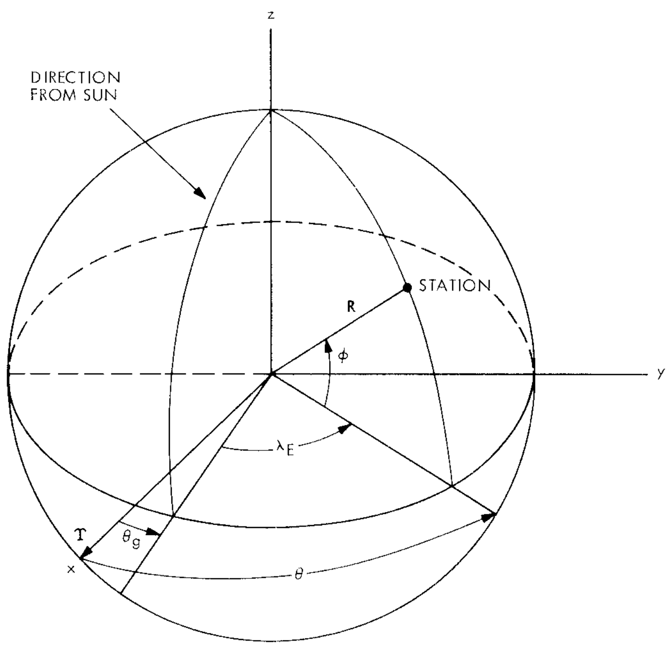


Fig. A-2. Station coordinates

# Appendix B

## The Duration of the Ionospheric Twilight

This appendix draws heavily on Figs. 3 and 4. Suppose that Earth rotates as indicated in Fig. 3. That is to say, that the spin axis of Earth is perpendicular to the x-y plane in the Sun-centered coordinate system (Fig. 1 of Ref. 1). It is clear from Fig. 3 that the angles $\alpha_1$ and $\alpha_2$ depicted are given by

$$\left.\begin{aligned} \alpha_1 &= \tan^{-1}\left(2\frac{h_{\max}+H}{R}\right)^{1/2} \\ \alpha_2 &= \tan^{-1}\left(2\frac{h_{\max}}{R}\right)^{1/2} \end{aligned}\right\} \tag{B-1}$$

In Eqs. (B-1), $R$ is the radius of Earth, $h_{\max}$ is the height of the maximum electron density of the F2 layer, and $H$ its scale height, parameters well described in Ref. 1. In order to obtain Eqs. (B-1), the assumption $R \gg h_{\max}$ was made. In this case, then, the duration between complete darkness and complete light is simply given by

$$\begin{aligned} \tau = {} & \frac{1}{\dot\theta}\tan^{-1}\left(2\frac{h_{\max}+H}{R}\right)^{1/2} \\ & - \frac{1}{\dot\theta}\tan^{-1}\left(2\frac{h_{\max}}{R}\right)^{1/2} \end{aligned} \tag{B-2}$$

where $\dot\theta$ is the spin rate of the Earth's rotation ($7.3 \cdot 10^{-5}$ rad/sec). Equation (B-2) gives the transition time between total darkness and complete light but only if $\delta_\odot = 0$ and $\theta_g = \pi/2$. In order to find out the general twilight or transition time between darkness and daylight in the upper ionosphere, we must look at Fig. 4. Here we depicted two sets of meridians, one set being Earth-fixed and the other set being centered in the Sun-fixed system (Fig. 1 of Ref. 1). With a little spherical geometry it can be deduced that in general the time of twilight in the upper ionosphere is given by

$$\begin{aligned} \dot\theta_{T_T} = \sin^{-1}\left\{\frac{\cos\delta_\odot}{\sin\theta_g}\left[\sin\left(\tan^{-1}\left(2\frac{h_{\max}+H}{R}\right)^{1/2}\right.\right.\right. \\ \left.\left.\left. - \tan^{-1}\left(2\frac{h_{\max}}{R}\right)^{1/2}\right)\right]\right\} \end{aligned} \tag{B-3}$$

Notice that if $\delta_\odot = 0$ and $\theta_g = \pi/2$, Eq. (B-3) reduces to Eq. (B-2). Equation (B-3) then determines the ionospheric twilight. We must also be quite aware of the fact that the paucity of scattering, because of the low density of matter (<$10^{14}$ atoms/cm$^3$), makes the derivation of Eq. (B-3) a purely geometrical consideration (shadows are sharp).

As a matter of fact the correct time of night is given by

$$t_{\text{night}} = t_N + \frac{1}{\dot\theta}\sin^{-1}\left\{\frac{\cos\delta_\odot}{\sin\theta_g}\sin\left[\tan^{-1}\left(2\frac{h_{\max}}{R}\right)^{1/2}\right]\right\} \tag{B-4}$$

where $t_N$ is obtained from Appendix A (Eq. A-19).

A simplification of Eq. (B-4) arises by noting that

$$\sin(\tan^{-1} a) = \frac{a}{\sqrt{1+a^2}} \tag{B-5}$$

and since $2h_{\max}/R \ll 1$ we obtain from Eq. (B-4)

$$t_{\text{night}} = t_N - \frac{1}{\dot\theta}\sin^{-1}\left\{\frac{\cos\delta_\odot}{\sin\theta_g}\left(2\frac{h_{\max}}{R}\right)^{1/2}\right\} \tag{B-6}$$

In a similar fashion Eq. (B-3) can be simplified by using trigonometric rules, viz,

$$\dot\theta_{T_T} = \sin^{-1}\left\{\frac{\cos\delta_\odot}{\sin\theta_g}\left[\left(2\frac{h_{\max}+H}{R}\right)^{1/2} - \left(2\frac{h_{\max}}{R}\right)^{1/2}\right]\right\} \tag{B-7}$$

## Appendix C

## Approximate Evaluation of the Integral (35).

Here we write down again the integral in question:

$$\phi(z) = \frac{1}{2}\int_{\infty}^{z-(h_{\max}/H)} dx\, e^{-x}\left(1 - \frac{(Hz+R)^2}{(Hx+R+h_{\max})^2}\right)^{-1/2} \tag{C-1}$$

With the substitution

$$\sin\theta = \frac{Hz+R}{Hx+R+h_{\max}} \tag{C-2}$$

the Integral (C-1) transforms to

$$\phi(z) = -\frac{a}{2}\exp\left[\frac{R+h_{\max}}{H}\right]\int_{0}^{\pi/2} d\theta\, \frac{\exp\left[-\dfrac{a}{\sin\theta}\right]}{\sin^2\theta} \tag{C-3}$$

with $a = z + (R/H)$. Since $z$ ranges from zero to infinity, the parameter $a$ is always large. The integral in Eq. (C-3) can be written as follows:

$$\int_0^{\pi/2} d\theta \cdots = \frac{\partial^2}{\partial a^2}\int_0^{\pi/2} d\theta \exp\left[-\frac{a}{\sin\theta}\right] \tag{C-4}$$

Observing the fact that $a \gg 1$, we may use the method of steepest descent. Accordingly we put (Taylor series about $\theta = \pi/2$)

$$\frac{a}{\sin\theta} \approx a\left[1 + \frac{1}{2}\left(\theta - \frac{\pi}{2}\right)^2\right] \tag{C-5}$$

and have

$$\int_0^{\pi/2} d\theta \exp\left[-\frac{a}{\sin\theta}\right] \approx e^{-a}\int_0^{\pi/2} d\theta \exp\left[-\frac{a}{2}\left(\theta - \frac{\pi}{2}\right)^2\right]$$

$$= \frac{1}{2}\sqrt{\frac{2\pi}{a}}\, e^{-a} \tag{C-6}$$

since the integration can be extended to infinity at the lower limit. It is expression (C-6) that is used in the text. The differentiation indicated in Eq. (C-4) gives

$$\frac{\partial^2}{\partial a^2}(a^{-1/2} e^{-a}) = \left(a^{-1/2} + a^{-3/2} + \frac{3}{4}a^{-5/2}\right)e^{-a} \approx a^{-1/2}e^{-a} \tag{C-7}$$

again because $a \gg 1$.