# A Note on Noisy Reference Detection

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*Telemetry systems supported by the DSN employ coherent detection of a bi-phase phase-shift keyed (PSK) waveform. The coherent reference for the detector is supplied by a bandpass limiter/phase-locked loop. Phase noise in this coherent reference is often a critical factor in establishing link performance. Lindsey has analyzed the performance of the DSN receivers under the extreme assumptions that (a, high-rate) the phase error of the coherent reference is constant over the symbol interval, and that (b, low-rate) the phase error of the coherent reference varies rapidly over the symbol interval. Blake/Lindsey and Tausworthe, subsequently developed techniques to approximate the DSN receiver performance between these extremes. Under close examination, however, it becomes apparent that with typical DSN parameters, the approximations used by Blake and Lindsey become suspect above 15–30 bits/s, and hence do not validly cover many of the interesting data rates. The interpolation proposed by Tausworthe depends upon approximating the log of the error probability by the first few terms of a Taylor series in the variational part of the decision variable. Again, there must be some ranges of defining parameters (not necessarily of practical interest) where his approximation becomes invalid. It is the intent of this article to develop a refined approximation to the performance of the DSN receivers with a (hopefully) wider validity range than previous techniques.*

## I. Introduction

Telemetry systems supported by the DSN employ coherent detection of a bi-phase PSK waveform. The coherent reference for the detector is supplied by a bandpass limiter/phase-locked loop. Phase noise in this coherent reference is often a critical factor in establishing link performance. Lindsey (Ref. 1) has analyzed the performance of the DSN receivers under the extreme assumptions that (a, high-rate) the phase error of the coherent reference is constant over the symbol interval, and that (b, low rate) the phase error of the coherent reference varies rapidly over the symbol interval. Blake and Lindsey (Ref. 2) and Tausworthe (Ref. 3) subsequently developed techniques to approximate the DSN receiver performance between these extremes. Under close examination, however, it becomes apparent that with typical DSN parameters, the approximations used by Blake and Lindsey (Ref. 2) become suspect above 15–30 bits/s, and hence do not validly cover many of the interesting data rates. The interpolation proposed by Tausworthe depends upon approximating the log of the error probability by the first few terms of a Taylor series in the variational part of the

decision variable. Again, there must be some ranges of defining parameters (not necessarily of practical interest) where his approximation becomes invalid. It is the intent of this article to develop a refined approximation to the performance of the DSN receivers with a (hopefully) wider validity range than previous techniques.

The system parameter which characterizes the high-, medium- and low-rate situations is the normalized data rate defined as the ratio of the data rate $\mathcal{R}$ to the two-sided phase-locked loop bandwidth $W_L$ at its operating point. Most of the significant performance variation occurs in the range of $\delta$ from 0.5 to 10. For the DSN receiver with its nominal design point loop bandwidth of 12 Hz, $W_L$ varies between 20 and 60 Hz at typical operating points, so the medium-rate approximate analysis is applicable to all data rates below about 500 bits/s.

## II. System Model

The system in which we are interested consists of a limiter-controlled phase-locked loop which tracks the carrier of the signal received from a spacecraft, followed by an integrate-and-dump correlation detector using the output of the phase-locked-loop for the coherent reference. For uncoded telemetry, the decision statistics emitted by the correlation detector can be represented by

$$D_t = \frac{1}{T} \int_t^{T+t} \cos\left(\omega_c t + \widehat{\phi}(t)\right) \cdot [A \, m_t \cos\left(\omega_c t + \phi_c(t)\right)$$
$$+ \, n(t)] \, dt \qquad (1a)$$

$$= A m_t \cdot \frac{1}{T} \int_t^{T+t} \cos\left(\widehat{\phi}(t) - \phi_c(t)\right) dt + N_t \qquad (1b)$$

This expression involves the implicit assumption that $n(t)$ is independent of the phase $\widehat{\phi}(t)$, and neglects losses due to bit timing errors, subcarrier synchronization errors, waveform distortions, etc. The modulation term $m_t$ is $\pm 1$. For coded telemetry, $m_t$ would be replaced in Eq. (1b) by the appropriate cross-correlation between received and tentative reference code words.

The critical feature of Eq. (1b) is its dependence upon the reference phase error $\phi_e(t) = \widehat{\phi}(t) - \phi_c(t)$, which is a band-limited low-pass random process whose bandwidth is the phase-locked-loop bandwidth. For a first-order tracking loop, $\phi_e$ has autocorrelation function

$$R_{\phi_e}(\tau) = \sigma_{\phi_e}^2 \exp\left(-2W_L |\tau|\right) \qquad (2)$$

Furthermore, if the signal-to-noise ratio in the operating loop bandwidth $\rho_L$ is large, $\phi_e(t)$ is essentially Gaussian

with $\sigma_{\phi_e}^2 = 1/\rho_L$. For moderate $\rho_L$ ($> 3$), quasi-linear loop theory shows that $\phi_e(t)$ is well approximated by a Gaussian process with $\sigma_{\phi_e}^2 = \exp\left(\sigma_{\phi_e}^2/2\right)/\rho_L$. In the following, we will use $\rho_L'$ to denote an equivalent quasi-linear loop signal-to-noise ratio (SNR), so that $\sigma_{\phi_e}^2 = 1/\rho_L'$. For the high-rate performance model, $W_L \cdot T << 1$, $\phi_e(t)$ is constant during the integration period, and the probability distribution of $D_t$ is well known (Ref. 1).

For the low-to-medium rate performance model, Blake and Lindsey propose the approximation

$$\left.\begin{array}{c} \cos \phi_e > 1 - \dfrac{1}{2} \phi_e^2 \\[4mm] D_t \sim A m_t \left\{1 - \dfrac{1}{2} \cdot \dfrac{1}{T} \int_0^T \phi_e^2(t) \, dt\right\} + N_t \end{array}\right\} \qquad (3)$$

The parameter

$$x = \frac{1}{T} \int_0^T \phi_e^2(t) \, dt$$

has a density function $g(x)$ which is well approximated by (Refs. 2 and 4)

$$g(x) = \rho_L' \left(\frac{\beta}{2\pi}\right)^{1/2} (\rho_L' x)^{-3/2} \exp\left\{-\frac{\beta}{2} [\rho_L' x - 2 + 1/\rho_L' x]\right\} \qquad (4)$$

The parameter $\beta \triangleq W_L T = 1/\delta$. As noted by Ref. 2, the approximation is very close for $\beta > 5$, but becomes invalid for $\beta < 1$. The density $g(x)$ has its maximum at

$$\rho_L' \cdot x = \left[1 + \left(\frac{3}{2\beta}\right)^2\right]^{1/2} - \left(\frac{3}{2\beta}\right)$$

and a zero at the origin.

Let $y = \rho_L' x$. The first few moments of $y$ can be readily calculated to be

$$\overline{y^0} = 1, \; \overline{y^1} = 1, \; \overline{y^2} = (1 + \beta)/\beta, \; \overline{y^3} = (3 + 3\beta + \beta^2)/\beta^2$$

From this, the variance of $y$ can be seen to be $V\{y\} = 1/\beta$. However, for very large $\delta$, small $\beta$, the distribution of $x$ is properly Chi-square with one degree of freedom, and thus the variance of $y$ should be 2. Clearly for small $\beta$, the true and approximate distributions are rapidly divergent. Suppose we can find some function $B(\;)$ such that letting $\beta' = B(\delta)$ forces equality of the second moments of $g(x)$ and the true density for $x$. Then clearly

$$\beta_\infty' = B(\infty) = \frac{1}{2}$$

At this value, $\overline{y^3} = 19$, as contrasted to the true value for a Chi-square (1) of 15. While $g(x, \beta')$ is a closer approximation to the true density than $g(x, \beta)$, its third (and higher) moments remain in error. Tausworthe (Ref. 3) has calculated the second central moment for

$$\zeta = \frac{1}{T} \int_0^T \cos \phi_e (\tau) \, d\tau$$

for use as an interpolation formula. We may use his calculation directly as $B(\ )$:

$$\beta' = B(\delta) = 1/\{\delta - (\delta^2/4)(1 - e^{-4/\delta})\} \qquad (5)$$

This calculation is identical to the second central moment calculated for

$$x = \frac{1}{T} \int_0^T \phi_c^2 (t) \, dt$$

There are a number of other plausible choices for an approximation to the density function for $x$, most of which fit reasonably at some values of $\delta$, and do not fit for others. The preferred choice among those considered is

$$h(y) = \sqrt{\frac{a}{\pi}} \exp(\sqrt{2ab}) \, y^{-\frac{1}{2}} \exp(-ay - b/y) \qquad (6)$$

where $y = \rho'_L x$. This density is fully defined by the parameters $a$ and $b$, which are determined by the requirement that the first two moments of $y$ match those of the true distribution:

$$\overline{y} = 1, \overline{y^2} = 1 + 1/B(\delta) \qquad (7)$$

The parameters $a$ and $b$ are thus determined:

$$\left.\begin{array}{l} a = \dfrac{B(\delta)}{4} (1 + \sqrt{1 + 4/B(\delta)}) \\[2mm] b = a - 1 + 1/4a, \ \sqrt{ab} = a - 1/2 \end{array}\right\} \qquad (8)$$

The third moment of $h(y)$ is readily calculated to be

$$\overline{y_h^3} = 1 + \frac{3}{2a} + \frac{3}{2a^2} + \frac{5}{8a^3} \qquad (9)$$

where $a$ is given by Eq. (8).

This should be compared against the third moment of $g(x, \beta')$, which is

$$\overline{y_{g'}^3} = 1 + 3/B(\delta) + 3/B(\delta)^2 \qquad (10)$$

and against the third moment of the actual distribution.

The $n$th order moments of the actual distribution can be calculated for small $n$ by reference to the definition in terms of an exponential-memory Gaussian process:

$$y = \frac{1}{T} \int_0^T (\phi_e (t)/\sigma_{\phi_e})^2 \, dt \qquad (11)$$

By definition,

$$\overline{y^3} = \frac{1}{T^3} \sigma_{\phi_e}^{-6} \iiint_0^T E \{\phi_e (t) \cdot \phi_e (s) \cdot \phi_e (z)\} \, ds \, dt \, dz$$

$$= \frac{1}{T^3} \iiint_0^T \{1 + 6 \exp(-4W_L |t - s|)$$

$$+ 8 \exp[-2W_L (|t - s| + |s - z|$$

$$+ |z - t|)]\} \, ds \, dt \, dz$$

$$= 1 + \frac{12}{\eta}\left[1 - \frac{1}{\eta}(1 - e^{-\eta})\right]$$

$$+ \frac{48}{\eta^2}\left[1 + e^{-\eta} - \frac{2}{\eta}(1 - e^{-\eta})\right] \qquad (12)$$

where

$$\eta = 4W_L T = 4/\delta$$

Figure 1 shows a comparison of the actual third moment of $y$ (line 1), the third moment of $g(x, \beta')$ (line 2), and the third moment of $h(y)$ (line 4). On this basis, $h(y)$ appears to provide the best approximation over the entire range of $\delta$.

## III. Detector Performance

For uncoded communications, the error probability is simply the probability that $m_t \cdot D_t$ is negative. Assuming that $N_t$ and $\cos(\phi_e(t))$ are independent, this is equal to the expected value (over the distribution of $x$) of the probability that $m_t \cdot D_t$ is negative conditioned on $x$:

$$P_e = \int_0^\infty f(x) \Pr\left\{1 - \frac{x}{2} < -N_t/A\right\} dx \qquad (13)$$

And if $R$ denotes the bit signal-to-noise ratio ($R = E_b/N_0 = S \cdot T/N_0$),

$$P_e = \int_0^\infty f(x) \, \mathrm{erfc}\left\{\sqrt{2R}\left(1 - \frac{x}{2}\right)\right\} dx \qquad (14)$$

This probability is shown in Fig. 2 for several different models, for $\delta$ in the range 0.1 to 10. The "+" on this figure

corresponds to the high-rate model calculation (Ref. 1). Line 1 was calculated using Tausworthe's interpolation formula, line 2 using $g(x, \beta')$, line 3 using $g(x, \beta)$, and line 4 using $h(x)$. Both the $h(x)$ calculations and Tausworthe's interpolation agree with the high-rate model for large $\delta$.

We are now faced with interpreting and evaluating these models. For $\delta$ above about 0.3, the performance for $g(x, \beta)$ is rapidly diverging from $g(x, \beta')$ as a result of the second-moment misfit. Below this point, there is little difference between these and the low-rate model results. Ergo, there is little reason to use the unmodified $g(x, \beta)$ model to interpolate for medium-rate performance.

In Fig. 1, the third moments of $g(x, \beta')$ and $h(x)$ are respectively above and below the actual third moment. If we believe that, in the presence of identical second moments, the value of the third moment is an indication of the weight of the distribution at large values, or values which tend to cause errors, then we should expect that the actual error probability would fall between these two models. In Fig. 2, the Tausworthe interpolation falls outside this interval for almost all values of $\delta$, implying that we should reject the hypothesis that Tausworthe's interpolator is accurate at the intermediate data rates. Further evidence to that effect can be obtained by interpreting the Tausworthe interpolation in terms of a density function on $\zeta$, the most straightforward choice being the interpolation between the known densities at the high-rate and low-rate extremes. The second and third moments of this fabricated density differ significantly from the moments of the actual distribution. The most reasonable candidate for an accurate approximation would appear to be calculations utilizing $h(x)$.

Figures 3, 4, and 5 compare these models for several other values of carrier loop SNR $(\rho_L)$, and bit SNR $(R)$. In the vicinity of $10^{-3}$ and higher error probability, typical of video missions, the calculated results differ little, and it is most reasonable to utilize the Tausworthe interpolation formula since it is simplest to compute. At lower error probability values, the models diverge significantly, and while these low error probabilities may not be interesting per se, for uncoded systems this divergence is interesting in that we may expect a similar divergence for *coded* systems at error probabilities which are of interest to deep space missions.

## IV. Summary

This article has compared several methods of calculating noisy reference telemetry performance at intermediate data rates, i.e., where the data rate to loop bandwidth ratio is on the order of 0.1 to 10. In general, at bit error probabilities of about $10^{-3}$, which have been typical of uncoded video missions, the interpolation formula proposed by Tausworthe (Ref. 3) provides acceptable answers. At low bit error probabilities, or for coded systems, evidence presented herein implies that the noisy reference calculation based upon the density function $h(y)$ in Eqs. (6) and (8) provides more accurate answers than previous techniques.

# References

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2. Lindsey, W. C., and Blake, I. F., "Effects of Phase-Locked Loop Dynamics on Phase-Coherent Communications," in *Supporting Research and Advanced Development*, Space Programs Summary 37-54, Vol. III, Jet Propulsion Laboratory, Pasadena, Calif., Dec. 31, 1968, pp. 192–195.

3. Tausworthe, R. C., "Efficiency of Noisy Reference Detection," in *Supporting Research and Advanced Development*, Space Programs Summary 37-54, Vol. III, Jet Propulsion Laboratory, Pasadena, Calif., Dec. 31, 1968, pp. 195–201.

4. Grenander, U., Pollack, H. O., and Slepian, D., "The Distribution of Quadratic Forms in Normal Variates . . .". *J. Soc. Ind. Appl. Math.*, Vol. 7, No. 4, pp. 374–401, Dec. 1959.
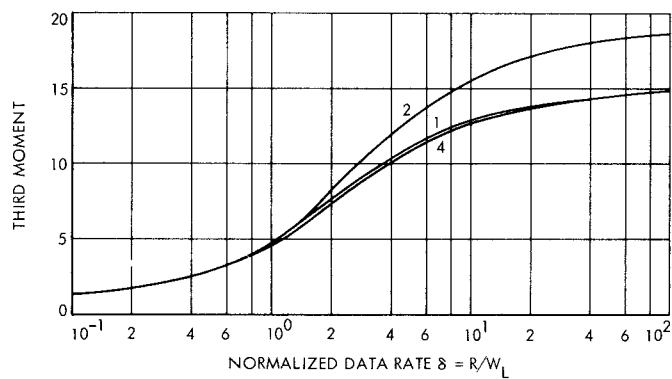
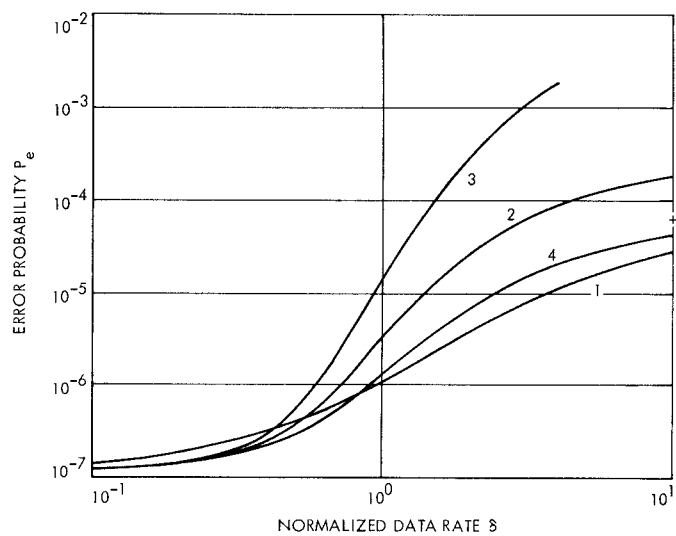Fig. 1. Third moments of distribution



Fig. 2. Calculated error probability vs normalized data rate
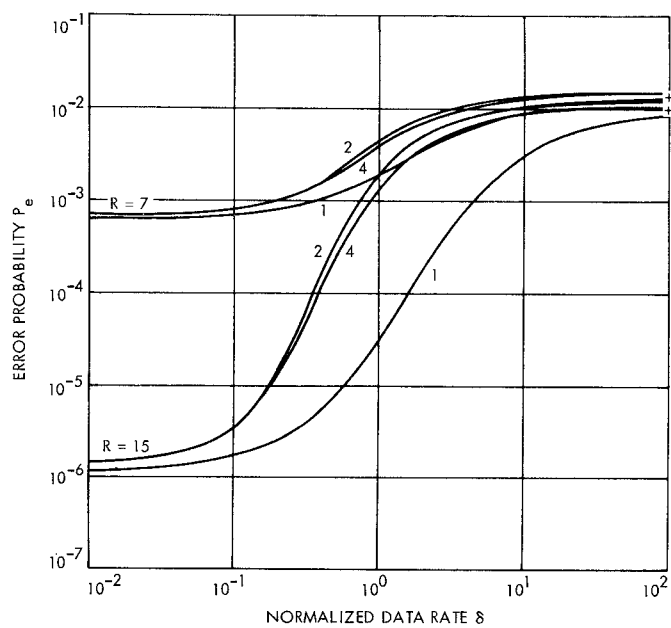for loop SNR = 10, bit SNR = 15



Fig. 3. Calculated error probability vs normalized data rate
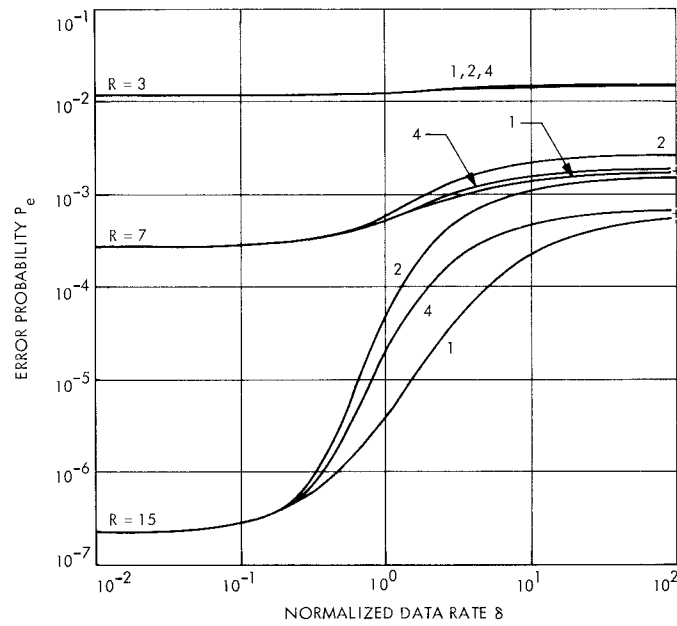for loop SNR = 4, bit SNR = 7 and 15

**Fig. 4. Calculated error probability vs normalized data rate for loop SNR = 7, bit SNR = 3, 7, and 15**
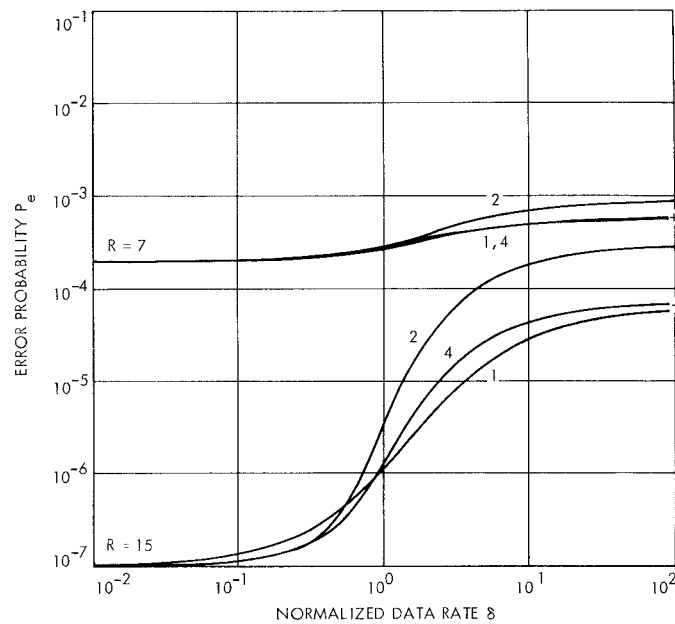


**Fig. 5. Calculated error probability vs normalized data rate for loop SNR = 10, bit SNR = 7 and 15**