Dual-Carrier Intermodulation Caused by a Zero-Memory Nonlinearity

C. A. Greenhall
Communications Systems Research Section

Intermodulation products of a dual carrier distorted by a zero-memory non-linearity F are calculated. Conversely, given certain of the intermodulation (IM) coefficients, the odd part F_o of the nonlinearity can be recovered. If the coefficients decrease fast enough, then F_o is analytic. The value of F_o is explicitly calculated for two cases of exponentially decreasing intermodulation products.

I. Introduction

This report is part of a study of intermodulation products (IMPs) which have been observed in dual-carrier tests at DSS 14 (Ref. 1). The main feature of the data on IMP amplitudes is that the amplitude of the nth IMP (counting away from the dual-carrier midfrequency) falls off exponentially with n. The data on the dependence of IMP amplitude on carrier power are much more tenuous.

It has been proposed that the IM distortion can be accounted for by nonlinear current–voltage characteristics of metal oxide contacts at bolted joints in the antenna structure. The present article describes analytically what sort of zero-memory characteristics could give rise to the observed IMP data. No physical assumptions are made.

The main mathematical tool is Chebyshev transform inversion, as described by N. Blachman (Ref. 2). This tool was created for the study of harmonic distortion, but we will see that it is also very useful for the study of IM distortion.

II. Formulas for Intermodulation Products

Given a zero-memory nonlinearity $y = F(x)$, $-L < x < L$, let the input be the symmetric dual carrier

$$x(t) = \frac{1}{2} v (\cos \omega_1 t + \cos \omega_2 t)$$

$$= v \cos \omega_+ t \cos \omega_- t$$

where

$$0 \leq v < L$$

$$0 < \omega_1 < \omega_2$$

$$\omega_+ = \frac{1}{2}(\omega_1 + \omega_2)$$

$$\omega_- = \frac{1}{2}(\omega_2 - \omega_1)$$

The output is then

$$y(t) = F(v \cos \omega_+ t \cos \omega_- t)$$

If $-L < v < L$ and F, let us say, is bounded and Borel measurable on $[-v, v]$, then the function $F(v \cos \theta \cos \phi)$ of the variables θ and ϕ has a double Fourier series:

$$F(v \cos \theta \cos \phi) \sim \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} c_{\mu\nu}(v) \cos(\mu\theta + \nu\phi)$$

where

$$c_{\mu\nu}(v) = (4\pi^2)^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(v \cos \theta \cos \phi) \cos(\mu\theta + \nu\phi) \, d\theta \, d\phi$$

The IMP of index $(\mu, \nu) \neq (0, 0)$ is

$$2c_{\mu\nu}(v) \cos(\mu\omega_+ + \nu\omega_-) t$$

The IMPs with which we are concerned are

$$2c_{1,2n+1}(v) \cos(\omega_2 + n(\omega_2 - \omega_1)) t, \qquad n = 1, 2, \cdots$$

(Note that $c_{1,2n} = 0$.)

Write f_o to mean the odd part of a function f; that is,

$$f_o(x) = \frac{1}{2}(f(x) - f(-x))$$

After exploiting symmetries, we get

$$c_{1\nu}(v) = (1/\pi^2) \int_{-\pi}^{\pi} d\phi \cos \nu\phi \int_{0}^{\frac{1}{2}\pi} d\theta \cos \theta \, F_o(v \cos \theta \cos \phi) \tag{1}$$

III. Inversion of the Mapping $F_o \rightarrow \{c_{1\nu}(v)\}$

We would like to deduce properties of F from properties of the $c_{1\nu}(v)$. Equation (1) shows that we can get information about only the odd part of F.

Let us adopt some notation for the inner integral of Eq. (1). For $n = 0, 1, 2, \cdots$, define an operator T_n by

$$(T_n f)(x) = (2/\pi) \int_{0}^{\frac{1}{2}\pi} f(x \cos \theta) \cos n\theta \, d\theta \tag{2}$$

Blachman (Ref. 2) calls this the nth-order Chebyshev transform. (However, he integrates from 0 to π; the definition above makes T_0 and T_1 one–one.) Then

$$c_{1\nu}(v) = (2\pi)^{-1} \int_{-\pi}^{\pi} (T_1 F_o)(v \cos \phi) \cos \nu\phi \, d\phi \tag{3}$$

is just half the νth cosine coefficient of $(T_1 F_o)(v \cos \phi)$; i.e.,

$$(T_1 F_o)(v \cos \phi) \sim 2 \sum_{\nu=1}^{\infty} c_{1\nu}(v) \cos \nu\phi \tag{4}$$

Thus, if we can sum the series in Eq. (4), we can get $T_1 F_o$. Then, as Ref. 2 shows, F_o can easily be obtained from $T_1 F_o$. Let us make precise the information we need here. Denote by B_v the space of bounded Borel measurable functions on $[-v, v]$, and by A_v the space of functions analytic in a neighborhood of $[-v, v]$. Then T_n takes B_v and A_v into themselves.

Proposition 1. (a) *If f, $g \in B_v$ and $T_1 f = T_1 g$ almost everywhere, then $f = g$ almost everywhere.*

(b) $T_1 : A_v \rightarrow A_v$ *has a two-sided inverse* $U_1 : A_v \rightarrow A_v$, *given by*

$$(U_1 f)(x) = \int_{0}^{\frac{1}{2}\pi} (t f(t))' (x \cos \theta) \, d\theta \tag{5}$$

(c) *If $f \in B_v$ and $T_1 f \in A_v$, then f is almost everywhere equal to the analytic function $U_1 T_1 f$.*

The notation in Eq. (5) means that the function $(t f(t))'$ is to be evaluated at $t = x \cos \theta$. Proposition 1 is proved in the Appendix.

We know now that if $T_1 F_o$, as determined by Eq. (4), is analytic on $[-v, v]$, then F_o is also analytic on $[-v, v]$, and can be got by applying U_1 to $T_1 F_o$.

Fortunately, the IMP data indicate that the $c_{1\nu}$ decay exponentially with ν. This is just the condition we need to show that $T_1 F_o$ is analytic on $[-v, v]$.

Proposition 2. *Let f be a bounded Borel measurable function on $[-1, 1]$. Then f is (essentially) analytic on $[-1, 1]$ if and only if the coefficients*

$$c_\nu = (2\pi)^{-1} \int_{-\pi}^{\pi} f(\cos \phi) e^{-i\nu\phi} \, d\phi \tag{6}$$

satisfy

$(\star)$ $|c_v| \leqq A \rho^{|v|}$, $-\infty < v < \infty$, *for some A and ρ with* $A > 0, 0 < \rho < 1$.

Proof. Let $p(w) = (w + w^{-1})/2$.

Let f be analytic in a neighborhood Ω of $[-1, 1]$. Define $g(w) = f(p(w))$, $w \epsilon \Omega$. Then g is analytic in an annulus $G_r = \{w: r^{-1} < |w| < r\}$, and so has a Laurent series that converges there. Since $g(e^{i\phi}) = f(\cos \phi)$, the c_v are the Laurent coefficients of g. Hence

$$\sum_{-\infty}^{\infty} |c_v w^v| < \infty \text{ for } r^{-1} < |w| < r$$

This implies $(\star)$.

Conversely, assume $(\star)$. Define

$$g(w) = \sum_{-\infty}^{\infty} c_v w^v$$

Then again g is analytic in some annulus G_r, and $g(e^{i\phi}) = f(\cos \phi)$. If $|w| = 1$ then $g(w) = g(w^{-1})$. By analytic continuation, this must hold everywhere in G_r. Since

$$(z + i\sqrt{1 - z^2})(z - i\sqrt{1 - z^2}) = 1$$

it follows that

$$g(z + i\sqrt{1 - z^2}) = g(z - i\sqrt{1 - z^2}) \tag{7}$$

for all $z \epsilon p(G_r)$, a region that contains $[-1, 1]$. Thus Eq. (7) defines a function $f^*(z)$ on $p(G_r)$ such that $f^* = f$ on $[-1, 1]$. The function f^* is analytic except possibly at 1 and -1, but since f^* is continuous, it is analytic at 1 and -1 also.

Letting $f(x) = (T_1 F_o)(vx)$ in Proposition 2 and then applying Proposition 1, we have shown

Proposition 3. *For a fixed dual-carrier amplitude v, assume that the IM coefficients $c_{1v}(v)$ of the zero-memory characteristic F decay at least as fast as an exponential $\rho^v, 0 < \rho < 1$. Then the odd part F_o is completely determined on $[-v, v]$ and is analytic on $[-v, v]$.*

The function F_o is obtained by summing the series in Eq. (4) and then applying the inverse Chebyshev transform U_1.

Proposition 3 is a worthwhile result if only because it gives us a condition for F_o to be smooth.

IV. Calculation of the Zero-Memory Characteristic Assuming Exponentially Decreasing IM Coefficients

The data point to the hypothesis

$$|c_{1v}(v)| = A\rho^v, \qquad v = 1, 3, 5, \cdots$$

where $A > 0$ and $0 < \rho < 1$. We have no information about the signs of the c_{1v}, that is, about the phases of the IMPs relative to the carrier components. Let us take two cases, namely

Case 1. $c_{1, 2n+1}(v) = A\rho^{2n+1}$

$$n = 0, 1, 2, \cdots$$

Case 2. $c_{1, 2n+1}(v) = (-1)^n A\rho^{2n+1}$

and calculate F_o in each case. Fortunately, each F_o is an elementary function. Put $f_o(x) = F_o(vx)$, $-1 \leqq x \leqq 1$. Then $(T_1 f_o)(x) = (T_1 F_o)(vx)$.

Take Case 1. By Eq. (4),

$$(T_1 f_o)(\cos \theta) = 2A \sum_{n=0}^{\infty} \rho^{2n+1} \cos(2n + 1)\theta$$

$$= 2A\rho(1 - \rho^2)\cos\theta/[(1 + \rho^2)^2 - 4\rho^2 \cos^2 \theta]$$

$$(T_1 f_o)(x) = Abx/(a^2 - x^2)$$

where

$$a = (1 + \rho^2)/(2\rho), \qquad b = (1 - \rho^2)/(2\rho) \tag{8}$$

By Proposition 1,

$$f_o(x) = (U_1 T_1 f_o)(x)$$

$$= 2Aa^2b \int_0^{1/2\pi} (a^2 - x^2 + x^2 \sin^2 \theta)^{-2} x \cos \theta \, d\theta$$

After doing the integration and returning to F_o, we get

$$F_o(x) = (Ab/a)\Phi(x/(av)), \quad -v \leqq x \leqq v \tag{9}$$

where

$$\Phi(x) = (1 - x^2)^{-3/2} \sin^{-1} x + (1 - x^2)^{-1} x \tag{10}$$

It makes sense to put $M = Ab/a$, $x_o = av$. Since $F_o(x) = M\Phi(x/x_o)$ for $-v \leqq x \leqq v$, we can compute $c_{1\nu}(v)$ in the form $A\rho^\nu$ for any lesser v by solving for A and ρ in terms of M and v/x_o.

Case 2 is similar, except that now

$$(T_1 f_o)(x) = Aax/(b^2 + x^2)$$

$$F_o(x) = (Aa/b)\Psi(x/(bv)), \quad -v \leqq x \leqq v \qquad (11)$$

$$\Psi(x) = (1 + x^2)^{-3/2}\sinh^{-1}x + (1 + x^2)^{-1}x \qquad (12)$$

We collect our calculations:

Proposition 4. *Assume Case 1 for a particular v. Then $F_o(x) = M\Phi(x/x_o)$, $-v \leqq x \leqq v$, where*

$$M = A(1 - \rho^2)/(1 + \rho^2)$$

$$x_o = v(1 + \rho^2)/(2\rho)$$

For all lesser v, Case 1 also holds, with

$$A = M(1 - (v/x_o)^2)^{-1/2}$$

$$\rho = (v/x_o)/[1 + (1 - (v/x_o)^2)^{1/2}]$$

The same is true for Case 2, but Φ is replaced by Ψ and the formulas are

$$M = A(1 + \rho^2)/(1 - \rho^2)$$

$$x_o = v(1 - \rho^2)/(2\rho)$$

$$A = M(1 + (v/x_o)^2)^{-1/2}$$

$$\rho = (v/x_o)/[1 + (1 + (v/x_o)^2)^{1/2}]$$

The two cases are qualitatively different, since Φ blows up at ± 1, but Ψ is analytic on the whole real line.

V. Conclusions

If we know the whole sequence of IM coefficients $c_{1\nu}(v)$ of a zero-memory nonlinearity F acting on a symmetric dual carrier of amplitude v, then we have complete knowledge of the odd part $F_o(x)$ for $-v \leqq x \leqq v$. If the $c_{1\nu}$ decay at least as fast as an exponential, then F_o is analytic on $[-v, v]$. We have calculated F_o explicitly for the cases $c_{1\nu} = A\rho^\nu$ and $c_{1\nu} = A(-1)^{1/2(\nu-1)}\rho^\nu$, which were suggested by data. We do not know whether the resulting F_o's have any physical meaning.

References

1. "Viking Dual Carrier Test Review," DJM-72-141, Jet Propulsion Laboratory, Pasadena, California, December 28, 1972 (JPL internal document).

2. Blachman, N. M., "Detectors, Bandpass Nonlinearities, and Their Optimization: Inversion of the Chebyshev Transform," *IEEE Transactions on Information Theory*, Vol. IT-17, 1971, pp. 397–405.

Appendix

Proof of Proposition 1

Lemma. Let $f \in B_v$. Then for $-v \leq x \leq v$,

$$(2/\pi) x \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} f(x \cos \theta \cos \phi) \cos \theta \, d\theta \, d\phi = \int_0^x f \quad (13)$$

Proof (Blachman). Think of θ and ϕ as being latitude and longitude on the first octant Ω of the unit sphere. Then $\cos \theta \cos \phi$ is a Cartesian coordinate ξ, $\cos \theta \, d\theta \, d\phi = d\sigma$, the element of area, and the left side of Eq. (13) is

$$(2/\pi) x \iint_\Omega f(x\xi) \, d\sigma \quad (14)$$

Now $d\sigma = d\xi \, d\alpha$, where α is longitude about a pole placed where the ξ-axis meets the sphere. Thus (14) equals

$$(2/\pi) x \int_0^1 d\xi \int_0^{\frac{1}{2}\pi} d\alpha \, f(x\xi) = x \int_0^1 f(x\xi) \, d\xi$$

$$= \int_0^x f$$

Define the operators M, J, and D:

$$(Mf)(x) = xf(x), (Jf)(x) = \int_0^x f, (Df)(x) = f'(x)$$

Then the following formulas hold:

$$M T_1 T_0 = M T_0 T_1 = (2/\pi) J \quad \text{on} \quad B_v \quad (15)$$

$$T_0 M = M T_1 \quad \text{on} \quad B_v \quad (16)$$

$$D T_0 = T_1 D \quad \text{on} \quad A_v \quad (17)$$

By Fubini's Theorem, Eq. (15) is just a restatement of the lemma. Equations (16) and (17) are quickly verified.

Proof of Proposition 1. If $f \in B_v$ and $T_1 f = 0$ almost everywhere, then by Eq. (15), $(2/\pi) Jf = M T_0 T_1 f = 0$. Hence $f = 0$ almost everywhere. This proves (a).

By definition, $U_1 = (\pi/2) T_0 DM$. On A_v

$$MU_1 T_1 = (\pi/2) M T_0 DM T_1$$

$$= (\pi/2) M T_0 D T_0 M$$

$$= (\pi/2) M T_0 T_1 DM$$

$$= JDM = M$$

and

$$MT_1 U_1 = (\pi/2) M T_1 T_0 DM$$

$$= JDM = M$$

Since we are dealing with analytic functions, (b) is proved.

Part (c) follows quickly from (a) and (b).

A different proof of Proposition 1(b) uses power series. See Ref. 2.