Capacity of Noncoherent Channels

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This article determines the wideband capacity limit for noncoherent Gaussian channels which are constrained to radiate signals with peak power equal to the average power. The capacity is a function of only two parameters, the predetection signal-to-noise ratio ST/N_0 and the number of signals M. It is shown that the capacity increases monotonically to a wideband limit as M increases. The role played by this limit for noncoherent Gaussian channels is similar to that played by the famous Shannon limit, S/N_0, for coherent channels. Numerical and graphical results are presented for parameters of interest. It was found that an excellent approximation to the wideband noncoherent limit is $S/N_0 \cdot ST/N_0/(2 + ST/N_0)$.

I. Introduction

Multi-frequency-shift-keying (MFSK) is a well-known communications technique (Ref. 1) which is particularly suited for channels with rapid random variations of the phase of the carrier signal. Phase instabilities of this kind are often due to turbulence in the electromagnetic propagation medium through which the signals pass, as, for example, the solar corona and the atmospheres of Venus, Jupiter, and Saturn. A useful parameter for characterizing the degree of phase instability is the predetection

signal-to-noise ratio $\alpha^2/2 = ST/N_0$, where S is the received signal power, N_0 is the one-sided spectral density of the Gaussian noise in the system, and T is the predetection correlation time used by the receiver. The critical quantity here is the predetection correlation time, which is a measure of the rapidity with which the carrier phase varies. A receiver, in order to average out or filter the received noise, will correlate for as long as possible, provided the carrier phase is constant. However, if the correlation time exceeds T, a random phase change in the carrier will

result in an averaging out of the desired signal as well as the noise.

The purpose of this article is to determine the ultimate, theoretical performance of MFSK systems in terms of the basic constraints required by practical engineering design. There are two: one is the already-mentioned pre-detection signal-to-noise ratio imposed by the channel; the other is that the peak power be equal to the average power S, as required by current transmitter technology. A subsidiary constraint is the system bandwidth W, which is determined, ultimately, by the complexity permitted in the receiver and the data rate of the system. Apart from these considerations, the bandwidth of an MFSK system determines the maximum number M of orthogonal signals that can be distinguished in time T, and the bandwidth is, by definition,

$$M = 2WT$$

Furthermore, the question of when an MFSK system may be considered "wideband," i.e., when the assumption that $W = \infty$ is a good approximation, depends on the predetection signal-to-noise ratio. It is shown that the wideband approximation is valid if $M \approx T \exp (ST/N_0)$ for $ST/N_0 >> 1$, which is not surprising to those familiar with MFSK. However, an $M = 16$ suffices when $ST/N_0 < 1$.

The main result is a proof that the wideband MFSK capacity, normalized to the famous wideband capacity of the coherent Gaussian channel $C_\infty = S/N_0 \log_2 e$ bits/s, where $e = 2.718 \cdot \cdot \cdot$ is the base of natural logarithms, is

$$C(\alpha)/C_\infty =$$
$$2\alpha^{-2} \int_0^\infty x \exp \left[-\frac{1}{2} (x^2 + \alpha^2) \right] I_0 (\alpha x) \ln I_0 (\alpha x) \, dx - 1$$
$$(1)$$

where $\alpha^2/2 = ST/N_0$ is the predetection signal-to-noise ratio, and

$$I_0 (\alpha x) = \frac{1}{\pi} \int_0^\pi e^{\alpha x \cos \theta} \, d\theta \qquad (2)$$

The capacity is plotted as a function of the predetection signal-to-noise ratio in Fig. 2, together with a useful curve-fitting approximation:

$$C(\alpha)/C_\infty > \frac{\alpha^2/2}{2 + \alpha^2/2} = \frac{ST/N_0}{2 + ST/N_0} \qquad (3)$$

Figure 3 exhibits capacity for finite bandwidths.

In terms of the minimum energy-to-noise ratio per bit $(E_b/N_0)_{min}$, the results are simply

$$(E_b/N_0)_{min} = [C(\alpha) C_\infty]^{-1} \log_e 2$$
$$\approx 0.7 (ST/N_0)/(2 + ST/N_0) \qquad (4)$$

The theoretical results are corroborated by experience at both high and low values of the predetection signal-to-noise ratio. As the predetection signal-to-noise increases, the wideband MFSK capacity approaches C_∞. This is as it should be, because, at high enough ST/N_0, it is possible to estimate and track out the random phase process and to use coherent signaling techniques. However, although MFSK signaling can achieve (in theory) the coherent limit as $ST/N_0 \to \infty$, coherent techniques (Ref. 1) become more practical since they do not require the exponential rise in bandwidth.

II. Formulation

A block diagram of an MFSK system is illustrated in Fig. 1. From a channel capacity point of view, the MFSK channel is characterized by transmission of one of M discrete inputs numbered $1, 2, \cdot \cdot \cdot, M$, and M continuous outputs $r_1, r_2, \cdot \cdot \cdot, r_M$, received every T seconds, which have the following probability distribution:

$$p(r_k \mid m) =$$
$$\begin{cases} r_k \exp \left(-\frac{1}{2} r_k^2 \right) & \text{for } k \neq m \\ r_k \exp \left[-\frac{1}{2} (r_k^2 + \alpha^2) \right] I_0 (\alpha r_k) & \text{for } k = m \end{cases}$$
$$(5)$$

where $p(r_k \mid m)$ is the conditional probability density of r_k given m (Ref. 2). The probability density of the M-vector $\mathbf{r} = (r_1, r_2, \cdot \cdot \cdot, r_M)$ is then

$$p(\mathbf{r} \mid m) = g(\mathbf{r}) \exp \left(-\frac{1}{2} \alpha^2 \right) I_0 (\alpha r_m) \qquad (6)$$

where

$$g(\mathbf{R}) = \prod_{k=1}^{M} R_k \exp \left(-\frac{1}{2} R_k^2 \right) \qquad (7)$$

The term $g(\mathbf{R})$ may be regarded as a joint probability density of M independent identically distributed random variables $R_1, R_2, \cdot \cdot \cdot, R_M$.

III. Information and Capacity

The channel capacity is

$$C_M(\alpha) = \frac{1}{T} \max_{\mathbf{P}} I_M(\alpha, \mathbf{P}) \log_2 e \qquad \text{bits/second} \qquad (8)$$

where

$$I_M(\alpha, \mathbf{P}) = \int \sum_{m=1}^{M} P_m p(\mathbf{r} \mid m) \ln\left[\frac{p(\mathbf{r} \mid m)}{\sum_{j=1}^{M} P_j p(\mathbf{r} \mid j)}\right] d\mathbf{r} \qquad (9)$$

and $\mathbf{P} = (P_1, P_2, \cdots, P_M)$ is the probability assignment on the input alphabet. After substitution for $P(\mathbf{r} \mid m)$ from Eq. (6) into Eq. (9), we have

$$I_M(\alpha, \mathbf{P}) =$$

$$\exp\left(-\frac{1}{2}\alpha^2\right)$$

$$\times \int_{\substack{r_i > 0 \\ i=1,\cdots,M}} g(\mathbf{R}) \sum_{m=1}^{M} P_m I_0(\alpha R_m) \ln \frac{I_0(\alpha R_m)}{\sum_{k=1}^{M} P_k I_0(\alpha R_k)} d\mathbf{R} \qquad (10)$$

Since $g(\mathbf{R})$ is a probability density, we can write, after splitting up the logarithm,

$$I_M(\alpha, \mathbf{P}) =$$

$$\exp\left(-\frac{1}{2}\alpha^2\right) E\left[\sum_{m=1}^{M} P_m I_0(\alpha R_m) \ln I_0(\alpha R_m)\right]$$

$$- \exp\left(-\frac{1}{2}\alpha^2\right) E\left[\sum_{m=1}^{M} P_m I_0(\alpha R_m) \ln \sum_{k=1}^{M} P_k I_0(\alpha R_k)\right] \qquad (11)$$

where $E[\ ]$ denotes expectation with respect to the probability density $g(\mathbf{R})$. We can further simplify the expression above by noting that the first term in Eq. (11) is independent of $\mathbf{P}$, because

$$\exp\left(-\frac{1}{2}\alpha^2\right) E\left[\sum_{m=1}^{M} P_m I_0(\alpha R_m) \ln I_0(\alpha R_m)\right]$$

$$= \exp\left(-\frac{1}{2}\alpha^2\right) E[I_0(\alpha R_1) \ln I_0(\alpha R_1)] \sum_{m=1}^{M} P_m$$

$$= \exp\left(-\frac{1}{2}\alpha^2\right) E[I_0(\alpha R_1) \ln I_0(\alpha R_1)]$$

$$= \int_0^\infty r \exp\left[-\frac{1}{2}(r^2 + \alpha^2)\right] I_0(\alpha r) \ln I_0(\alpha r) \, dr \qquad (12)$$

We are now ready to maximize the information $I_M(\alpha, \mathbf{P})$ and thereby to determine the capacity of the MFSK channel.

IV. Maximization of Information and Wideband Capacity

All of the results of this section are summarized in Theorem 1. Let

$$I_M(\alpha) \overset{\Delta}{=} \sup_{\mathbf{P}} I_M(\alpha, \mathbf{P})$$

$$S_M = \sum_{m=1}^{M} I_0(\alpha R_M)$$

Note that $E[I_0(\alpha R_m)] = \mu$ for all $m = 1, 2, \cdots, M$, where

$$\mu = \int_0^\infty x \exp(-x^2/2) I_0(\alpha x) \, dx = \exp(\alpha^2/2) \qquad (13)$$

THEOREM 1.

(a) $I_M(\alpha) = \exp(-\alpha^2/2)$

$$\times \left\{ E[I_0(\alpha R_1) \ln I_0(\alpha R_1)] - E\left[\frac{S_M}{M} \ln \frac{S_M}{M}\right] \right\} \qquad (14)$$

(b) $I_M(\alpha) \geqslant I_{M-1}(\alpha) \qquad (15)$

(c) $I(\alpha) = \lim_{M \to \infty} I_M(\alpha)$

$$= \exp(-\alpha^2) \{ E[I_0(\alpha R_1) \ln I_0(\alpha R_1)] - \mu \ln \mu \} \qquad (16)$$

$$= \int_0^\infty r \exp\left(-\frac{r^2 + \alpha^2}{2}\right) I_0(\alpha r) \ln I_0(\alpha r) \, dr - \frac{\alpha^2}{2} \qquad (17)$$

Division of Eq. (17) by T and $C_\infty = S/N_0$ nats[1]/second yields the wideband capacity result stated in Eq. (1).

Proof:

(a) Maximization. Since the first term in Eq. (11) is independent of P, we maximize $I_M(\alpha, \mathbf{P})$ by minimizing the second term

$$G(\mathbf{P}) = E\left[\sum_{m=1}^{M} P_m I_0(\alpha R_M) \ln \sum_{k=1}^{M} P_k I_0(\alpha R_k)\right] \qquad (18)$$

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[1]One bit = 0.693 nats.

We observe that $G(\mathbf{P})$ is the expectation of the convex function $x \ln x$; therefore, $G(\mathbf{P})$ is also convex, and so any critical point $\mathbf{P}_0$ will produce the global minimum. To determine such a critical point, we set

$$\frac{\partial}{\partial P_i}\left[G(\mathbf{P}) - \lambda \left(\sum_{m=1}^{M} P_m - 1 \right) \right] = 0 \qquad (19)$$

where λ is a Lagrange multiplier accounting for the constraint on the sum of the probabilities being unity. The result is

$$E\left[I_0(\alpha R_i) \ln \sum_{m=1}^{M} P_m I_0(\alpha R_m) + I_0(\alpha R_i) \right] = \lambda$$

or

$$E\left[I_0(\alpha R_i) \ln \sum_{m=1}^{M} P_m I_0(\alpha R_m) \right] = \text{constant for } i \leq M$$

$$(20)$$

Setting

$$P_1 = P_2 = \cdots = P_M = \frac{1}{M}$$

produces a probability vector which does the job.

(b) Monotonicity. By the foregoing, $G(\mathbf{P}_0) \leq G(\mathbf{P}')$. Take P' such that

$$P_1' = P_2' = \cdots = P_{M-1}' = \frac{1}{M-1}$$

and $P_M' = 0$. Obviously P' is a critical point for I_{M-1} but not necessarily for I_M; consequently,

$$I_M(\alpha) \equiv I_M(\alpha, \mathbf{P}_0) \geq I_M(\alpha, \mathbf{P}') \equiv I_{M-1}(\alpha) \qquad (21)$$

(c) Wideband capacity. Here we must prove that

$$\lim_{M \to \infty} E\left[\frac{S_M}{M} \ln \frac{S_M}{M} \right] = \mu \ln \mu$$

Since S_M is the sum of M identically distributed random variables

$$I_0(\alpha R_1), I_0(\alpha R_2) \cdots I_0(\alpha R_M)$$

we have, by the strong law of large numbers (Ref. 3),

$$\lim_{M \to \infty} \frac{S_M}{M} = \mu \qquad \text{a.e.}$$

and also

$$\lim_{M \to \infty} \frac{S_M}{M} \ln \frac{S_M}{M} = \mu \ln \mu \qquad \text{a.e.}$$

Consequently, one is inclined to conjecture that

$$\lim_{M \to \infty} E\left[\frac{S_M(\alpha)}{M} \ln \frac{S_M(\alpha)}{M} \right] = \mu \ln \mu = \frac{\alpha^2}{2} \exp(\alpha^2/2)$$

We use martingale theory (Ref. 3) to establish this.

Definition:

A sequence of random variables $V_1, V_2, \cdots$ forms a (sub-) martingale with respect to the increasing sequence of Borel fields $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$ if and only if

(i) V_n is measurable with respect to $\mathcal{G}_n$.

(ii) $E[|V_n|] < \infty$.

(iii) $E[V_n | \mathcal{G}_{n-1}] (\geq) = V_n$. \qquad a.e.

Let $X_1, X_2, \cdots$ be identically distributed random variables having finite mean. Let $S_n = X_1 + \cdots X_n$. Let $\beta_n = \beta(S_n, S_{n+1}, \cdots)$, the Borel field generated by $S_n, S_{n+1}, \cdots$. Let $Z_n = S_n/n$. The term Z_n is measurable with respect to β_n, while $\{ \cdots, \beta_n, \beta_{n-1}, \cdots, \beta_1 \}$ is a sequence of increasing Borel fields.

LEMMA 1.

$\{ \cdots, Z_n, Z_{n-1}, \cdots, Z_1 \}$ is a martingale with respect to $\{ \cdots, \beta_n, \beta_{n-1}, \cdots, \beta_1 \}$.

Proof:

(i) By construction, Z_n is measurable with respect to β_n.

(ii) $E[|Z_n|] = E\left[\frac{|S_n|}{n} \right] \leq \sum_{j=1}^{n} E\left[\frac{|X_j|}{n} \right]$

$= E[|X_1|] < \infty.$

(iii) By permutability of the X_j's,

$$E[X_j | S_n] = E[X_1 | S_n] \qquad \text{a.e.}$$

Thus

$$S_n = E[S_n | S_n] = E\left[\sum_{j=1}^{n} X_j \Big| S_n \right]$$

$$= \sum_{i=1}^{n} E[X_j | S_n] = nE[X_1 | S_n] \qquad \text{a.e.}$$

or

$$E[X_1 | S_n] = S_n/n \qquad \text{a.e.}$$

Now consider

$$E[Z_{n-1} \mid Z_n] = E\left[\frac{S_{n-1}}{n-1} \Big| S_n\right] = \frac{1}{n-1} E\left[\sum_{j=1}^{n-1} X_j \mid S_n\right]$$

$$= \frac{1}{n-1} \sum_{j=1}^{n-1} E[X_j \mid S_n]$$

$$= \frac{1}{n-1} \sum_{j=1}^{n-1} \frac{S_n}{n} = \frac{S_n}{n} = Z_n \qquad \text{a.e.}$$

Equivalently,

$$E[Z_{n-1} \mid \beta_n] = Z_n \qquad \text{a.e.}$$

and so the lemma holds.

COROLLARY 1.

If ϕ is a function which is convex and continuous on a convex set containing the range of X_1 and if

$$E[|\phi(X_1)|] < \infty$$

then $\{\phi(S_n/n)\}_\infty$ is a submartingale and

$$E\left[\phi\left(\frac{S_n}{n}\right)\right] \text{ decreases monotonically to } \phi(E[X_1])$$

Applying this to the function $\phi(X) = X \ln X$ with $X = I_0(\alpha r_1)$ yields the result desired in Theorem 1(c).

Proof:

Since

$$\left\{ \cdots, \frac{S_n}{n}, \frac{S_{n-1}}{n-1}, \cdots, S_1 \right\}$$

is a martingale,

$$\left\{ \cdots, \phi\left(\frac{S_n}{n}\right), \phi\left(\frac{S_{n-1}}{n-1}\right), \cdots, \phi(S_1) \right\}$$

is a submartingale. By the martingale convergence theorem, there exists a unique random variable w, measurable with respect to $\beta_\infty = \bigcap_n \beta_n$, such that

$$\lim_{n\to\infty} \phi\left(\frac{S_n}{n}\right) = w \qquad \text{a.e.}$$

and

$$\lim_{n\to\infty} E\,\phi\left(\frac{S_n}{n}\right) = E[w]$$

By the strong law of large numbers,

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_1] \qquad \text{a.e.}$$

Since ϕ is continuous,

$$\lim_{n\to\infty} \phi\left(\frac{S_n}{n}\right) = \phi\left(\lim_{n\to\infty} \frac{S_n}{n}\right) = \phi(E[X_1]) \qquad \text{a.e.}$$

Hence

$$E\left[\phi\left(\frac{S_n}{n}\right)\right] \to E[\phi(E[X_1])] = \phi(E[X_1])$$

Moreover, since

$$\left\{ \cdots, \phi\left(\frac{S_n}{n}\right), \cdots, \phi(S_1) \right\}$$

is a submartingale,

$$E\,\phi\left(\frac{S_n}{n}\right) \leq E\left[\phi\left(\frac{S_{n-1}}{n-1}\right)\right]$$

Therefore

$$E\left[\phi\left(\frac{S_n}{n}\right)\right]$$

downverges to $\phi(E[x_1])$.

V. Selection of the Number of Signals M

In the preceding sections, it was established that the capacity $C_M(\alpha)$ of an MFSK system increases monotonically to the limit $C(\alpha)$ as the number of signals M is increased. Practical systems must utilize a finite input alphabet not only because the bandwidth $W = M/2T$ is finite, but, more important, because the system complexity, as measured by the number of arithmetical operations, must be kept small. Since $C(\alpha)$ is finite, we know that $C_M(\alpha)$ can be pushed as close to this limit as necessary. For all practical purposes, however, it is sufficient to determine an M such that $C_M/C(\alpha)$ is near 100%. The purpose of this section is to determine useful analytical

bounds on M. It is necessary to consider the case when $\alpha^2/2 = ST/N_0$ is small separately from the case when ST/N_0 becomes large.

A. Behavior of $C_M(\alpha)$ vs M when ST/N_0 is small

Subtracting Eq. (14) from Eq. (16) yields

$$I(\alpha) - I_M(\alpha) = \frac{1}{\mu}\left\{ E\left[\frac{S_M}{M}\ln\frac{S_M}{M}\right] - \mu\ln\mu \right\}$$

$$= \frac{1}{\mu}\left\{ E\left[\frac{S_M}{M}\ln\frac{S_M}{M} - \frac{1}{2}\left(\frac{S_M}{M}\right)^2 + \frac{1}{2}\left(\frac{S_M}{M}\right)^2\right] - \mu\ln\mu \right\} \quad (22)$$

But

$$\phi(x) = x\ln x - \frac{1}{2}x^2$$

is convex and continuous for $x > 1$; consequently,

$$E[\phi(x)] \leq \phi(E[x])$$

Applying this result to the equation above produces

$$I(\alpha) - I_M(\alpha) \leq \frac{1}{\mu}\left\{ \mu\ln\mu - \frac{1}{2}\mu^2 + \frac{1}{2M^2}E[S_M^2] - \mu\ln\mu \right\}$$

But the expected value of the square of a sum of identically distributed random variables is

$$E[S_M^2] = M\sigma^2 + M^2\mu^2$$

where

$$\sigma^2 = E[I_0^2(\alpha r)] - \mu^2 = \int_0^\infty I_0^2(\alpha r)\, r \exp\left(-\frac{r^2}{2}\right) dr - \exp\alpha^2$$

$$= \sum_{n=0}^\infty \frac{(\alpha^2/2)^n}{n!}\left[\binom{2n}{n} - 2^n\right]$$

$$= \frac{\alpha^4}{4}\left(1 + \alpha^2 + \frac{9}{10}\alpha^4 + \cdots\right)$$

Therefore,

$$I(\alpha) - I_M(\alpha) \leq \frac{\sigma^2}{2\mu M} \quad (23)$$

Finally, since

$$C_M(\alpha)/C(\alpha) = I_M(\alpha)/I(\alpha)$$

we obtain

$$1 \geq \frac{C_M(\alpha)}{C(\alpha)} \geq 1 - \frac{\sigma^2}{2\mu M I(\alpha)} \quad (24)$$

Using our curve-fitting approximation

$$I(\alpha) \gtrsim \frac{\alpha^4}{8}\frac{1}{1 + \alpha^2/4}$$

and

$$\mu = \exp(\alpha^2/2) \approx 1 + \alpha^2/2 + \alpha^4/8 + \cdots$$

we obtain

$$1 \geq \frac{C_M(\alpha)}{C(\alpha)} \geq 1 - \frac{1 + 3/4\,\alpha^2 + 0(\alpha^4)}{M} \qquad \text{for } \alpha^2 < 1$$

Thus, as $\alpha^2/2 \to 0$,

$$C_M(\alpha)/C(\alpha) \to (M-1)/M$$

Achieving 99% of capacity requires an $M \geq 100$ even when $\alpha^2/2 \to 0$.

B. Behavior of $C_M(\alpha)/C(\alpha)$ With M When ST/N_0 Is Large

The lower bound on $C_M(\alpha)/C(\alpha)$ derived for small α is inadequate when α becomes large, because σ^2 increases as $\exp 2\alpha^2$, and this would require M to grow as $\exp(1.5\,\alpha^2)$, which is too large. We do know, however, that $\ln M$ must grow at least as fast as $I(\alpha) \approx \alpha^2/2$ for large α because the information at the input of the channel must be at least as high as the capacity.

Rewriting Eq. (22), we have

$$I - I_M = E\left[\frac{S_M}{\mu M}\ln\frac{S_M}{\mu M}\right]$$

which, by symmetry, is

$$I - I_M = E\left[\frac{X_M}{\mu}\ln\left(\frac{S_{M-1} + X_M}{\mu M}\right)\right] \quad [X_M = I_0(\alpha r_M)]$$

By conditional expectation,

$$I - I_M = E\left[\frac{X_M}{\mu}E\left[\ln\left(\frac{S_{M-1} + X_M}{\mu M}\right)\Big| X_M\right]\right]$$

By Jensen's inequality,

$$I - I_M \leq E\left[\frac{X_M}{\mu} \ln\left(\frac{E\left[S_{M-1}\right] + X_M}{\mu M}\right)\right] = E\left[\frac{X}{\mu} \ln\left(1 + \left(\frac{X}{\mu} - 1\right)\middle/ M\right)\right]$$

$$= \int_0^\infty r \exp\left[-\frac{1}{2}\left(r^2 + \alpha^2\right)\right] I_0\left(\alpha r\right) \ln\left(1 + \frac{I_0\left(\alpha r\right) \exp\left(-\alpha^2/2\right) - 1}{M}\right) dr$$

$$< \int_0^\infty r \exp\left[-\frac{1}{2}\left(r^2 + \alpha^2\right)\right] I_0\left(\alpha r\right) \ln\left[1 + I_0\left(\alpha r\right) \exp\left(-\alpha^2/2 - \ln M\right)\right] \tag{25}$$

Further analytical approximation can now be used to show that

$$(I - I_M)/I = 0\left(\frac{1}{\alpha}\right)$$

as $\alpha \to \infty$ when

$$\ln M = \frac{\alpha^2}{2} + K\alpha$$

so that, ultimately, when α is very large, taking $M \approx \exp\left(\alpha^2/2\right)$ will suffice to achieve $C_M\left(\alpha\right) \approx C\left(\alpha\right)$. To demonstrate this, note that

$$I_0\left(\alpha r\right) \leq e^{\alpha r} \tag{26}$$

and let

$$\ln M = \frac{1}{2} \alpha^2 + K\alpha \tag{27}$$

$$I - I_M < \int_0^\infty r I_0\left(\alpha r\right) \exp\left[-\frac{1}{2}\left(r^2 + \alpha^2\right)\right] \ln\left(1 + \exp\left(\alpha r - \alpha^2 - K\alpha\right)\right) dr$$

$$< \int_0^{\alpha+K} r I_0\left(\alpha r\right) \exp\left[-\frac{1}{2}\left(r^2 + \alpha^2\right)\right] \ln 2 \, dr$$

$$+ \int_{\alpha+K}^\infty r I_0\left(\alpha r\right) \exp\left[-\frac{1}{2}\left(r^2 + \alpha^2\right)\right]\left(\alpha r - \alpha^2 - K\alpha + \ln 2\right) dr$$

Combining the $\ln 2$ terms yields

$$I - I_M < \ln 2 + \int_{\alpha+K}^\infty r I_0\left(\alpha r\right) \exp\left[-\frac{1}{2}\left(r^2 + \alpha^2\right)\right]\left(\alpha r - \alpha^2 - K\alpha\right) dr$$

$$< \ln 2 + \alpha \int_{\alpha+K}^\infty r \exp\left[-\frac{1}{2}\left(r - \alpha\right)^2\right]\left(r - \alpha - K\right) dr$$

$$= \ln 2 + \alpha \int_K^\infty \left(r + \alpha\right)\left(r - K\right) \exp\left[-\frac{1}{2} r^2\right] dr$$

$$\leq \ln 2 + \frac{\alpha}{K}\left(1 + \frac{\alpha}{K}\right) \exp\left(-\frac{K^2}{2}\right) \quad \text{for } K > 1$$

$$\frac{I - I_M}{I} \leq \left(1 + \frac{4}{\alpha^2}\right)\left[\frac{2 \ln 2}{\alpha^2} + \frac{2}{\alpha K} \exp\left(-\frac{K^2}{2}\right) + \frac{2}{K^2} \exp\left(-\frac{K^2}{2}\right)\right] \tag{28}$$

Thus

$$\lim_{\alpha \to \infty} \frac{C_M}{C\left(\alpha\right)} \geq 1 - \frac{2}{K^2} \exp\left(-K^2/2\right) \tag{29}$$

This shows that the number of signals required to achieve a $C_M\left(\alpha\right)$ close to $C\left(\alpha\right)$ when α is large grows as

$$M \sim \exp\left[\alpha^2/2\left(1 + 2K/\alpha\right)\right] \tag{30}$$

VI. Conclusion

In summary, the MFSK system capacity behaves roughly as follows:

$$C_M = \frac{S}{N_0}\left(\frac{ST/N_0}{2 + ST/N_0}\right)$$

$$\times \begin{cases} 1 - \dfrac{1}{M} & \text{if } ST/N_0 < 1 \\[2ex] \dfrac{\ln M}{ST/N_0} & \text{if } ST/N_0 >> M >> 1 \\[2ex] 1 - \dfrac{\ln 2}{ST\,N_0} & \text{if } M \gtrsim ST/N_0 >> 1 \end{cases}$$

(31)

The middle term was obtained from the fact that $I_M \approx \ln M$ for $\alpha^2/2 >> \ln M$.

We observe that MFSK systems capable of receiving a reasonably small number of orthogonal signals, such as the $M = 64$ MFSK receiver under construction for the DSN, have the potential of operating above 90% of $C(\alpha)$ at predetection signal-to-noise ratios of less than about 2. This does not mean that systems with a limited number of signals are necessarily inefficient at high ST/N_0. The efficiency can be maintained by reducing T and increasing the rate proportionately, as is common practice even in the case of coherent systems. However, it must be remembered that $C(\alpha)$ is itself an increasing function of α, and this, together with code efficiency, must be taken into account when such tradeoffs are made.

Comparing these results with the hard-decision MFSK channel of Ref. 4, we note that the unquantized capacity always increases with M as opposed to first increasing and then decreasing as in the hard-quantized channel. This leads us to conjecture that unavoidable quantization in practical receivers ultimately destroys the monotonically increasing property in $C_M(\alpha)$, as observed in Ref. 5.

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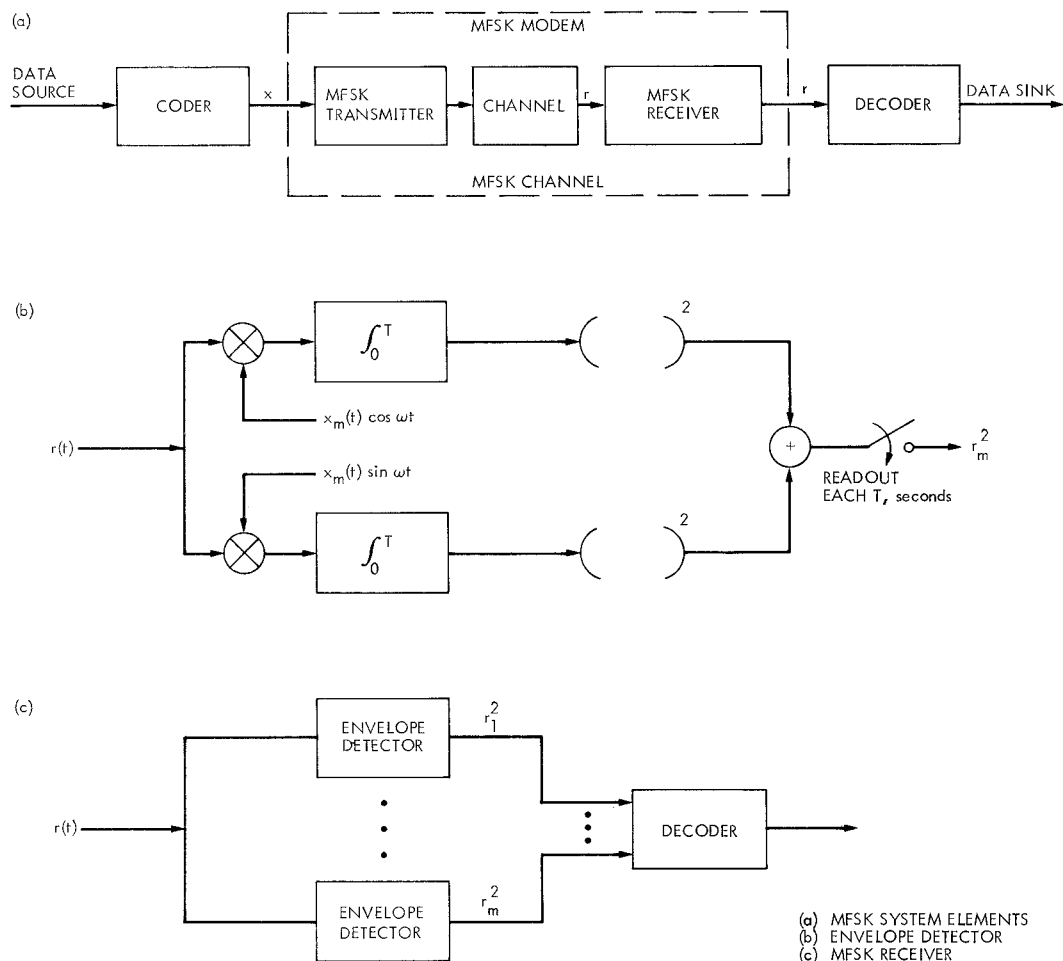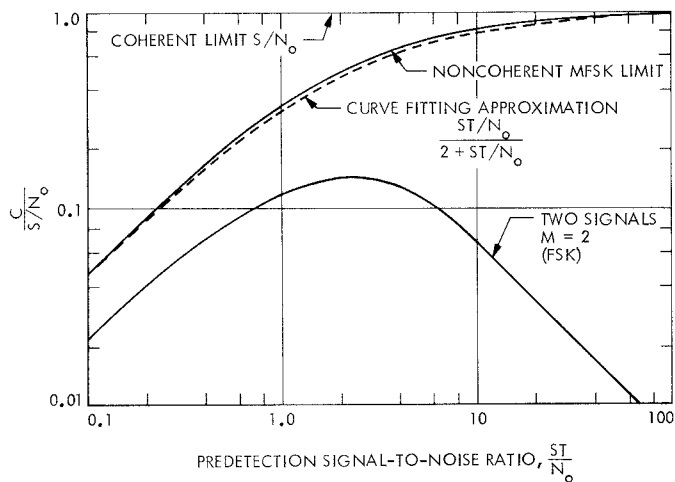
Fig. 1. MFSK channel

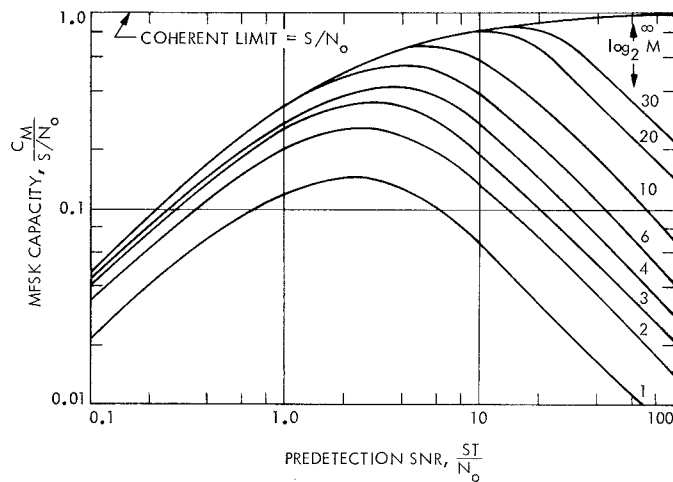Fig. 2. Wideband MFSK capacity as a function of predetection SNR

Fig. 3. Capacity of MFSK systems as a function of alphabet size (bandwidth) and predetection SNR