Derivation of a General Expression for Ionospheric Range Corrections Valid for Arbitrary Solar Zenith Angles, Azimuths, Elevation Angles and Station Locations

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A general expression is derived for the electron density profile as a function of latitude and longitude for that part of the Earth which is in direct sunlight including dawn and dusk. This expression allows one to determine by standard means the range correction for arbitrary ray path directions. It is also shown that the naive application of the Chapman ionosphere entails range correction errors which for low elevation angles (<20 deg) and large solar zenith angles (>40 deg) cannot be tolerated. Numerical calculations are displayed showing the dependence of the range correction on the pertinent parameters.

I. Introduction

It is well known that for high frequencies ω of a radio wave, the susceptibility of a plasma $\epsilon - 1$ is given by

$$\epsilon - 1 = -\frac{\omega_p^2}{\omega^2} \tag{1}$$

where $\omega_p = 4\pi\, e^2\, N/m$ is the plasma frequency proportional to the electron number density N (Ref. 1). A radio beam traversing the ionosphere for the purpose of tracking a satellite or a spacecraft will accordingly be affected by the plasma of the ionosphere.[1] It is also known (Refs. 3 and 4) that the quantity most important for the determination of the range correction is the total electron content:

$$I = \int N(s)\, ds \tag{2}$$

where the integral is taken over the ray path.

In order to successfully calibrate for range errors, I must therefore be known. Usually I is determined by means of Faraday rotation measurements (Ref. 5) along the ray path between a geostationary satellite and an observer on Earth. But, to calibrate the range to a spacecraft, the total electron content I in the ray path connecting the spacecraft with the observer must be known. In order to be able to determine this quantity, a mapping

[1] If the cyclotron frequency $\omega_c \ll \omega$, magnetic effects are negligible (Ref. 2) and Eq. (1) is a very good approximation.

from one ray path to the other must be performed. This can be done only with a model of the ionosphere. In the next Section we will derive such a model based on principles laid down and explored by Chapman many years ago (Ref. 6).

This three-parameter model will be applied subsequently to the determination of the ionospheric range correction for any arbitrary ray path arbitrarily located on the sunlit face of the Earth.

II. The Ionospheric Electron Profile

For small zenith angles, the Chapman electron density profile is given by the expression

$$N_e(Z) = N_{e\,\max} \exp\left\{\frac{1}{2}\left[1 - Z - \sec\chi \exp(-Z)\right]\right\} \tag{3}$$

where

$$Z = \frac{h - h_{\max}}{H}$$

$N_{e\,\max}$ = maximum electron concentration at altitude $h = h_{\max}$ and, for $\chi = 0$

χ = angle between the direction of the Sun's rays and the zenith

H = scale height of the ionosphere

From first principles it can be shown that the following expressions hold (see the derivation on pages 6 and 7);

$$\sigma_A N_0 H = \exp\left(\frac{h_{\max}}{H}\right) \tag{4a}$$

$$\left(\frac{\sigma_i N_0 S_\infty}{\alpha h\bar{\nu}}\right)^{1/2} = N_{e\,\max} \exp\left(\frac{1}{2} + \frac{1}{2}\frac{h_{\max}}{H}\right) \tag{4b}$$

relating the empirical quantities $N_{\max}$, $h_{\max}$ to basic quantities as ionization cross sections, etc. These relations are needed for our derivation of the electron profile. The meaning of the various terms in Eqs. (4) is:

σ_A = absorption cross section for ultraviolet radiation in air, cm^2

N_0 = number density of air molecules on ground ($h = 0$), cm^{-3}

S_∞ = intensity of the Sun's ultraviolet radiation outside the Earth's atmosphere, 10^{-7} J cm^{-2} s^{-1}

α = recombination coefficient of electrons with ions, cm^3 s^{-1}

$h\bar{\nu}$ = average photon energy, 10^{-7} J

All quantities above are averages over the ionizing part of the solar spectrum. As is seen from Eq. (3), for $\chi = 90$ deg $N_e(z)$ becomes identically zero, which obviously means that Eq. (3) breaks down at large χ. We will now give a derivation of the electron profile applicable for all χ. To be sure, a derivation for this situation has been given by Chapman long ago. But since that derivation is somewhat pedantic and not generally well known, we felt a concise and short derivation at this place is not out of order, all the more so since the profile derived here encompasses *all* zenith angles of the Sun χ and is valid in three dimensions, unlike most derivations, which are essentially two-dimensional.

The reasons for formerly confining oneself to only two-dimensional considerations are certain symmetries which allow one to map the electron density profile into three dimensions only if it is known along the Sun's meridian. This will be explained later. For computational ease, a three-dimensional mapping is far superior. To proceed with the derivation, the coordinate system used in this study will be defined first (Fig. 1). It is a right-handed Cartesian coordinate system with the x-axis pointing toward the Sun and the y-axis lying in the equatorial plane of the Earth.

Note that if the declination of the Sun is $\delta \neq 0$, the z-axis of this coordinate system does not coincide with the spin axis of the Earth. Figures 2a and 2b give cross-sections through the x–y and the x–z planes, respectively, of Fig. 1 for clarification. A polar coordinate system associated with the Cartesian system just introduced shall be designated by r, θ_0, ϕ_0, where these symbols have their usual meaning. Later we shall link these coordinates to the usual geographical coordinates with the help of the declination δ of the Sun and the universal time (UT) (see Section III).

In order to find out the number of electrons at r, θ_0, and ϕ_0, we must first know how much radiation is absorbed at r, how much of this radiation ionizes, how much recombines, and finally use the stationarity condition $\dot{N}_e = 0$ to eventually obtain the electron profile. The intensity of radiation absorbed at the point r is:

$$dS = \sigma_A N S d\ell \tag{5}$$

where $d\ell$ is a line element along a Sun's ray at position r, θ_0, ϕ_0; S is the solar radiation flux; and N is the number density of the neutral species assumed to obey an exponential law; viz.,

$$N = N_0 \exp\left(-\frac{r - R}{H}\right) \tag{6}$$

From Figs. 2a and 2b it is easy to deduce that the line element $d\ell$ can be expressed by

$$d\ell = dr\left(1 - \frac{y_0^2 + z_0^2}{r^2}\right)^{-1/2} \tag{7}$$

From Eqs. (5), (6), and (7) it follows that

$$\frac{dS}{S} = \sigma_A N_0 \exp\left(-\frac{r - R}{H}\right)\left(1 - \frac{y_0^2 + z_0^2}{r^2}\right)^{-1/2} dr \tag{8}$$

and since $S = S_\infty$ at $r = \infty$, we have

$$S = S_\infty \exp\left\{\sigma_A N_0 \int_\infty^r \exp\left(-\frac{x - R}{H}\right)\left(1 - \frac{y_0^2 + z_0^2}{x^2}\right)^{-1/2} dx\right\} \tag{9}$$

The number of ionized pairs created per second at r is now given by

$$\dot{n} = \sigma_i N \frac{S}{h\bar{\nu}} \tag{10}$$

The rate of electron production $\dot{n}$ and the rate of electron recombination $\alpha N_i N_e$, where N_i is the ion concentration, determines the net rate of change of the electron density:

$$\dot{N}_e = \dot{n} - \alpha N_i N_e \tag{11}$$

Assuming neutrality $N_i = N_e$ and stationarity $\dot{N}_e = 0$, we obtain

$$N_e = \sqrt{\frac{\dot{n}}{\alpha}} \tag{12}$$

the desired result.

From Eqs. (12), (10), and (9), there results

$$N_e = \left(\frac{\sigma_i S_\infty N_0}{\alpha h\bar{\nu}}\right)^{1/2} \exp\left\{\frac{1}{2}\left[-\frac{r - R}{H} + \sigma_A N_0 \int_\infty^r \exp\left(-\frac{x - R}{H}\right)\left(1 - \frac{y_0^2 + z_0^2}{x^2}\right)^{-1/2} dx\right]\right\} \tag{13}$$

We now wish to link the first-principle quantities σ_i, σ_A etc. to the three parameters which are universally adopted for a Chapman ionosphere, to wit, $N_{e\,\max}$, $h_{\max}$, and H. In Eq. (3), sec $\chi = 0$ corresponds to $y_0 = z_0 = 0$ in Eq. 13). Therefore, comparing Eq. (3) with Eq. (13), in this case, reveals that

$$\exp\left(\frac{h_{\max}}{H}\right) = \sigma_A N_0 H \tag{14a}$$

$$N_{e\,\max} \exp\left[\frac{1}{2}\left(1 + \frac{h_{\max}}{H}\right)\right] = \left(\frac{\sigma_i S_\infty N_0}{\alpha h\bar{\nu}}\right)^{1/2} \tag{14b}$$

in order that Eqs. (3) and (13) agree. Using Eqs. (14), we finally obtain for the electron density profile in standard notation.

$$N_e = N_{e\,\max} \exp\left\{\frac{1}{2}\left[1 - Z + \int_\infty^Z e^{-x}\left(1 - \frac{y_0^2 + z_0^2}{(Hx + R + h_{\max})^2}\right)^{-1/2} dX\right]\right\} \tag{15}$$

where

$$Z = \frac{r - R - h_{\max}}{H} \tag{16}$$

Finally, if we wish to determine the electron density profile radially above ground as a function of altitude Z at the arbitrary position θ_0 and ϕ_0, we must put

$$\left.\begin{aligned} y_0 &= (HZ + R + h_{\max}) \sin\theta_0 \sin\phi_0 \\ z_0 &= (HZ + R + h_{\max}) \cos\theta_0 \end{aligned}\right\} \tag{17}$$

and insert these expressions into Eq. (15). The range of Z extends from $Z = -h_{\max}/H$ at the surface of the Earth to infinity.

The great circle $\phi_0 = 0$, which intersects the x-axis and the z-axis as depicted in Fig. 1, is of course not unique in the sense that any great circle, provided it intersects the x-axis at $y = z = 0$, is equally good. The geometry of Fig. 1 is invariant with respect to rotations about the x-axis. If therefore the electron profile is known for all θ_0 and $\phi_0 = 0$, it is known throughout the lit hemisphere. In fact, the electron profile at $\phi_0 = 0$ and θ_0 is the same as that at $\theta_0 = \theta_1$ and $\phi_0 = \phi_1$ if

$$\sin\theta_0 = \sin\theta_1 \cos\phi_1 \tag{18}$$

Two-dimensional calculations as they have been done exclusively in the past (r, θ, ϕ = const.), would have been quite adequate. But since we are going to link the x, y, z system of Fig. 1 to an Earth-fixed geographical coordinate system to determine range corrections from arbitrary station locations into arbitrary directions, which are conveniently expressed in Earth-fixed coordinates, it was felt that a mapping of the electron density using one common coordinate system is advantageous.

Concluding this section, we note that no mention is made of the solar zenith angle χ in Eq. (15). For $\chi = 0$, Eq. (15) is identical with Eq. (3). But for all other zenith angles $\chi \neq 0$, Eq. (3) is only an approximation to the exact result of Eq. (15) (exact within the assumptions of this theory) and becomes progressively worse as χ increases. As a matter of fact, for $\chi = 90$ deg, Eq. (3) breaks down completely. However, Eq. (15) is still perfectly valid in this case. A glance at Fig. 1 shows that $\chi = 90$ deg corresponds to $\phi_0 = \pi/2$ or $\phi_0 = 3\pi/2$, in which case the integral of Eq. (15) is well defined and incidentally independent of θ_0 as it should be from symmetry.

The preceding derivation was made under the assumption that the three parameters $N_{e\,\max}$, $h_{\max}$, and H are global in the sense that they are independent of location. This is, of course, most certainly not true. However, the geometrical considerations here and in the next section are not affected by this unfortunate circumstance. The three parameters just mentioned may then be considered functions of θ_0 and ϕ_0.

In the next section we shall derive the range correction for an arbitrary ray path originating from an arbitrary location on Earth expressed in Earth-fixed coordinates as long as the station is located on the sunlit side of the Earth. We shall also dwell on the determination of the parameters determining the electron profile and show the errors made when using Eq. (3) instead of Eq. (15).

III. The Range Correction

It has been shown (Ref. 4) that the range correction due to a tenuous plasma is given by

$$\Delta\rho = \frac{2\pi e^2}{m\,\omega^2}\int_R^\infty dr \left(1 - \frac{R^2\cos^2\gamma}{r^2}\right)^{-1/2} N_e(s(r)) \tag{19}$$

The symbols in Eq. (19) have their usual meaning: $s(r)$ is the unperturbed (straight) ray path between the Earth-bound station and the distant space craft, expressed as a function of r, the distance from the center of the Earth. In the coordinate system displayed in Fig. 1, an arbitrary ray path or a line joining a station's antenna with the spacecraft may be obtained by the following expression:

$$\begin{aligned}\underset{\sim}{e}_{\text{ray}} = {} & (\sin\gamma\sin\theta_0\cos\phi_0 - \cos\gamma\cos\alpha\cos\theta_0\cos\phi_0 - \cos\gamma\sin\alpha\sin\phi_0)\underset{\sim}{e}_x \\ & + (\sin\gamma\sin\theta_0\sin\phi_0 - \cos\gamma\cos\alpha\cos\theta_0\sin\phi_0 + \cos\gamma\sin\alpha\cos\phi_0)\underset{\sim}{e}_y \\ & + (\sin\gamma\cos\theta_0 + \cos\gamma\cos\alpha\sin\theta_0)\underset{\sim}{e}_z\end{aligned} \tag{20}$$

In Eq. (18), θ_0 and ϕ_0 are the polar coordinates, as introduced earlier. On the other hand, γ is the elevation angle (the angle between the line of sight and the tangential plane at the surface of the Earth). The angle α is the azimuth of the line of sight. To be specific, α is the angle between the projection of the line of sight on the tangential plane and a meridian. We must emphasize that the coordinate system of Fig. 1 is geared to the Sun, in the sense that the x-axis points toward the Sun. Presently we shall give the coordinate transformation from the system of Fig. 1 to a geographical (Earth-fixed) system. The quantities $\underset{\sim}{e}$ in Eq. (19) are unit vectors along their specified directions. Before effectuating the coordinate transformation just mentioned, we shall give the range correction Eq. (18) concisely in the coordinate system as explained by Fig. 1.

Defining:

$$s = -R\sin\gamma + \sqrt{r^2 - R^2\cos^2\gamma} \tag{21}$$

we merely have to insert the following expressions for y_0 and z_0 into Eq. (15) in conjunction with Eq. (19) to obtain the range correction, to wit:

$$y_0 = s(\sin\gamma \sin\theta_0 \sin\phi_0 - \cos\gamma \cos\alpha \cos\theta_0 \sin\phi_0 + \cos\gamma \sin\alpha \cos\phi_0) + R\sin\theta_0 \sin\phi_0 \tag{22a}$$

and

$$z_0 = s(\sin\gamma \cos\theta_0 + \cos\gamma \cos\alpha \sin\theta_0) + R\cos\theta_0 \tag{22b}$$

If therefore the station location θ_0, ϕ_0 is known in this coordinate system and if the elevation angle γ as well as the azimuth α is also known, it is then an easy matter to determine the range correction using Eq. (19). However, coordinates are usually given in an Earth-fixed geographical coordinate system. We must therefore perform a transformation from the coordinate system of Fig. 1 to the conventional geographical system. Figure 3 will help to do this. There we show the old coordinate system x, y, z (x pointing toward the Sun and y lying in the equatorial plane) together with the new coordinate system X, Y, Z, where X and Y lie in the Earth's equatorial plane, and X defines the Greenwich meridian. The angle T between the Greenwich meridian and the direction x toward the Sun is called the universal time. The declination δ of the Sun is also known. From Fig. 3 it is clear that the connection between x, y, z and X, Y, Z is given by

$$\left.\begin{aligned} X &= x\cos T\cos\delta x - \sin Ty - \cos T\sin\delta z \\ Y &= y\sin T\cos\delta x + \cos Ty - \sin T\sin\delta z \\ Z &= z\sin\delta x + \cos\delta z \end{aligned}\right\} \tag{23}$$

Inverting Eqs. (22) yields

$$\left.\begin{aligned} x &= X\cos T\cos\delta X + \sin T\cos\delta Y + \sin\delta Z \\ y &= Y - \sin TX + \cos TY \\ z &= Z - \cos T\sin\delta X - \sin T\sin\delta Y + \cos\delta Z \end{aligned}\right\} \tag{24}$$

Let us introduce polar coordinates θ_G and ϕ_G, where θ_G is the geographical colatitude ranging from 0 to π and ϕ_G is the geographical longitude ranging from 0 (at Greenwich) to 2π (again at Greenwich). The other angles of the ray path, γ and α in the old coordinate system, are also partially affected by the coordinate transformation. It is clear that γ is unaffected; however α goes over into α_G. Presently, with the aid of Eqs. (23) and (24), we will give the connections between the angles θ_G, ϕ_G, α_G, and θ_0, ϕ_0, and α. They are obtained by expressing x,y,z and XYZ in their respective polar coordinates, and read

$$\cos\theta_G = \sin\delta\sin\theta_0\cos\phi_0 + \cos\delta\cos\theta_0 \tag{25a}$$

$$\begin{aligned} \sin\theta_G\cos\phi_G = &-\cos T\sin\delta\cos\theta_0 \\ &+ \cos T\cos\delta\sin\theta_0\cos\phi_0 \\ &- \sin T\sin\theta_0\sin\phi_0 \end{aligned} \tag{25b}$$

$$\begin{aligned} \sin\theta_G\sin\phi_G = &-\sin T\sin\delta\cos\theta_0 \\ &+ \sin T\cos\delta\sin\theta_0\cos\phi \\ &+ \cos T\sin\theta_0\sin\phi_0 \end{aligned} \tag{25c}$$

These equations may be used to obtain θ_0 and ϕ_0 once T and δ as well as the station location θ_G and ϕ_G are known. The three equations (25) are of course not independent. Any two of these equations may be used for the determination of θ_0 and ϕ_0. It is at the discretion of the programmer which equations to choose. Again, from Eqs. (20) and (24), it is easy to find expressions which determine the azimuth α once α_G, the azimuth given in an Earth-fixed system, is known. Here we display two equations which determine $\cos\alpha$ and $\sin\alpha$ as long as θ_0 and ϕ_0 have been computed from Eqs. (25):

$$\begin{aligned} \sin\gamma\cos\theta_G + \cos\gamma\cos\alpha_G\sin\theta_G = {}& \sin\delta(\sin\gamma\sin\theta_0\cos\phi_0 - \cos\gamma\cos\alpha\cos\theta_0\cos\phi_0 - \cos\gamma\sin\alpha\sin\theta_0) \\ &+ \cos\delta(\sin\gamma\cos\theta_0 + \cos\gamma\cos\alpha\sin\theta_0) \end{aligned} \tag{26a}$$

and

$$\begin{aligned} &\sin\gamma\sin\theta_G\cos\phi_G \\ &- \cos\gamma\cos\alpha_G\cos\theta_G\cos\phi_G \\ &- \cos\gamma\sin\alpha_G\sin\phi_G = \cos T\cos\delta(\sin\gamma\sin\theta_0\cos\phi_0 - \cos\gamma\cos\alpha\cos\theta_0\cos\phi_0 - \cos\gamma\sin\alpha\sin\phi_0) \\ &\qquad - \sin T(\sin\gamma\sin\theta_0\sin\phi_0 - \cos\gamma\cos\alpha\cos\theta_0\sin\phi_0 + \cos\gamma\sin\alpha\cos\phi_0) \\ &\qquad - \cos T\sin\delta(\sin\gamma\cos\theta_0 + \cos\gamma\cos\alpha\sin\theta_0) \end{aligned} \tag{26b}$$

It is now clear how to proceed in order to determine the range correction, Eq. (19). Once the parameters of the ray path are known it is not difficult to trace the analysis back with the help of the equations given above and actually compute the range correction for all contingencies. Suppose the colatitude θ_G and the longitude ϕ_G of a station are known, suppose also that the universal time T and the declination of the sun δ are known at a particular epoch.

It is then possible to determine the angles θ_0 and ϕ_0 with the help of Eqs. (25). Furthermore, the elevation angle γ and the azimuth α_G of the ray path are also known. It is then possible to determine α from Eqs. (26). Once θ_0, ϕ_0, and α (γ is obviously invariant) are ascertained in the manner just described, Eqs. (22) must be used in conjunction with Eqs. (19) and (15) to obtain the range correction.

After having determined all geometrical quantities, the three parameters of the ionosphere profile (the scale height H, the altitude at which the maximum electron density occurs, h_{max}, and the maximum electron density $N_{e\,max}$) must be determined. Faraday rotation measurements determine the total electron content, in essence the integral of expression (15) over the ray path between the observing station and a satellite. Ionosonde measurements determine $N_{e\,max}$ and h_{max} (Ref. 8).

Once h_{max} and $N_{e\,max}$ are known, it is possible to determine the scale height H from the total electron content as found from Faraday rotation measurements in conjunction with Eq. (15). Concluding this section, we display a few plots based on the theory developed on the preceding pages. Displayed are comparisons between the original Chapman equation (Eq. 3) and the more general expression, Eq. (15), as applied to the range correction, Eq. (19). The azimuth α is set equal to zero in all computations.

Figure 4 shows, for an elevation angle of $\gamma = 0$ deg, that for all solar zenith angles, except around 25 deg, the range equivalent difference between Eq. (15) and Eq. (3) is larger than 1 m. The Deep Space Network range accuracy is specified to be better than 1 m.

It is therefore clear that the Chapman ionosphere is inadequate. Figure 5 shows a similar comparison for an elevation angle of $\gamma = 15$ deg. The situation is somewhat better, however; at a solar zenith angle χ of 50 deg, the difference between the exact expression of Eq. (15) and the naive Chapman formulation is again an intolerable meter and becomes worse for larger χ.

Figure 6 displays the range correction as a function of the elevation angle γ rather than the solar zenith angle χ as in the preceding figures. Here it is seen that for large elevation angles (near zenith ranging) the discrepancy between the Chapman formulation (Eq. 3) and the exact Eq. (15)[2] is practically nonexistent. However, for elevation angles below 30 deg, the difference between the two formulations is again of the order of meters.

The conclusion is therefore inescapably that the Chapman electron density profile as given by Eq. (3) is inapplicable when solar zenith angles are large (>40 deg) and elevation angles are small (<30 deg).

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[2]Within the context of the underlying model.

References

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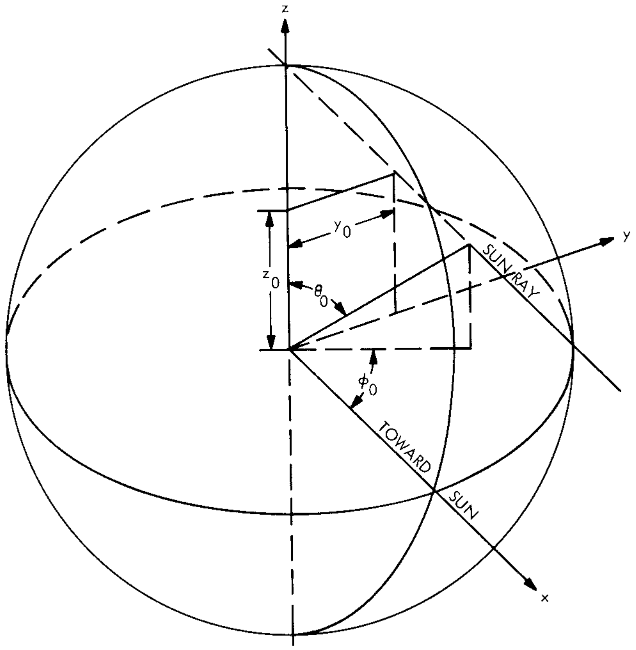

Fig. 1. Geometry of the system

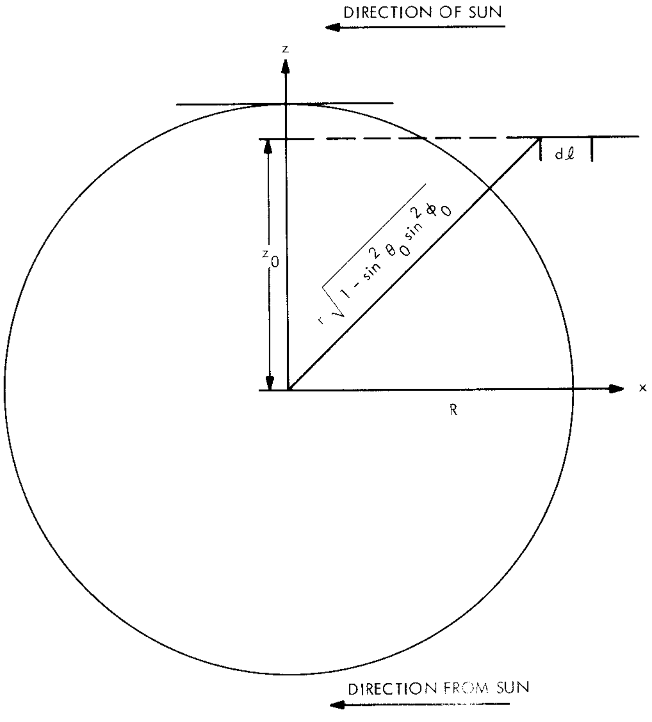

Fig. 2a. Cross section through the x–z plane of Fig. 1

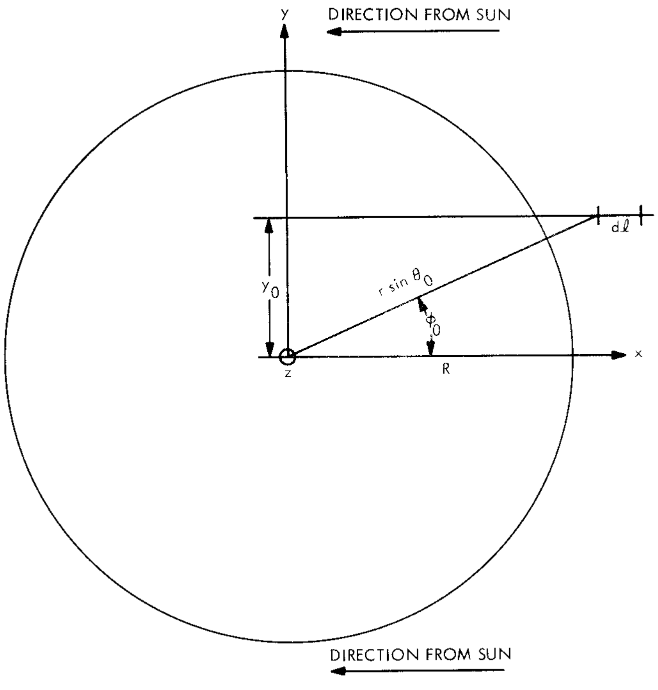

Fig. 2b. Cross section through the x–y plane of Fig. 1

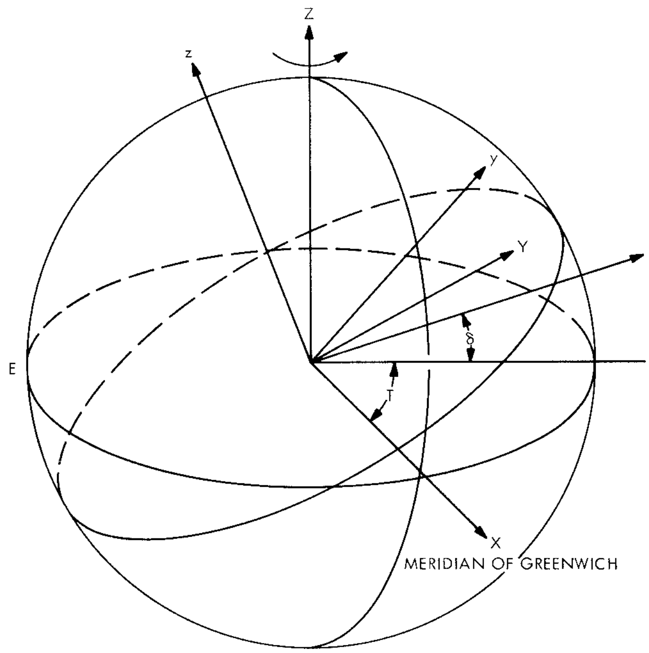

Fig. 3. Transformation from the Sun-oriented to the geographical coordinate system

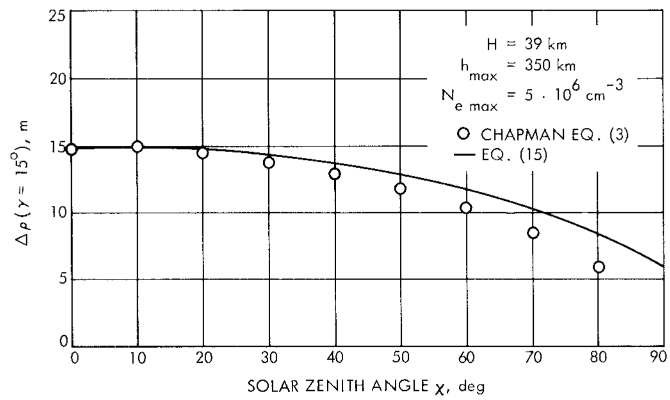

Fig. 5. Solar zenith angle vs range equivalent difference, for elevation angle of 15 deg

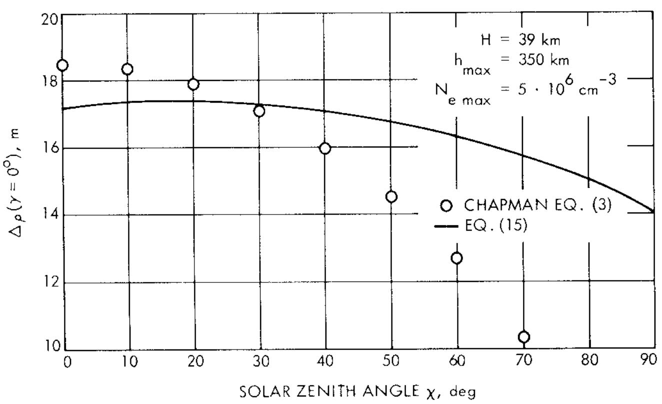

Fig. 4. Solar zenith angle vs range equivalent difference, for elevation angle of 0 deg

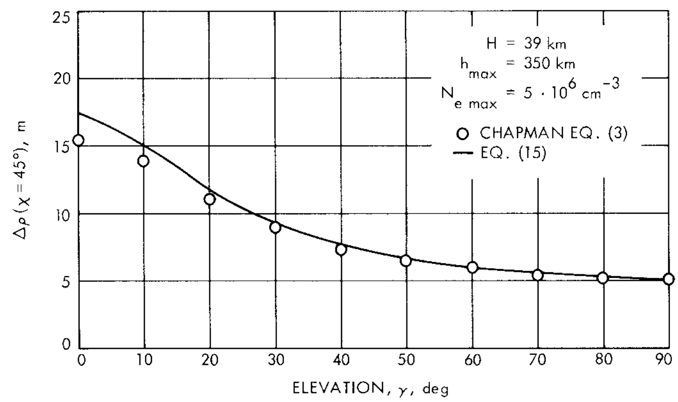

Fig. 6. Elevation angle vs range equivalent difference, for solar zenith angle of 45 deg